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Outer-Connected Semitotal Domination in Graphs

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Abstract. In this paper, we introduce and initiate the study of outer-connected semitotal domination in graphs. Given a graph G without isolated vertices, a set S of vertices of G is a semitotal dominating set if every vertex outside of S is adjacent to a vertex in S and every vertex in S is of distance at most 2 units from another vertex in S. A semitotal dominating set S of G is an outer-connected semitotal dominating set if either $S = V(G)$ or $S \neq V(G)$ satisfying the property that the subgraph induced by $V(G) \setminus S$ is connected. The smallest cardinality $\tilde{\gamma}_{t2}(G)$ of an outer-connected semitotal dominating set is the outer-connected semitotal domination number of G. First, we determine the specific values of $\tilde{\gamma}_{t2}(G)$ for some special graphs and characterize graphs G for specific (small) values of $\tilde{\gamma}_{t2}(G)$. Finally, we investigate the outer-connected semitotal dominating sets in the join, corona, and composition of graphs and, as a consequence, we determine their corresponding outer-connected semitotal domination numbers.

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1. Introduction

In 2014, the concept of semitotal domintion was introduced and investigated by W. Goddard, M. Henning, and C. McPillan (see [6]). Accordingly, it strengthens the concept of domination but relaxes the concept of total domination. Semitotal domination was further studied by M. Henning and A. Marcon (see [8]) in 2014 and 2016, by G. Hao and W. Zhuang (see [7]) in 2018, and by I. Aniversario *et al.* [1] in 2019.

In this present paper, inspired by the work of J. Cyman [4], on outer-connected domination, we introduce and initiate the study of the outer-connected semitotal domination in graphs. We investigate the concept in some special graphs and in graphs under some binary operations, such as the join, corona, and lexicographic products of graphs.

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2. Terminology and Notation

The symbols $V(G)$ and $E(G)$ denote the *vertex set* and *edge set*, respectively, of a graph G. For $S \subseteq V(G)$, $|S|$ is the cardinality of S. In particular, $|V(G)|$ and $|E(G)|$ are the *order* and *size*, respectively, of G. The *induced subgraph* $\langle S\rangle$ is the graph with vertex set S and such that $uv \in E(\langle S\rangle)$ if and only if $u, v \in S$ and $uv \in E(G)$. All graph terminologies that are not introduced but are being used here are adopted from [2].

Given two graphs G and H with disjoint vertex sets, the *join* $G+H$ of graphs G and H, is the graph with vertex-set $V(G+H) = V(G)\cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex of the i^{th} copy of H. The *lexicographic product or composition* of G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

For vertex u of G, all vertices adjacent to u constitute the set $N_G(u)$ called the *open neighborhood* of u. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. If $S \subseteq V(G)$, the *open neighborhood* of S in G is the set $N_G(S) = \cup_{u\in S}N_G(u)$. The *closed neighborhood* of S in G is the set $N_G[S] = N_G(S)\cup S$. We define $N_G(S) = \cup_{v\in S}N_G(v)$ and $N_G[S] = S \cup N_G(S)$. A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. Thus, S is a *dominating set* of G if and only if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. A set $S \subseteq V(G)$ is a *total dominating set* of G if for every $v \in V(G)$, there exists $u \in S$ such that $uv \in E(G)$. The minimum cardinality of a dominating set (resp. total dominating set) of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the *domination number* (resp. *total domination number*) of G. A dominating set (resp. total dominating set) S of G with $|S| = \gamma(G)$ (resp. $|S| = \gamma_t(G)$) is called a γ*-set* (resp. γ_t*-set*) of G. The authors always refer to [3] for the introduction and more comprehensive discussion of the development of the concept of domination in graphs.

A set $S \subseteq V(G)$ of a *graph* $G = (V, E)$ is called an *outer-connected dominating set* of G if the following hold: (i) S is a dominating set of G, and (ii) either $S = V(G)$ or the induced subgraph $\langle V(G) \setminus S\rangle$ of $V(G) \setminus S$ is connected. The cardinality of a minimum outer-connected dominating set of G is called the *outer-connected domination number* of G, and is denoted by $\tilde{\gamma}(G)$. For a graph G without isolated vertices, a set $S \subseteq V(G)$ is a *total outer-connected dominating set* if S is a total dominating set of G and the subgraph induced by $V(G) \setminus S$ is connected. The minimum cardinality of a total outer-connected dominating set in G is the *total outer-connected domination number* denoted by $\tilde{\gamma}_t(G)$. We refer to [4] and [5] for the introduction and results concerning outer-connected domination and total outer-connected domination, respectively, that are of interest in this study.

Suppose that G has no isolated vertices. A set $S \subseteq V(G)$ is a *semitotal dominating set* of G if S is a dominating set in G such that for every $x \in S$ there exists $y \in S \setminus \{x\}$ for which $d_G(x, y) \leq 2$. The smallest cardinality of a semitotal dominating set in G, denoted by $\gamma_{t2}(G)$, is called a *semitotal domination number* of G. A semitotal dominating set of G with cardinality $\gamma_{t2}(G)$ is called a γ_{t2}-set. Some results on semitotal domination in graphs are found in [1, 6–8].

A semitotal dominating set S is an *outer-connected semitotal dominating set* of G if either $S = V(G)$ or $S \neq V(G)$ and the induced subgraph $\langle V(G) \setminus S\rangle$ is connected. The smallest cardinality of an outer-connected semitotal dominating set in G, denoted by $\tilde{\gamma}_{t2}(G)$, is called the *outer-connected semitotal domination number* of G. An outer-connected semitotal dominating set in G with cardinality $\tilde{\gamma}_{t2}(G)$, is called a $\tilde{\gamma}_{t2}$-set.

For the purposes of this study, we write for $v \in V(G)$,

$$N_G^2(v) \ = \ \{u \in V(G) \setminus \{v\} : d_G(u, v) \leq 2\},$$

and write for $S \subseteq V(G)$,

$$N_G^2(S) = \cup_{v\in S} N_G^2(v).$$

3. Results

Observe that an outer-connected semitotal dominating set is both a semitotal dominating set and an outer-connected dominating set. On the other hand, a total outer-connected dominating set is an outer-connected semitotal dominating set. Thus,

$$\max\{2, \gamma_{t2}(G), \tilde{\gamma}(G)\} \leq \tilde{\gamma}_{t2}(G) \leq \tilde{\gamma}_t(G). \tag{1}$$

Strict inequalities in Equation 1 can be attained for a graph. To see this, consider the graph G in Figure 1. It can be verified that $\{a, b, c, d\}$, $\{x, y, z, w\}$, $\{x, y, z, w, a, b\}$ and $\{a, b, c, d, x, y, z\}$ are γ_{t2}-set, $\tilde{\gamma}$-set, $\tilde{\gamma}_{t2}$-set and $\tilde{\gamma}_t$-set, respectively. Thus, $\tilde{\gamma}(G) = 4 = \gamma_{t2}(G)$, $\tilde{\gamma}_{t2}(G) = 6$ and $\tilde{\gamma}_t(G) = 7$.

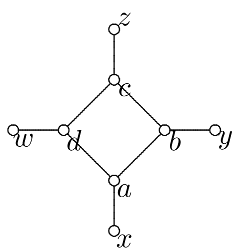

Figure 1: Graph G with $\max\{2, \gamma_{t2}(G), \tilde{\gamma}(G)\} < \tilde{\gamma}_{t2}(G) < \tilde{\gamma}_t(G)$

Proposition 1. *For path P_n on $n \geq 2$ vertices*

$$\tilde{\gamma}_{t2}(P_n) = \begin{cases} 2, & \text{if } n = 2 \\ n-1, & \text{if } 3 \leq n \leq 5 \\ n-2, & \text{if } n \geq 6. \end{cases}$$

Proof. The case where $n = 2$ is obvious. Assume $n \geq 3$. Let $S \subseteq V(P_n)$, be $\tilde{\gamma}_{t2}$ -set of P_n with $S \neq V(P_n)$. Then $P = \langle V(Pn) \setminus S\rangle$ is a path. Suppose that $[x, y, z]$ is a geodesic in P. Then $y \notin N_{P_n}[S]$, a contradiction. Thus, $|V(P)| = 1$ or 2. Consequently, $\tilde{\gamma}_{t2}(P_n) = |S| \geq n-2$. It is can readily be verified that if $3 \leq n \leq 5$, $|S| = n-1$. That is, $\tilde{\gamma}_{t2}(P_n) = n-1$. Suppose that $n \geq 6$. Put $P_n = [x_1, x_2, \ldots, x_n]$. Since $S = \{x_1, x_2, x_5, x_6, \ldots, x_n\}$ is an outer-connected semitotal dominating set of P_n, $\tilde{\gamma}_{t2}(P_n) \leq |S| = n-2$. Therefore, $\tilde{\gamma}_{t2}(P_n) = n-2$. □

Proposition 2. *For cycle C_n on $n \geq 3$ vertices*

$$\tilde{\gamma}_{t2}(C_n) = \begin{cases} 2, & \text{if } n = 3 \\ n-2, & \text{if } n \geq 4. \end{cases}$$

Proof. The case for $n = 3$ is trivial. Assume that $n \geq 4$, and say $C = [x_1, x_2, \ldots, x_n, x_1]$. Since $\{x_3, x_4, \ldots, x_n\}$ is an outer-connected semitotal dominating set of C_n, $\tilde{\gamma}_{t2}(C_n) \leq n-2$. Following similar arguments used above, if $S \subseteq V(C_n)$ is a $\tilde{\gamma}_{t2}$- set of C_n and $P = \langle V(C_n) \setminus S \rangle$, then P is a path with $1 \leq |V(P)| \leq 2$. Consequently, $\tilde{\gamma}_{t2}(C_n) = |S| \geq n-2$. □

Proposition 3. *For complete multipartite graph $K_{n_1,n_2,\ldots,n_t}$ of order $n = n_1+n_2+\ldots+n_t$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_t$ and $t \geq 2$,*

$$\tilde{\gamma}_{t2}(K_{n_1,n_2,\ldots,n_t}) = \begin{cases} n_2, & \text{if } t = 2 \text{ and } n_1 = 1, n_2 \geq 2 \\ 2, & \text{else.} \end{cases}$$

Proof. Put $G = K_{n_1,n_2,\ldots,n_t}$, and let $U_1, U_2, \ldots, U_t$ be the partite sets of G. First, observe that if $n_t = 1$, then $G = K_t$ and $\tilde{\gamma}_{t2}(G) = 2$. Assume that $n_t \geq 2$. We consider the following cases:

Case 1: Suppose that $n_1 = 1$. We consider the following subcases:

Subcase 1.1: If $t = 2$ and $n_2 \geq 2$, then $G = K_{1,n_2}$, a star of order $n \geq 3$. If $S \subseteq V(G)$ is an outer-connected semitotal dominating set of G, then either $|S| = n-1 = n_2$ or $|S| = n$. Thus, $\tilde{\gamma}_{t2}(G) = n_2$.

Subcase 1.2: Suppose that $t \geq 3$ and $n_1 = n_2 = 1$ such that G is not a path. Pick $u \in U_2$ and $v \in U_3$. Then $S = \{u, v\}$ is an outer-connected semitotal dominating set of G. Thus, $\tilde{\gamma}_{t2}(G) = 2$.

Subcase 1.3: Suppose that $t \geq 3$ and $n_2 \geq 2$. Pick $u \in U_1$ and $v \in U_2$. Then $S = \{u, v\}$ is an outer-connected semitotal dominating set of G. Thus, $\tilde{\gamma}_{t2}(G) = 2$.

Case 2: Suppose that $t \geq 2$ and $n_k \geq 2$ for all $k \in \{1, 2, \ldots, t\}$. Pick $u \in U_1$ and $v \in U_2$. Then $S = \{u, v\}$ is an outer-connected semitotal dominating set of G. □

Proposition 4. *Let G be a connected graph of order $n \geq 2$. Then*

(*i*) $\tilde{\gamma}_{t2}(G) = 2$ *if and only if G can be obtained from a connected graph H of order $n-2$ by adding to H vertices u and v such that* $d_G(u, v) = 1$ or 2 *and $\{u, v\}$ dominates $V(H)$.*

(ii) $\tilde{\gamma}_{t2}(G) = n$ *if and only if* $G = K_2$.

Proof. Statement (i) is clear. By Proposition 3, if $n = 2$, then $\tilde{\gamma}_{t2}(K_n) = 2 = n$. Assume that $\tilde{\gamma}_{t2}(G) = n$. Suppose that $n \geq 3$. By Proposition 3, $G \neq K_n$. Let $[u, v, z]$ be a geodesic in G. Put $S = V(G) \setminus \{v\}$. Then S is a semitotal dominating set with $V(G) \setminus S = \{v\}$. Thus S is an outer-connected semitotal dominating set of G. Consequently, $\tilde{\gamma}_{t2}(G) \leq n - 1$, a contradiction. Therefore, $n = 2$. This proves (ii). □

Proposition 5. *Let* G *be any graph with nontrivial components* $C_1, C_2, \ldots, C_k$ *of orders* $n_1, n_2, \ldots, n_k$, *respectively. Then*

$$\tilde{\gamma}_{t2}(G) = \min\{\tilde{\gamma}_{t2}(C_j) + \sum_{i=1, i\neq j}^{k} n_i : j = 1, 2, \ldots, k\}.$$

Proof. Put $\alpha = \min\{\tilde{\gamma}_{t2}(C_j) + \sum_{i=1,i\neq j}^{k} n_i : j = 1, 2, \ldots, k\}$. Let $j \in \{1, 2, \ldots, k\}$, and choose a $\tilde{\gamma}_{t2}$-set S_j of C_j. Since $S = \left(\cup_{i=1;i\neq j}^{k} V(C_i)\right) \cup S_j$ is an outer-connected semitotal dominating set of G, $\tilde{\gamma}_{t2}(G) \leq |S| = \tilde{\gamma}_{t2}(C_j) + \sum_{i=1,i\neq j}^{k} n_i$. Since j is arbitrary, $\tilde{\gamma}_{t2}(G) \leq \alpha$.

Let $S \subseteq V(G)$ be an outer-connected semitotal dominating set of G. Since S is a semitotal dominating set of G, $S_j = S \cap V(C_j)$ is a semitotal dominating set of C_j for all $j = 1, 2, \ldots, k$. First, we claim that there exists $j \in \{1, 2, \ldots, k\}$ for which $S_i = V(C_i)$ for all i except possibly when $i = j$. Suppose that, to the contrary, there exist distinct $i, j \in \{1, 2, \ldots, k\}$ such that $S_i \neq V(C_i)$ and $S_j \neq V(C_j)$. Pick $u \in V(C_i) \setminus S_i$ and $v \in V(C_j) \setminus S_j$. Observe that $\langle V(G) \setminus S\rangle$ does not have a path joining u and v, a contradiction and thus, the claim is established. This means that, $S = S_j \cup \left(\cup_{i=1,i\neq j}^{k} V(C_i)\right)$ for some j. Next, we claim that S_j is an outer-connected semitotal dominating set of C_j. If $S = V(G)$, then $S_j = V(C_j)$ and we are done. Suppose that $S \neq V(G)$. Since S_j is a semitotal dominating set, it is left to verify that $\langle V(C_j) \setminus S_j\rangle$ is connected. But since $V(C_j) \setminus S_j = V(G) \setminus S$, the conclusion follows. Thus,

$$|S| = |S_j| + \sum_{i=1,i\neq j}^{k} n_i \geq \tilde{\gamma}_{t2}(C_j) + \sum_{i=1,i\neq j}^{k} n_i \geq \alpha.$$

Since S is arbitrary, $\tilde{\gamma}_{t2}(G) \geq \alpha$. □

Proposition 6. *Let* G *be a nontrivial graph.*

(i) *If* G *is connected, then* $\tilde{\gamma}_{t2}(G + K_1) = 2$.

(ii) *If* G *is disconnected with components* $C_1, C_2, \ldots, C_k$ *of orders* $n_1, n_2, \ldots, n_k$,

respectively, satisfying that $n_1 \leq n_2 \leq \cdots \leq n_k$, *then*

$$\tilde{\gamma}_{t2}(G + K_1) = \min\{\sum_{j=1}^{k} \gamma(C_j), 2 + \sum_{j=1}^{k-1} n_j\}.$$

Proof. Put $V(K_1) = \{u\}$. To prove (i), suppose that G is connected. Since G is nontrivial, G contains at least two vertices which are not cutvertices. Pick a non-cutvertex v of G. Then $S = \{u, v\}$ is an outer-connected semitotal dominating set of $G + K_1$. By Equation 1, $\tilde{\gamma}_{t2}(G + K_1) = 2$.

To prove (ii), suppose that G is disconnected with components C_1, C_2, $\ldots$, C_k of orders n_1, n_2, $\ldots$, n_k, respectively, and satisfying that $n_1 \leq n_2 \leq \cdots \leq n_k$. If $n_k = 1$, then $G + K_1$ is a star, and the result follows from (i). It is worth noting that in this case,

$$\sum_{j=1}^{k} \gamma(C_j) = k = |V(G)|.$$

Assume $n_k \geq 2$. First, let $S_j \subseteq V(C_j)$ be a γ-set of C_j for all $j = 1, 2, \ldots, k$. Then $S = \cup_{j=1}^{k} S_j$ is a semitotal dominating set of $G + K_1$. Since $u \in V(G + K_1) \setminus S$, S is an outer-connected semitotal dominating set of $G + K_1$. Thus,

$$\tilde{\gamma}_{t2}(G + K_1) \leq |S| = \sum_{j=1}^{k} \gamma(C_j).$$

Next, let $S = \left(\cup_{j=1}^{k-1} V(C_j)\right) \cup \{u, v\}$, where $v \in V(C_k)$ which is a non-cutvertex of C_k. Then S is an outer-connected dominating set of $G + K_1$. This means that

$$\tilde{\gamma}_{t2}(G + K_1) \leq 2 + \sum_{j=1}^{k-1} n_j.$$

Thus,

$$\tilde{\gamma}_{t2}(G + K_1) \leq \min\{\sum_{j=1}^{k} \gamma(C_j), 2 + \sum_{j=1}^{k-1} n_j\}.$$

Now, to get the other inequality, let $S \subseteq V(G + K_1)$ be a $\tilde{\gamma}_{t2}$-set of $G + K_1$. Then $S_j = S \cap V(C_j)$ is a dominating set of C_j for all $j \in \{1, 2, \ldots, k\}$. If $u \notin S$, then

$$\tilde{\gamma}_{t2}(G + K_1) = |S| = \sum_{j=1}^{k} |S_j| \geq \sum_{j=1}^{k} \gamma(C_j).$$

Suppose that $u \in S$. Since $\langle V(G + K_1) \setminus S \rangle$ is connected, $V(G + K_1) \setminus S = V(C_j) \setminus S_j$ for some j. Moreover, since S is a $\tilde{\gamma}_{t2}$-set of G, $j = k$. Thus,

$$S = \left(\cup_{i=1}^{k-1} V(C_i)\right) \cup S_k \cup \{u\}$$

so that

$$\tilde{\gamma}_{t2}(G + K_1) = |S| = 1 + |S_k| + \sum_{j=1}^{k-1} n_j \geq 2 + \sum_{j=1}^{k-1} n_j.$$

Therefore,

$$\tilde{\gamma}_{t2}(G + K_1) \geq \min\{\sum_{j=1}^{k} \gamma(C_j), 2 + \sum_{j=1}^{k-1} n_j\}.$$

□

Suppose that $C_j = K_1$ for all $j \in \{1, 2, \ldots, k-1\}$ in Proposition 3. If $\gamma(C_k) = 1$, then

$$\sum_{j=1}^{k} \gamma(C_j) < 2 + \sum_{j=1}^{k-1} n_j.$$

On the other hand, if $\gamma(C_k) \geq 3$, then

$$\sum_{j=1}^{k} \gamma(C_j) > 2 + \sum_{j=1}^{k-1} n_j,$$

and attain equality if $\gamma(C_k) = 2$.

Theorem 1. [1] *Let G and H be nontrivial graphs, and $S \subseteq V(G + H)$. Then S is a semitotal dominating set in $G + H$ if and only if one of the following holds:*

(i) $S \subseteq V(G)$ *is a nonsingleton dominating set in G;*

(ii) $S \subseteq V(H)$ *is a nonsingleton dominating set in H;*

(iii) $S \cap V(G) \neq \varnothing$ and $\mathrm{S} \cap \mathrm{V(G)} \neq \varnothing$.

Theorem 2. *Let G and H be any nontrivial graphs, and $S \subseteq V(G + H)$, then S is an outer-connected semitotal dominating set in $G+H$ if and only if one of the following holds:*

(i) $S \subseteq V(G)$ *and one of the following holds:*

- *(a)* $S = V(G)$ *and H is connected;*
- *(b)* $S \neq V(G)$ *and S is a nonsingleton dominating set in G.*

(ii) $S \subseteq V(H)$ *and one of the following holds:*

- *(a)* $S = V(H)$ *and G is connected;*

(b) *$S \neq V(H)$ and S is a nonsingleton dominating set in H.*

(iii) *$S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$ such that if $S \neq V(G+H)$, then one of the following holds:*

(a) *$V(G) \subseteq S$ and $\langle V(H) \setminus S\rangle$ is connected;*

(b) *$V(H) \subseteq S$ and $\langle V(G) \setminus S$ is a connected;*

(c) *$V(G) \setminus S \neq \varnothing$ and $V(H) \setminus S \neq \varnothing$.*

Proof. Assume that S is an outer-connected semitotal dominating set of $G+H$. Suppose that $S \subseteq V(G)$. If $S = V(G)$, then $H = \langle V(G+H) \setminus S\rangle$ is connected, and $(i)(a)$ holds. Suppose that $S \neq V(G)$. Since S is a semitotal dominating set of $G+H$, S is a nonsingleton dominating set of G, and $(i)(b)$ holds. Similarly, if $S \subseteq V(H)$, then (ii) holds. Now, assume that $S_G = S \cap V(G) \neq \varnothing$ and $S_H = S \cap V(H) \neq \varnothing$. Suppose further that $S \neq V(G+H)$. Statement $(iii)(a)$ follows from the fact that if $V(G) \subseteq S$, then $\langle V(G+H) \setminus S\rangle = \langle V(H) \setminus S_H\rangle$ is connected. Similarly, if $V(H) \subseteq S$, then $(iii)(b)$ holds. If both $(iii)(a)$ and $(iii)(b)$ do not hold, then necessarily, $(iii)(c)$ holds.

Conversely, suppose that $S \neq V(G+H)$ satisfying condition (i). Then $S \subseteq V(G)$ and is a nonsingleton dominating set of G. By Theorem 1, S is semitotal dominating set of $G+H$. If $S = V(G)$, then $\langle V(G+H) \setminus S\rangle = H$, which by $(i)(a)$ is connected. Suppose that $S \neq V(G)$. Then $\langle V(G+H) \setminus S\rangle = \langle (V(G) \setminus S) \cup V(H)\rangle$ is clearly connected. This makes S an outer-connected semitotal dominating set of $G+H$. Similarly, if (ii) holds, then S is an outer-connected semitotal dominating set of $G+H$. Finally, suppose that (iii) holds. By Theorem 1, S is a semitotal dominating set of $G+H$. If $V(G) \subseteq S$, then $\langle V(G+H) \setminus S\rangle = \langle V(H) \setminus S\rangle$, which is connected by $(iii)(a)$. Similarly, if $(iii)(b)$ holds, then S is an outer-connected semitotal dominating set of $G+H$. Following similar arguments as above, if $(iii)(c)$ holds then S is an outer-connected semitotal dominating set of $G+H$. □

Corollary 1. *For all nontrivial graphs G and H, $\tilde{\gamma}_{t2}(G+H) = 2$.*

Proof. Pick $u \in V(G)$ and $v \in V(H)$. By Theorem 2, $S = \{u, v\}$ is an outer-connected semitotal dominating set of $G+H$. Thus, $\tilde{\gamma}_{t2}(G+H) \leq 2$. Finally, by (1), $\tilde{\gamma}_{t2}(G+H) = 2$. □

Theorem 3. *Let G be a nontrivial connected graph and $S \subseteq V(G \circ K_1)$. Then S is an outer-connected semitotal dominating set of $G \circ K_1$ if and only if one of the following holds for S:*

(i) *$S = V(G \circ K_1) \setminus V(K_1^v)$ for some $v \in V(G)$;*

(ii) *$S = A \cup \left(\cup_{v\in V(G)} V(K_1^v)\right)$, where $A \subseteq V(G)$ is an outer-connected dominating set of G.*

Proof. Put $V(K_1) = \{x\}$. Then $V(K_1^v) = \{x^v\}$. Assume that S is an outer-connected semitotal dominating set of $G \circ K_1$. We consider two cases:

Case 1: Suppose that $x^v \notin S$ for some $v \in V(G)$. Since S is a dominating set of $G \circ K_1$, $v \in S$. Since $x^v \in V(G \circ K_1) \setminus S$ and $\langle V(G \circ K_1) \setminus S\rangle$ is connected, $V(G \circ K_1) \setminus S = \{x^v\}$. That is, $S = V(G \circ K_1) \setminus \{x^v\}$. In this case, (i) holds.

Case 2: Suppose that $x^v \in S$ for all $v \in V(G)$. Define $A = S \cap V(G)$. Then

$$S = A \cup \left(\cup_{v \in V(G)} \{x^v\}\right).$$

We claim that A is an outer-connected dominating set of G. First, let $v \in V(G) \setminus A$. Then $x^v \in S$. Since S is a semitotal dominating set of $G \circ K_1$, there exists $u \in S$ for which $d_{G \circ K_1}(x^v, u) \leq 2$. Because $v \notin S$, $u \in A \cap N_G(v)$. Since v is arbitrary, A is a dominating set of G. Note further that , $V(G) \setminus A = V(G \circ K_1) \setminus S$. Thus, A is an outer-connected dominating set of G. In this case, (ii) holds.

Conversely, obviously, if condition (i) holds for S, then S is an outer-connected semitotal dominating set of $G \circ K_1$. Now, suppose that condition (ii) holds for S. Since $\cup_{v \in V(G)} \{x^v\}$ is a dominating set of $G \circ K_1$, S is a dominating set of $G \circ K_1$. Let $u \in S$. We consider the following cases:

Case 1: Suppose that $u = x^v$ for some $v \in V(G)$. If $v \in S$, then we pick v for $d_G(u, v) \leq 2$. Suppose that $v \notin S$. Since A is a dominating set of G and $v \in V(G) \setminus A$, there exists $w \in A \subseteq S$ such that $wv \in E(G) \subseteq E(G \circ K_1)$. Since $d_{G \circ K_1}(u, w) = 2$, w is the desired vertex.

Case 2: Suppose that $u \in V(G)$. In this case, we pick $x^u \in S$. Note that $ux^u \in E(G \circ K_1)$.

The above cases show that S is a semitotal dominating set of $G \circ K_1$. Finally, since A is an outer-connected dominating set of G, $\langle V(G) \setminus A\rangle = \langle V(G \circ K_1) \setminus S\rangle$ is connected. Therefore, S is an outer-connected semitotal dominating set of $G \circ K_1$. □

Corollary 2. *For nontrivial connected graph G of order n,*

$$\tilde{\gamma}_{t2}(G \circ K_1) = n + \tilde{\gamma}(G).$$

Proof. In view of Theorem 3,

$$\tilde{\gamma}_{t2}(G \circ K_1) = \min\{2n - 1, n + \tilde{\gamma}(G)\} = n + \tilde{\gamma}(G).$$

□

Theorem 4. *Let G and H be nontrivial connected graphs, and let $S \subseteq V(G \circ H)$. Then S is an outer-connected semitotal dominating set of $G \circ H$ if and only if one of the following holds for S:*

(*i*) *There exists* $v \in V(G)$ *and* $B \subseteq V(H^v)$ *such that*

$$S = (V(G \circ H) \setminus V(H^v)) \cup B,$$

where either $B = V(H^v)$ *or* $\langle V(H^v) \setminus B\rangle$ *is connected.*

(*ii*)

$$S = A \cup (\cup_{x \in A} V(H^x)) \cup \left(\cup_{x \in V(G)\setminus A} S_x\right), \tag{2}$$

where $A \subseteq V(G)$ *and* $S_x \subseteq V(H^x)$ *for all* $x \in V(G) \setminus A$ *satisfying the following:*

(*a*) $\langle V(G) \setminus A\rangle$ *is connected;*

(*b*) *For each* $x \in V(G) \setminus A$*,* S_x *is a dominating set of* H^x*. Moreover, if* $|S_x| = 1$*, then* $A \cap N_G(x) \neq \varnothing$*.*

Proof. Assume that S is an outer-connected semitotal dominating set of $G \circ H$. If $S = V(G \circ H)$, then (i) holds. In what follows, we assume that $S \neq V(G \circ H)$. We consider two cases:

Case 1: Suppose that $V(G) \subseteq S$. Since $\langle V(G \circ H) \setminus S\rangle$ is connected, there exists $v \in V(G)$ and $B \subseteq V(H^v)$ such that $V(G \circ H) \setminus S = V(H^v) \setminus B$. That is,

$$S = (V(G \circ H) \setminus V(H^v)) \cup B$$

and $\langle V(H^v) \setminus B\rangle$ is connected.

Case 2: Suppose that $V(G) \nsubseteq S$. Put $A = S \cap V(G)$. If $A = \varnothing$, then (2) trivially holds with $S_x = S \cap V(H^x)$ for all $x \in V(G)$. Suppose that $A \neq \varnothing$. Since $\langle V(G \circ H) \setminus S\rangle$ is connected and $V(G) \setminus A \neq \varnothing$, $V(H^x) \subseteq S$ for all $x \in A$. Put $S_x = S \cap V(H^x)$ for all $x \in V(G) \setminus A$. Then Equation (2) holds for S. Statement $(ii)(a)$ follows immediately from the connectedness of $\langle V(G \circ H) \setminus S\rangle$. Now, let $x \in V(G) \setminus A$ and $u \in V(H^x) \setminus S_x$. Since S is a dominating set of $G \circ H$ and $u \notin S$, there exists $w \in S$ for which $uw \in E(G \circ H)$. Since $x \notin S$, $w \neq x$ so that $w \in S_x$. This means that S_x is a dominating set of H^x. Suppose further that $|S_x| = 1$, say $S_x = \{u\}$. Because S is a semitotal dominating set of $G \circ H$, there exists $w \in S \setminus \{u\}$ such that $d_{G \circ H}(u, w) \leq 2$. Since $w \notin S_x$, $w \in A \cap N_G(x)$, and Statement $(ii)(b)$ holds.

Conversely, suppose that condition (i) holds. Since $V(G) \subseteq S$, S is a semitotal dominating set of $G \circ H$. Moreover, $V(G \circ H) \setminus S = V(H^v) \setminus B$ so that S is an outer-connected semitotal dominating set of $G \circ H$. Now, suppose that condition (ii) holds for S. By condition $(ii)(b)$, S is a dominating set of $G \circ H$. Suppose that $A = \varnothing$. Then $S = \bigcup_{x \in V(G)} S_x$, and by condition $(ii)(b)$, S_x is a nonsingleton dominating set of H^x for all $x \in V(G)$. Note that for each $x \in V(G)$, $d_{G \circ H}(u, v) \leq 2$ for all $u, v \in S_x$. It follows that S is a semitotal dominating set of $G \circ H$. Further, since $V(G) \subseteq V(G \circ H) \setminus S$, $\langle V(G \circ H) \setminus S\rangle$ is connected. Finally, suppose that $A \neq \varnothing$. Let $x \in S$. If $x \in A$, then $V(H^x) \subseteq S$. Pick $u \in V(H^x)$.

Then we have $u \in S$ and $d_{G\circ H}(x,u) = 1$. If $x \in V(H^v)$ for some $v \in A$, then v is the desired vertex in S for which $d_{G\circ H}(x,v) \leq 2$. Next, suppose that $x \in V(H^v)$ for some $v \in V(G) \setminus A$. If $|S_v| \geq 2$, then pick $u \in S_v \setminus \{x\}$. Then $u \in S$ and $d_{G\circ H}(x,u) \leq 2$. Lastly, suppose that $|S_v| = 1$, i.e., $S_v = \{x\}$. By condition $(ii)(b)$, the exists $z \in A \cap N_G(v)$. Then $z \in S$ and $d_{G\circ H}(x,z) = 2$. We have shown that S is a semitotal dominating set of $G \circ H$. Condition $(ii)(a)$ implies further that S is an outer-connected semtiotal dominating set of $G \circ H$. □

Corollary 3. *Let G and H be nontrivial connected graphs of orders n and m, respectively.*

(i) *If $\gamma(H) = 1$, then $\tilde{\gamma}_{t2}(G \circ H) \leq \min\{2n, n + m\tilde{\gamma}(G)\}$.*

(ii) *If $\gamma(H) \geq 2$, then*

$$\tilde{\gamma}_{t2}(G \circ H) \leq \min\{n\gamma(H), \tilde{\gamma}(G)\,(1 + m - \gamma(H)) + n\gamma(H)\}$$

Proof. Let $A \subseteq V(G)$ be a $\tilde{\gamma}$-set of G. For each $v \notin A$, let $S_v \subseteq V(H)$ be a γ-set of H^v. Define

$$S = A \cup \left(\bigcup_{v\in A} V(H^v)\right) \cup \left(\bigcup_{v\in V(G)\setminus A} S_v\right).$$

Since S satisfies Theorem 4(ii), S is an outer-connected semitotal dominating set of $G \circ H$. Thus,

$$\begin{aligned}
\tilde{\gamma}_{t2}(G \circ H) &\leq |S| \\
&= |A| + m|A| + (n - |A|)\,\gamma(H) \\
&= \tilde{\gamma}(G) + m\tilde{\gamma}(G) + (n - \tilde{\gamma}(G))\gamma(H) \\
&= \tilde{\gamma}(G)\,(1 + m - \gamma(H)) + n\gamma(H).
\end{aligned}$$

In particular, if $\gamma(H) = 1$, then $\tilde{\gamma}_{t2}(G \circ H) \leq n + m\tilde{\gamma}(G)$.

To complete the desired results, for the case where $\gamma(H) = 1$, let $y \in V(H)$ for which $N_H[y] = V(H)$. Pick a $z \in V(H) \setminus \{y\}$ and define $S_v = \{z, y\}$ for all $v \in V(G)$. On the other hand, if $\gamma(H) \geq 2$, then choose $S_v \subseteq V(H^v)$ to be a γ-set of H^v for all $v \in V(G)$. In any case, $S = \bigcup_{v\in V(G)} S_v$ satisfies Theorem 4(ii). Thus, S is an outer-connected semitotal dominating set of $G \circ H$. This means that if $\gamma(H) = 1$, then

$$\tilde{\gamma}_{t2}(G \circ H) \leq |S| = 2n.$$

For $\gamma(H) \geq 2$,

$$\tilde{\gamma}_{t2}(G \circ H) \leq |S| = n\gamma(H).$$

□

Corollary 4. *Let G and H be nontrivial connected graphs of orders n and m, respectively.*

(*i*) *If* $\gamma(H) = 1$, *then* $\tilde{\gamma}_{t2}\left((G + K_1) \circ H\right) = \min\{2n + 2, n + m + 1\}$.

(*ii*) *If* $\gamma(H) \geq 2$, *then*

$$\tilde{\gamma}_{t2}\left((G + K_1) \circ H\right) = \min\{(n + 1)\gamma(H), \tilde{\gamma}(G)\left(1 + m - \gamma(H)\right) + n\gamma(H)\}$$

Proof. Put $K = (G + K_1) \circ H$, and $\alpha = \min\{2n + 2, n + m + 1\}$. By Corollary 3, with $\tilde{\gamma}(G + K_1) = 1$, we have $\tilde{\gamma}_{t2}(K) \leq \alpha$.

Now, let $S \subseteq V(K)$ be an outer-connected semitotal dominating set of K. Since S is a dominating set of K, $S \cap V(H^x + x) \neq \varnothing$ for all $x \in V(G + K_1)$. First, suppose that $S \cap V(G + K_1) = \varnothing$. By Theorem 4(*ii*), $S \cap V(H^x)$ is a nonsingleton dominating set of $H^x + x$ for all $x \in V(G + K_1)$. This means that $|S| \geq 2(n + 1) = 2n + 2 \geq \alpha$. Next, suppose that $S \cap V(G + K_1) \neq \varnothing$. Clearly, if $V(G + K_1) \subseteq S$, then $|S| \geq 2n + 2$. Assume that $V(G+K_1) \setminus S \neq \varnothing$. Let $w \in S \cap V(G+K_1)$. Since S is an outer-connected semitotal dominating set of K and $V(G + K_1) \setminus S \subseteq V(K) \setminus S$, $V(H^w) \subseteq S$. This means that

$$|S| \geq |V(H^w + w)| + \sum_{x \in V(G+K_1) \setminus \{w\}} |S \cap V(H^x + x)| \geq m + 1 + n \geq \alpha.$$

Since S is arbitrary, $\tilde{\gamma}_{t2}(K) \geq \alpha$.

Similar arguments will prove (*ii*). □

It is worth noting that the wheel graphs and the fan graphs are among the graphs represented by $G + K_1$ in Corollary 4.

Theorem 5. [1] *Let* G *and* H *be nontrivial connected graphs, and let* $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G[H])$. *Then* C *is a semitotal dominating set in* $G[H]$ *if and only if one of the following holds:*

(*i*) S *is a total dominating set in* G;

(*ii*) S *is a semitotal dominating set in* G *and for each* $x \in S \setminus N_G(S), T_x$ *is a dominating set in* H;

(*iii*) S *is a dominating set in* G, *such that,* T_x *is a dominating set in* H *for each* $x \in S \setminus N_G(S)$, and $|T_x| \geq 2$ for each $x \in S \setminus N_G^2(S)$.

For $C \subseteq V(G[H])$, define $\overline{C}_G = \{x \in V(G) : (x, y) \notin C \text{ for some } y \in V(H)\}$.

Theorem 6. *Let* G *be a nontrivial connected graph and* $n \geq 2$, *and* $C = \cup_{x \in S}\left(\{x\} \times T_x\right) \neq V(G[K_n])$. *Then* C *is an outer-connected semitotal dominating set of* $G[K_n]$ *if and only if each of the following holds:*

(*i*) *One of the following holds:*

(*a*) S *is a semitotal dominating set in* G.

(b) *S is a dominating set in G such that $|T_x| \geq 2$ for each $x \in S \setminus N_G^2(S)$.*

(ii) *Exactly one of the following holds:*

(a) *$\overline{C}_G = \{x\}$ for some $x \in V(G)$.*

(b) *$|\overline{C}_G| \geq 2$ and for each distinct $u, v \in \overline{C}_G$, G has a u-v geodesic P for which either $S \cap V(P) = \varnothing$ or $|T_x| < n$ for each $x \in S \cap V(P)$.*

Proof. Assume that C is an outer-connected semitotal dominating set of $G[K_n]$. Since C is a semitotal dominating set of $G[K_n]$ and every nonempty subset of $V(K_n)$ is a dominating set of K_n, (i) holds by Theorem 5. To show (ii), if $|\overline{C}_G| = 1$, then $(ii)(a)$ holds. Suppose that $|\overline{C}_G| \geq 2$, and let $u, v \in \overline{C}_G$ with $u \neq v$. Pick $z, w \in V(K_n)$ such that $(u, z), (v, w) \notin C$. Since $\langle V(G[K_n]) \setminus C\rangle$ is connected, there exists a (u, z)-(v, w) geodesic $[(u, z) = (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) = (v, w)]$ in $G[K_n]$ such that $(x_k, y_k) \notin C$ for all $k = 1, 2, \ldots, n$. It implies that for some $k_1 < k_2 < \cdots < k_r$ in $\{1, 2, \ldots, n\}$, $P = [u = x_{k-1}, x_{k_2}, \ldots, x_{k_r} = v]$ is a u-v geodesic in G. If $V(P) \cap S = \varnothing$, then we are done. Suppose that $V(P) \cap S \neq \varnothing$, and let $x_{k_j} \in S \cap V(P)$. Necessarily, $y_{k_j} \notin T_{x_{k_j}}$. Thus, $T_{x_{k_j}} \neq V(K_n)$. This shows that $(ii)(b)$ holds.

Conversely, condition (i) implies that C is a semitotal dominating set of $G[K_n]$ by Theorem 5. If condition $(ii)(a)$ holds, then

$$V(G[K_n]) \setminus C = \begin{cases} \{x\} \times V(K_n), & \text{if } x \notin S \\ \{x\} \times (V(K_n) \setminus T_x), & \text{if } x \in S, \end{cases}$$

and C is an outer-connected semitotal dominating set of $G[H]$. Now, suppose that condition $(ii)(b)$ holds. Let $(u, z), (v, w) \in V(G[K_n]) \setminus C$ be distinct.

Case 1: $u \neq v$

Since $u, v \in \overline{C}_G$, G has a u-v geodesic $P = [u = x_1, x_2, \ldots, x_n = v]$ as being described in $(ii)(b)$. If $S \cap V(P) = \varnothing$, then for any $y \in V(K_n)$, $[(u, z), (x_2, y), (x_3, y), \ldots, (x_{n-1}, y), (v, w)]$ is a (u, z)-(v, w) path in $\langle V(G[K_n]) \setminus C\rangle$. Suppose that $S \cap V(P) \neq \varnothing$. Put $y_1 = z$ and $w = y_n$. For each $k \in \{2, \ldots, n-1\}$, pick any $y_k \in V(K_n)$ whenever $x_k \notin S$; otherwise, pick $y_k \in V(K_n) \setminus T_{x_k}$. Then $[(u, z) = (x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n) = (v, w)]$ is a (u, z)-(v, w) path in $\langle V(G[K_n]) \setminus C\rangle$.

Case 2: $u = v$

Pick $x \in \overline{C}_G \setminus \{u\}$. Let $P = [x = x_1, x_2, \ldots, x_{n-1}, x_n = u]$ be a x-u geodesic in G as described in condition $(ii)(b)$. In particular, $x_{n-1} \in \overline{C}_G$. Pick $y \in V(K_n)$ such that $(x_{n-1}, y) \notin C$. Then $[(u, z), (x_{n-1}, y), (v, w)]$ is a (u, z)-(v, w) path in $\langle V(G[K_n]) \setminus C\rangle$.

The above cases imply that $\langle V(G[K_n]) \setminus C\rangle$ is connected. Therefore, C is an outer-connected semitotal dominating set of $G[K_n]$. □

Now, we provide proof for the following lemma, which is very useful to get the desired result in this section. The lemma is given without proof in [1].

Lemma 1. [1] *If G is a nontrival connected graph and $S \subseteq V(G)$ is a dominating set in G, then*

$$\gamma_{t2}(G) \leq 2|S \setminus N_G^2(S)| + |S \cap N_G^2(S)|.$$

Proof. Let $S \subseteq V(G)$ be a dominating set of G. For each $x \in S \setminus N_G^2(S)$, pick $u_x \in V(G)$ such that $xu_x \in E(G)$. Then $S^* = S \cup \{u_x : x \in S \setminus N_G^2(G)\}$ is a semitotal dominating set of G. Thus,

$$\gamma_{t2}(G) \leq |S^*| = |S \cap N_G^2(S)| + 2|S \setminus N_G^2(S)|.$$

□

Corollary 5. *Let G be a nontrivial connected graph and $n \geq 2$. Then*

$$\tilde{\gamma}_{t2}(G[K_n]) = \gamma_{t2}(G).$$

Proof. Let $S \subseteq V(G)$ be a γ_{t2}-set of G. Choose $v \in V(K_n)$ and define $C = S \times \{v\}$. Since conditions $(i)(a)$ and $(ii)(b)$ of Theorem 6 hold for C, C is an outer-connected semitotal dominating set of $G[K_n]$. Consequently, $\tilde{\gamma}_{t2}(G[K_n]) \leq |S| = \gamma_{t2}(G)$.

Let $C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[K_n])$ be an outer-connected semitotal dominating set of $G[K_n]$. By Theorem 6, S is a dominating set of G. If S is a semitotal dominating set of G, then

$$\gamma_{t2}(G) \leq |S| \leq \sum_{x \in S} |T_x| = |C|.$$

Suppose that S is not a semitotal dominating set in G. Let $S_1 = S \setminus N_G^2(S)$ and $S_2 = S \cap N_G^2(S)$. By Theorem 6,

$$C = \left(\bigcup_{x \in S_1} (\{x\} \times T_x) \right) \cup \left(\bigcup_{x \in S_2} (\{x\} \times T_x) \right),$$

where $|T_x| \geq 2$ for all $x \in S_1$. Thus,

$$\begin{aligned} |C| &= \sum_{x \in S_1} |T_x| + \sum_{x \in S_2} |T_x| \\ &\geq 2|S_1| + |S_2| \\ &= 2|S \setminus N_G^2(S)| + |S \cap N_G^2(S)|. \end{aligned}$$

By Lemma 1, $\gamma_{t2}(G) \leq |C|$.
Since C is arbitrary, $\gamma_{t2}(G) \leq \gamma_{t2}(G[K_n])$. □

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