Outer-Connected Semitotal Domination in Graphs

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Abstract. In this paper, we introduce and initiate the study of outer-connected semitotal domination in graphs. Given a graph G without isolated vertices, a set S of vertices of G is a semitotal dominating set if every vertex outside of S is adjacent to a vertex in S and every vertex in S is of distance at most 2 units from another vertex in S. A semitotal dominating set S of G is an outer-connected semitotal dominating set if either S = V(G) or S ≠ V(G) satisfying the property that the subgraph induced by V(G) \ S is connected. The smallest cardinality ˜γt₂(G) of an outer-connected semitotal dominating set is the outer-connected semitotal domination number of G. First, we determine the specific values of ˜γt₂(G) for some special graphs and characterize graphs G for specific (small) values of ˜γt₂(G). Finally, we investigate the outer-connected semitotal dominating sets in the join, corona, and composition of graphs and, as a consequence, we determine their corresponding outer-connected semitotal domination numbers.

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1. Introduction

In 2014, the concept of semitotal domination was introduced and investigated by W. Goddard, M. Henning, and C. McPillan (see [6]). Accordingly, it strengthens the concept of domination but relaxes the concept of total domination. Semitotal domination was further studied by M. Henning and A. Marcon (see [8]) in 2014 and 2016, by G. Hao and W. Zhuang (see [7]) in 2018, and by I. Aniversario et al. [1] in 2019.

In this present paper, inspired by the work of J. Cyman [4], on outer-connected domination, we introduce and initiate the study of the outer-connected semitotal domination in graphs. We investigate the concept in some special graphs and in graphs under some binary operations, such as the join, corona, and lexicographic products of graphs.
2. Terminology and Notation

The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively, of a graph $G$. For $S \subseteq V(G)$, $|S|$ is the cardinality of $S$. In particular, $|V(G)|$ and $|E(G)|$ are the order and size, respectively, of $G$. The induced subgraph $\langle S \rangle$ is the graph with vertex set $S$ and such that $uv \in E(\langle S \rangle)$ if and only if $u, v \in S$ and $uv \in E(G)$. All graph terminologies that are not introduced but are being used here are adopted from [2].

Given two graphs $G$ and $H$ with disjoint vertex sets, the join $G+H$ of graphs $G$ and $H$, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. The corona of $G$ and $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$, and then joining the $i^{th}$ vertex of $G$ to every vertex of the $i^{th}$ copy of $H$. The lexicographic product or composition of $G$ and $H$, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

For vertex $u$ of $G$, all vertices adjacent to $u$ constitute the set $N_G(u)$ called the open neighborhood of $u$. The closed neighborhood of $u$ in $G$ is the set $N_G[u] = N_G(u) \cup \{u\}$. If $S \subseteq V(G)$, the open neighborhood of $S$ in $G$ is the set $N_G(S) = \cup_{u \in S} N_G(u)$. The closed neighborhood of $S$ in $G$ is the set $N_G[S] = N_G(S) \cup S$. We define $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = S \cup N_G(S)$. A set $S \subseteq V(G)$ is a dominating set of $G$ if $N_G[S] = V(G)$. Thus, $S$ is a dominating set of $G$ if and only if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. A set $S \subseteq V(G)$ is a total dominating set of $G$ if for every $v \in V(G)$, there exists $u \in S$ such that $uv \in E(G)$. The minimum cardinality of a dominating set (resp. total dominating set) of $G$, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the domination number (resp. total domination number) of $G$. A dominating set (resp. total dominating set) $S$ of $G$ with $|S| = \gamma(G)$ (resp. $|S| = \gamma_t(G)$) is called a $\gamma$-set (resp. $\gamma_t$-set) of $G$. The authors always refer to [3] for the introduction and more comprehensive discussion of the development of the concept of domination in graphs.

A set $S \subseteq V(G)$ of a graph $G = (V, E)$ is called an outer-connected dominating set of $G$ if the following hold: (i) $S$ is a dominating set of $G$, and (ii) either $S = V(G)$ or the induced subgraph $\langle V(G) \setminus S \rangle$ of $V(G) \setminus S$ is connected. The cardinality of a minimum outer-connected dominating set of $G$ is called the outer-connected domination number of $G$, and is denoted by $\tilde{\gamma}(G)$. For a graph $G$ without isolated vertices, a set $S \subseteq V(G)$ is a total outer-connected dominating set if $S$ is a total dominating set of $G$ and the subgraph induced by $\langle V(G) \setminus S \rangle$ is connected. The minimum cardinality of a total outer-connected dominating set in $G$ is the total outer-connected domination number denoted by $\tilde{\gamma}_t(G)$. We refer to [4] and [5] for the introduction and results concerning outer-connected domination and total outer-connected domination, respectively, that are of interest in this study.

Suppose that $G$ has no isolated vertices. A set $S \subseteq V(G)$ is a semitotal dominating set of $G$ if $S$ is a dominating set in $G$ such that for every $x \in S$ there exists $y \in S \setminus \{x\}$ for which $d_G(x, y) \leq 2$. The smallest cardinality of a semitotal dominating set in $G$, denoted by $\gamma_{12}(G)$, is called a semitotal domination number of $G$. A semitotal dominating set of $G$ with cardinality $\gamma_{12}(G)$ is called a $\gamma_{12}$-set. Some results on semitotal domination in graphs
are found in [1, 6–8].

A semitotal dominating set $S$ is an outer-connected semitotal dominating set of $G$ if either $S = V(G)$ or $S \neq V(G)$ and the induced subgraph $(V(G) \setminus S)$ is connected. The smallest cardinality of an outer-connected semitotal dominating set in $G$, denoted by $\tilde{\gamma}_{t2}(G)$, is called the outer-connected semitotal domination number of $G$. An outer-connected semitotal dominating set in $G$ with cardinality $\tilde{\gamma}_{t2}(G)$, is called a $\tilde{\gamma}_{t2}$-set.

For the purposes of this study, we write for $v \in V(G)$,

$$N^2_G(v) = \{u \in V(G) \setminus \{v\} : d_G(u, v) \leq 2\},$$

and write for $S \subseteq V(G)$,

$$N^2_G(S) = \bigcup_{v \in S} N^2_G(v).$$

3. Results

Observe that an outer-connected semitotal dominating set is both a semitotal dominating set and an outer-connected dominating set. On the other hand, a total outer-connected dominating set is an outer-connected semitotal dominating set. Thus,

$$\max\{2, \gamma_{t2}(G), \tilde{\gamma}(G)\} \leq \tilde{\gamma}_{t2}(G) \leq \tilde{\gamma}_{t}(G). \quad (1)$$

Strict inequalities in Equation 1 can be attained for a graph. To see this, consider the graph $G$ in Figure 1. It can be verified that $\{a, b, c, d\}$, $\{x, y, z, w\}$, $\{x, y, z, w, a, b\}$ and $\{a, b, c, d, x, y, z\}$ are $\gamma_{t2}$-set, $\tilde{\gamma}$-set, $\tilde{\gamma}_{t2}$-set and $\tilde{\gamma}_{t}$-set, respectively. Thus, $\tilde{\gamma}(G) = 4 = \gamma_{t2}(G)$, $\tilde{\gamma}_{t2}(G) = 6$ and $\tilde{\gamma}_{t}(G) = 7$.

![Figure 1: Graph G with max{2, \gamma_{t2}(G), \tilde{\gamma}(G)} < \tilde{\gamma}_{t2}(G) < \tilde{\gamma}_{t}(G)](image)

**Proposition 1.** For path $P_n$ on $n \geq 2$ vertices

$$\tilde{\gamma}_{t2}(P_n) = \begin{cases} 
2, & \text{if } n = 2 \\
3, & \text{if } 3 \leq n \leq 5 \\
n - 2, & \text{if } n \geq 6.
\end{cases}$$

**Proof.** The case where $n = 2$ is obvious. Assume $n \geq 3$. Let $S \subseteq V(P_n)$, be $\tilde{\gamma}_{t2}$-set of $P_n$ with $S \neq V(P_n)$. Then $P = (V(Pn) \setminus S)$ is a path. Suppose that $[x, y, z]$ is a
geodesic in $P$. Then $y \notin N_{P_n}[S]$, a contradiction. Thus, $|V(P)| = 1$ or 2. Consequently, \( \tilde{\gamma}_{t2}(P_n) = |S| \geq n - 2 \). It is can readily be verified that if $3 \leq n \leq 5$, $|S| = n - 1$. That is, $\tilde{\gamma}_{t2}(P_n) = n - 1$. Suppose that $n \geq 6$. Put $P_n = [x_1, x_2, \ldots, x_n]$. Since $S = \{x_1, x_2, x_5, x_6, \ldots, x_n\}$ is an outer-connected semitotal dominating set of $P_n$, $\tilde{\gamma}_{t2}(P_n) \leq |S| = n - 2$. Therefore, $\tilde{\gamma}_{t2}(P_n) = n - 2$. \hfill\(\square\)

**Proposition 2.** For cycle $C_n$ on $n \geq 3$ vertices

\[
\tilde{\gamma}_{t2}(C_n) = \begin{cases} 2, & \text{if } n = 3 \\ n - 2, & \text{if } n \geq 4. \end{cases}
\]

**Proof.** The case for $n = 3$ is trivial. Assume that $n \geq 4$, and say $C = [x_1, x_2, \ldots, x_n, x_1]$. Since $\{x_3, x_4, \ldots, x_n\}$ is an outer-connected semitotal dominating set of $C_n$, $\tilde{\gamma}_{t2}(C_n) \leq n - 2$. Following similar arguments used above, if $S \subseteq V(C_n)$ is a $\tilde{\gamma}_{t2}$-set of $C_n$ and $P = (V(C_n) \setminus S)$, then $P$ is a path with $1 \leq |V(P)| \leq 2$. Consequently, $\tilde{\gamma}_{t2}(C_n) = |S| \geq n - 2$. \hfill\(\square\)

**Proposition 3.** For complete multipartite graph $K_{n_1,n_2,\ldots,n_t}$ of order $n = n_1 + n_2 + \ldots + n_t$, where $1 \leq n_1 \leq n_2 \leq \ldots \leq n_t$ and $t \geq 2$,

\[
\tilde{\gamma}_{t2}(K_{n_1,n_2,\ldots,n_t}) = \begin{cases} n_2, & \text{if } t = 2 \text{ and } n_1 = 1, n_2 \geq 2 \\ 2, & \text{else}. \end{cases}
\]

**Proof.** Put $G = K_{n_1,n_2,\ldots,n_t}$, and let $U_1, U_2, \ldots, U_t$ be the partite sets of $G$. First, observe that if $n_t = 1$, then $G = K_t$ and $\tilde{\gamma}_{t2}(G) = 2$. Assume that $n_t \geq 2$. We consider the following cases:

**Case 1:** Suppose that $n_1 = 1$. We consider the following subcases:

**Subcase 1.1:** If $t = 2$ and $n_2 \geq 2$, then $G = K_{1,n_2}$, a star of order $n \geq 3$. If $S \subseteq V(G)$ is an outer-connected semitotal dominating set of $G$, then either $|S| = n - 1 = n_2$ or $|S| = n$. Thus, $\tilde{\gamma}_{t2}(G) = n_2$.

**Subcase 1.2:** Suppose that $t \geq 3$ and $n_1 = n_2 = 1$ such that $G$ is not a path. Pick $u \in U_2$ and $v \in U_3$. Then $S = \{u, v\}$ is an outer-connected semitotal dominating set of $G$. Thus, $\tilde{\gamma}_{t2}(G) = 2$.

**Subcase 1.3:** Suppose that $t \geq 3$ and $n_2 \geq 2$. Pick $u \in U_1$ and $v \in U_2$. Then $S = \{u, v\}$ is an outer-connected semitotal dominating set of $G$. Thus, $\tilde{\gamma}_{t2}(G) = 2$.

**Case 2:** Suppose that $t \geq 2$ and $n_k \geq 2$ for all $k \in \{1, 2, \ldots, t\}$. Pick $u \in U_1$ and $v \in U_2$. Then $S = \{u, v\}$ is an outer-connected semitotal dominating set of $G$. \hfill\(\square\)

**Proposition 4.** Let $G$ be a connected graph of order $n \geq 2$. Then

(i) $\tilde{\gamma}_{t2}(G) = 2$ if and only if $G$ can be obtained from a connected graph $H$ of order $n - 2$ by adding to $H$ vertices $u$ and $v$ such that $d_G(u, v) = 1$ or 2 and $\{u, v\}$ dominates $V(H)$. 


(ii) \( \bar{\gamma}_{t2}(G) = n \) if and only if \( G = K_2 \).

**Proof.** Statement (i) is clear. By Proposition 3, if \( n = 2 \), then \( \bar{\gamma}_{t2}(K_n) = 2 = n \). Assume that \( \bar{\gamma}_{t2}(G) = n \). Suppose that \( n \geq 3 \). By Proposition 3, \( G \neq K_n \). Let \([u, v, z] \) be a geodesic in \( G \). Put \( S = V(G) \setminus \{v\} \). Then \( S \) is a semitotal dominating set with \( V(G) \setminus S = \{v\} \). Thus \( S \) is an outer-connected semitotal dominating set of \( G \). Consequently, \( \bar{\gamma}_{t2}(G) \leq n - 1 \), a contradiction. Therefore, \( n = 2 \). This proves (ii). \( \Box \)

**Proposition 5.** Let \( G \) be any graph with nontrivial components \( C_1, C_2, \ldots, C_k \) of orders \( n_1, n_2, \ldots, n_k \), respectively. Then

\[
\bar{\gamma}_{t2}(G) = \min \{ \bar{\gamma}_{t2}(C_j) + \sum_{i=1, i \neq j}^{k} n_i : j = 1, 2, \ldots, k \}.
\]

**Proof.** Put \( \alpha = \min \{ \bar{\gamma}_{t2}(C_j) + \sum_{i=1, i \neq j}^{k} n_i : j = 1, 2, \ldots, k \} \). Let \( j \in \{1, 2, \ldots, k\} \), and choose a \( \bar{\gamma}_{t2} \)-set \( S_j \) of \( C_j \). Since \( S = \left( \bigcup_{i=1, i \neq j}^{k} V(C_i) \right) \cup S_j \) is an outer-connected semitotal dominating set of \( G \), \( \bar{\gamma}_{t2}(G) \leq |S| = \bar{\gamma}_{t2}(C_j) + \sum_{i=1, i \neq j}^{k} n_i \). Since \( j \) is arbitrary, \( \bar{\gamma}_{t2}(G) \leq \alpha \).

Let \( S \subseteq V(G) \) be an outer-connected semitotal dominating set of \( G \). Since \( S \) is a semitotal dominating set of \( G \), \( S_j = S \cap V(C_j) \) is a semitotal dominating set of \( C_j \) for all \( j = 1, 2, \ldots, k \). First, we claim that there exists \( j \in \{1, 2, \ldots, k\} \) for which \( S_i = V(C_i) \) for all \( i \) except possibly when \( i = j \). Suppose that, to the contrary, there exist distinct \( i, j \in \{1, 2, \ldots, k\} \) such that \( S_i \neq V(C_i) \) and \( S_j \neq V(C_j) \). Pick \( u \in V(C_i) \setminus S_i \) and \( v \in V(C_j) \setminus S_j \). Observe that \( V(G) \setminus S \), the conclusion follows. Thus,

\[
|S| = |S_j| + \sum_{i=1, i \neq j}^{k} n_i \geq \bar{\gamma}_{t2}(C_j) + \sum_{i=1, i \neq j}^{k} n_i \geq \alpha.
\]

Since \( S \) is arbitrary, \( \bar{\gamma}_{t2}(G) \geq \alpha \). \( \Box \)

**Proposition 6.** Let \( G \) be a nontrivial graph.

(i) If \( G \) is connected, then \( \bar{\gamma}_{t2}(G + K_1) = 2 \).

(ii) If \( G \) is disconnected with components \( C_1, C_2, \ldots, C_k \) of orders \( n_1, n_2, \ldots, n_k \),
respectively, satisfying that $n_1 \leq n_2 \leq \cdots \leq n_k$, then

$$\tilde{\gamma}_t(G + K_1) = \min\{\sum_{j=1}^{k} \gamma(C_j), 2 + \sum_{j=1}^{k-1} n_j\}.$$  

Proof. Put $V(K_1) = \{u\}$. To prove (i), suppose that $G$ is connected. Since $G$ is nontrivial, $G$ contains at least two vertices which are not cutvertices. Pick a non-cutvertex $v$ of $G$. Then $S = \{u, v\}$ is an outer-connected semitotal dominating set of $G + K_1$. By Equation 1, $\tilde{\gamma}_t(G + K_1) = 2$.

To prove (ii), suppose that $G$ is disconnected with components $C_1, C_2, \ldots, C_k$ of orders $n_1, n_2, \ldots, n_k$, respectively, and satisfying that $n_1 \leq n_2 \leq \cdots \leq n_k$. If $n_k = 1$, then $G + K_1$ is a star, and the result follows from (i). It is worth noting that in this case,

$$\sum_{j=1}^{k} \gamma(C_j) = k = |V(G)|.$$

Assume $n_k \geq 2$. First, let $S_j \subseteq V(C_j)$ be a $\gamma$-set of $C_j$ for all $j = 1, 2, \ldots, k$. Then $S = \bigcup_{j=1}^{k} S_j$ is a semitotal dominating set of $G + K_1$. Since $u \in V(G + K_1) \setminus S$, $S$ is an outer-connected semitotal dominating set of $G + K_1$. Thus,

$$\tilde{\gamma}_t(G + K_1) \leq |S| = \sum_{j=1}^{k} \gamma(C_j).$$

Next, let $S = \left(\bigcup_{j=1}^{k-1} V(C_j)\right) \cup \{u, v\}$, where $v \in V(C_k)$ which is a non-cutvertex of $C_k$. Then $S$ is an outer-connected dominating set of $G + K_1$. This means that

$$\tilde{\gamma}_t(G + K_1) \leq 2 + \sum_{j=1}^{k-1} n_j.$$  

Thus,

$$\tilde{\gamma}_t(G + K_1) \leq \min\{\sum_{j=1}^{k} \gamma(C_j), 2 + \sum_{j=1}^{k-1} n_j\}.$$  

Now, to get the other inequality, let $S \subseteq V(G + K_1)$ be a $\tilde{\gamma}_t$-set of $G + K_1$. Then $S_j = S \cap V(C_j)$ is a dominating set of $C_j$ for all $j \in \{1, 2, \ldots, k\}$. If $u \notin S$, then

$$\tilde{\gamma}_t(G + K_1) = |S| = \sum_{j=1}^{k} |S_j| \geq \sum_{j=1}^{k} \gamma(C_j).$$

Suppose that $u \in S$. Since $V(G + K_1) \setminus S$ is connected, $V(G + K_1) \setminus S = V(C_j) \setminus S_j$ for
some \(j\). Moreover, since \(S\) is a \(\tilde{\gamma}_{t2}\)-set of \(G\), \(j = k\). Thus,

\[
S = \left( \bigcup_{i=1}^{k-1} V(C_i) \right) \cup S_k \cup \{u\}
\]

so that

\[
\tilde{\gamma}_{t2}(G + K_1) = |S| = 1 + |S_k| + \sum_{j=1}^{k-1} n_j \geq 2 + \sum_{j=1}^{k-1} n_j.
\]

Therefore,

\[
\tilde{\gamma}_{t2}(G + K_1) \geq \min\{\sum_{j=1}^{k} \gamma(C_j), 2 + \sum_{j=1}^{k-1} n_j\}.
\]

\[
\text{Suppose that } C_j = K_1 \text{ for all } j \in \{1, 2, \ldots, k-1\} \text{ in Proposition 3. If } \gamma(C_k) = 1, \text{ then}
\]

\[
\sum_{j=1}^{k} \gamma(C_j) < 2 + \sum_{j=1}^{k-1} n_j.
\]

On the other hand, if \(\gamma(C_k) \geq 3\), then

\[
\sum_{j=1}^{k} \gamma(C_j) > 2 + \sum_{j=1}^{k-1} n_j,
\]

and attain equality if \(\gamma(C_k) = 2\).

**Theorem 1.** [1] Let \(G\) and \(H\) be nontrivial graphs, and \(S \subseteq V(G + H)\). Then \(S\) is a semitotal dominating set in \(G + H\) if and only if one of the following holds:

(i) \(S \subseteq V(G)\) is a nonsingleton dominating set in \(G\);

(ii) \(S \subseteq V(H)\) is a nonsingleton dominating set in \(H\);

(iii) \(S \cap V(G) \neq \emptyset\) and \(S \cap V(G) \neq \emptyset\).

**Theorem 2.** Let \(G\) and \(H\) be any nontrivial graphs, and \(S \subseteq V(G + H)\), then \(S\) is an outer-connected semitotal dominating set in \(G + H\) if and only if one of the following holds:

(i) \(S \subseteq V(G)\) and one of the following holds:

(a) \(S = V(G)\) and \(H\) is connected;

(b) \(S \neq V(G)\) and \(S\) is a nonsingleton dominating set in \(G\).

(ii) \(S \subseteq V(H)\) and one of the following holds:

(a) \(S = V(H)\) and \(G\) is connected;
(b) $S \neq V(H)$ and $S$ is a nonsingleton dominating set in $H$.

(iii) $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$ such that if $S \neq V(G + H)$, then one of the following holds:

(a) $V(G) \subseteq S$ and $\langle V(H) \setminus S \rangle$ is connected;

(b) $V(H) \subseteq S$ and $\langle V(G) \setminus S \rangle$ is a connected;

(c) $V(G) \setminus S \neq \emptyset$ and $V(H) \setminus S \neq \emptyset$.

Proof. Assume that $S$ is an outer-connected semitotal dominating set of $G + H$. Suppose that $S \subseteq V(G)$. If $S = V(G)$, then $H = (V(G + H) \setminus S)$ is connected, and (i)(a) holds. Suppose that $S \neq V(G)$. Since $S$ is a semitotal dominating set of $G + H$, $S$ is a nonsingleton dominating set of $G$, and (i)(b) holds. Similarly, if $S \subseteq V(H)$, then (ii) holds. Now, assume that $S_G = S \cap V(G) \neq \emptyset$ and $S_H = S \cap V(H) \neq \emptyset$. Suppose further that $S \neq V(G + H)$. Statement (iii)(a) follows from the fact that if $V(G) \subseteq S$, then $(V(G + H) \setminus S) = (V(H) \setminus S_H)$ is connected. Similarly, if $V(H) \subseteq S$, then (iii)(b) holds. If both (iii)(a) and (iii)(b) do not hold, then necessarily, (iii)(c) holds.

Conversely, suppose that $S \neq V(G + H)$ satisfying condition (i). Then $S \subseteq V(G)$ and is a nonsingleton dominating set of $G$. By Theorem 1, $S$ is semitotal dominating set of $G + H$. If $S = V(G)$, then $(V(G + H) \setminus S) = H$, which by (i)(a) is connected. Suppose that $S \neq V(G)$. Then $(V(G + H) \setminus S) = ((V(G) \setminus S) \cup V(H))$ is clearly connected. This makes $S$ an outer-connected semitotal dominating set of $G + H$. Similarly, if (ii) holds, then $S$ is an outer-connected semitotal dominating set of $G + H$. Finally, suppose that (iii) holds. By Theorem 1, $S$ is a semitotal dominating set of $G + H$. If $V(G) \subseteq S$, then $(V(G + H) \setminus S) = (V(H) \setminus S)$, which is connected by (iii)(a). Similarly, if (iii)(b) holds, then $S$ is an outer-connected semitotal dominating set of $G + H$. Following similar arguments as above, if (iii)(c) holds then $S$ is an outer-connected semitotal dominating set of $G + H$.

Corollary 1. For all nontrivial graphs $G$ and $H$, $\gamma_{t_2}(G + H) = 2$.

Proof. Pick $u \in V(G)$ and $v \in V(H)$. By Theorem 2, $S = \{u, v\}$ is an outer-connected semitotal dominating set of $G + H$. Thus, $\gamma_{t_2}(G + H) \leq 2$. Finally, by (1), $\gamma_{t_2}(G + H) = 2$.

Theorem 3. Let $G$ be a nontrivial connected graph and $S \subseteq V(G \circ K_1)$. Then $S$ is an outer-connected semitotal dominating set of $G \circ K_1$ if and only if one of the following holds for $S$:

(i) $S = V(G \circ K_1) \setminus V(K_1^v)$ for some $v \in V(G)$;

(ii) $S = A \cup (\bigcup_{v \in V(G)} V(K_1^v))$, where $A \subseteq V(G)$ is an outer-connected dominating set of $G$. 
Proof. Put \( V(K_1) = \{ x \} \). Then \( V(K_1^x) = \{ x^v \} \). Assume that \( S \) is an outer-connected semitotal dominating set of \( G \circ K_1 \). We consider two cases:

**Case 1:** Suppose that \( x^v \not\in S \) for some \( v \in V(G) \). Since \( S \) is a dominating set of \( G \circ K_1 \), \( v \in S \). Since \( x^v \in V(G \circ K_1) \setminus S \) and \( V(G \circ K_1) \setminus S \) is connected, \( V(G \circ K_1) \setminus S = \{ x^v \} \). That is, \( S = V(G \circ K_1) \setminus \{ x^v \} \). In this case, (i) holds.

**Case 2:** Suppose that \( x^v \in S \) for all \( v \in V(G) \). Define \( A = S \cap V(G) \). Then

\[
S = A \cup \{ \cup_{v \in V(G)} \{ x^v \} \}.
\]

We claim that \( A \) is an outer-connected dominating set of \( G \). First, let \( v \in V(G) \setminus A \). Then \( x^v \in S \). Since \( S \) is a semitotal dominating set of \( G \circ K_1 \), there exists \( u \in S \) for which \( d_{G \circ K_1}(x^v, u) \leq 2 \). Because \( v \not\in S \), \( u \in A \cap N_G(v) \). Since \( v \) is arbitrary, \( A \) is a dominating set of \( G \). Note further that, \( V(G) \setminus A = V(G \circ K_1) \setminus S \). Thus, \( A \) is an outer-connected dominating set of \( G \). In this case, (ii) holds.

Conversely, obviously, if condition (i) holds for \( S \), then \( S \) is an outer-connected semitotal dominating set of \( G \circ K_1 \). Now, suppose that condition (ii) holds for \( S \). Since \( \cup_{v \in V(G)} \{ x^v \} \) is a dominating set of \( G \circ K_1 \), \( S \) is a dominating set of \( G \circ K_1 \). Let \( u \in S \). We consider the following cases:

**Case 1:** Suppose that \( u = x^v \) for some \( v \in V(G) \). If \( v \in S \), then we pick \( v \) for \( d_G(u, v) \leq 2 \). Suppose that \( v \not\in S \). Since \( A \) is a dominating set of \( G \) and \( v \in V(G) \setminus A \), there exists \( w \in A \subseteq S \) such that \( wv \in E(G) \subseteq E(G \circ K_1) \). Since \( d_{G \circ K_1}(u, w) = 2 \), \( w \) is the desired vertex.

**Case 2:** Suppose that \( u \in V(G) \). In this case, we pick \( x^u \in S \). Note that \( ux^u \in E(G \circ K_1) \).

The above cases show that \( S \) is a semitotal dominating set of \( G \circ K_1 \). Finally, since \( A \) is an outer-connected dominating set of \( G \), \( V(G) \setminus A = V(G \circ K_1) \setminus S \) is connected. Therefore, \( S \) is an outer-connected semitotal dominating set of \( G \circ K_1 \).

\[
\text{Corollary 2. For nontrivial connected graph } G \text{ of order } n,
\]

\[
\tilde{\gamma}_{t_2}(G \circ K_1) = n + \tilde{\gamma}(G).
\]

**Proof.** In view of Theorem 3,

\[
\tilde{\gamma}_{t_2}(G \circ K_1) = \min \{ 2n - 1, n + \tilde{\gamma}(G) \} = n + \tilde{\gamma}(G).
\]

\[
\text{Theorem 4. Let } G \text{ and } H \text{ be nontrivial connected graphs, and let } S \subseteq V(G \circ H). \text{ Then } S \text{ is an outer-connected semitotal dominating set of } G \circ H \text{ if and only if one of the following holds for } S:
\]


(i) There exists \( v \in V(G) \) and \( B \subseteq V(H^v) \) such that
\[
S = (V(G \circ H) \setminus V(H^v)) \cup B,
\]
where either \( B = V(H^v) \) or \( (V(H^v) \setminus B) \) is connected.

(ii) \[
S = A \cup (\bigcup_{x \in A} V(H^x)) \cup \left( \bigcup_{x \in V(G) \setminus A} S_x \right),
\]
where \( A \subseteq V(G) \) and \( S_x \subseteq V(H^x) \) for all \( x \in V(G) \setminus A \) satisfying the following:

(a) \( \{V(G) \setminus A\} \) is connected;
(b) For each \( x \in V(G) \setminus A \), \( S_x \) is a dominating set of \( H^x \). Moreover, if \( |S_x| = 1 \), then \( A \cap N_G(x) \neq \emptyset \).

Proof. Assume that \( S \) is an outer-connected semitotal dominating set of \( G \circ H \). If \( S = V(G \circ H) \), then (i) holds. In what follows, we assume that \( S \neq V(G \circ H) \). We consider two cases:

**Case 1:** Suppose that \( V(G) \subseteq S \). Since \( \langle V(G \circ H) \setminus S \rangle \) is connected, there exists \( v \in V(G) \) and \( B \subseteq V(H^v) \) such that \( V(G \circ H) \setminus S = V(H^v) \setminus B \). That is,
\[
S = (V(G \circ H) \setminus V(H^v)) \cup B
\]
and \( (V(H^v) \setminus B) \) is connected.

**Case 2:** Suppose that \( V(G) \not\subseteq S \). Put \( A = S \cap V(G) \). If \( A = \emptyset \), then (2) trivially holds with \( S_x = S \cap V(H^x) \) for all \( x \in V(G) \). Suppose that \( A \neq \emptyset \). Since \( \langle V(G \circ H) \setminus S \rangle \) is connected and \( V(G) \setminus A \neq \emptyset \), \( V(H^x) \subseteq S \) for all \( x \in A \). Put \( S_x = S \cap V(H^x) \) for all \( x \in V(G) \setminus A \). Then Equation (2) holds for \( S \). Statement (ii)(a) follows immediately from the connectedness of \( \langle V(G \circ H) \setminus S \rangle \). Now, let \( x \in V(G) \setminus A \) and \( u \in V(H^x) \setminus S_x \). Since \( S \) is a dominating set of \( G \circ H \) and \( u \notin S \), there exists \( w \in S \) for which \( uw \in E(G \circ H) \). Since \( x \notin S, w \neq x \) so that \( w \in S_x \). This means that \( S_x \) is a dominating set of \( H^x \). Suppose further that \( |S_x| = 1 \), say \( S_x = \{u\} \). Because \( S \) is a semitotal dominating set of \( G \circ H \), there exists \( w \in S \setminus \{u\} \) such that \( d_{G \circ H}(u, w) \leq 2 \). Since \( w \notin S_x, w \in A \cap N_G(x) \), and Statement (ii)(b) holds.

Conversely, suppose that condition (i) holds. Since \( V(G) \subseteq S \), \( S \) is a semitotal dominating set of \( G \circ H \). Moreover, \( V(G \circ H) \setminus S = V(H^v) \setminus B \) so that \( S \) is an outer-connected semitotal dominating set of \( G \circ H \). Now, suppose that condition (ii) holds for \( S \). By condition (ii)(b), \( S \) is a dominating set of \( G \circ H \). Suppose that \( A = \emptyset \). Then \( S = \bigcup_{x \in V(G)} S_x \), and by condition (ii)(b), \( S_x \) is a nonsingleton dominating set of \( H^x \) for all \( x \in V(G) \). Note that for each \( x \in V(G) \), \( d_{G \circ H}(u, v) \leq 2 \) for all \( u, v \in S_x \). It follows that \( S \) is a semitotal dominating set of \( G \circ H \). Further, since \( V(G) \subseteq V(G \circ H) \setminus S \), \( \langle V(G \circ H) \setminus S \rangle \) is connected. Finally, suppose that \( A \neq \emptyset \). Let \( x \in S \). If \( x \in A \), then \( V(H^x) \subseteq S \). Pick \( u \in V(H^x) \).
Then we have $u \in S$ and $d_{G \circ H}(x, u) = 1$. If $x \in V(H^v)$ for some $v \in A$, then $v$ is the desired vertex in $S$ for which $d_{G \circ H}(x, v) \leq 2$. Next, suppose that $x \in V(H^v)$ for some $v \in V(G) \setminus A$. If $|S_v| \geq 2$, then pick $u \in S_v \setminus \{x\}$. Then $u \in S$ and $d_{G \circ H}(x, u) \leq 2$. Lastly, suppose that $|S_v| = 1$, i.e., $S_v = \{x\}$. By condition (ii)(b), the exists $z \in A \cap N_G(v)$. Then $z \in S$ and $d_{G \circ H}(x, z) = 2$. We have shown that $S$ is a semitotal dominating set of $G \circ H$. Condition (ii)(a) implies further that $S$ is an outer-connected semitotal dominating set of $G \circ H$. 

**Corollary 3.** Let $G$ and $H$ be nontrivial connected graphs of orders $n$ and $m$, respectively.

(i) If $\gamma(H) = 1$, then $\bar{\gamma}_{t2}(G \circ H) \leq \min\{2n, n + m\gamma(G)\}$.

(ii) If $\gamma(H) \geq 2$, then

$$
\bar{\gamma}_{t2}(G \circ H) \leq \min\{n\gamma(H), \bar{\gamma}(G) (1 + m - \gamma(H)) + n\gamma(H)\}
$$

**Proof.** Let $A \subseteq V(G)$ be a $\bar{\gamma}$-set of $G$. For each $v \notin A$, let $S_v \subseteq V(H)$ be a $\gamma$-set of $H^v$. Define

$$
S = A \cup \left( \bigcup_{v \in A} V(H^v) \right) \cup \left( \bigcup_{v \in V(G) \setminus A} S_v \right).
$$

Since $S$ satisfies Theorem 4(ii), $S$ is an outer-connected semitotal dominating set of $G \circ H$. Thus,

$$
\bar{\gamma}_{t2}(G \circ H) \leq |S| = |A| + m|A| + (n - |A|) \gamma(H)
$$

$$
= \bar{\gamma}(G) + m\bar{\gamma}(G) + (n - \bar{\gamma}(G))\gamma(H)
$$

$$
= \bar{\gamma}(G) (1 + m - \gamma(H)) + n\gamma(H).
$$

In particular, if $\gamma(H) = 1$, then $\bar{\gamma}_{t2}(G \circ H) \leq n + m\bar{\gamma}(G)$.

To complete the desired results, for the case where $\gamma(H) = 1$, let $y \in V(H)$ for which $N_H[y] = V(H)$. Pick a $z \in V(H) \setminus \{y\}$ and define $S_v = \{z, y\}$ for all $v \in V(G)$. On the other hand, if $\gamma(H) \geq 2$, then choose $S_v \subseteq V(H^v)$ to be a $\gamma$-set of $H^v$ for all $v \in V(G)$. In any case, $S = \bigcup_{v \in V(G)} S_v$ satisfies Theorem 4(ii). Thus, $S$ is an outer-connected semitotal dominating set of $G \circ H$. This means that if $\gamma(H) = 1$, then

$$
\bar{\gamma}_{t2}(G \circ H) \leq |S| = 2n.
$$

For $\gamma(H) \geq 2$,

$$
\bar{\gamma}_{t2}(G \circ H) \leq |S| = n\gamma(H).
$$

**Corollary 4.** Let $G$ and $H$ be nontrivial connected graphs of orders $n$ and $m$, respectively.
If \( \gamma(H) = 1 \), then \( \gamma_{2}((G + K_{1}) \circ H) = \min\{2n + 2, n + m + 1\} \).

(ii) If \( \gamma(H) \geq 2 \), then
\[
\gamma_{2}((G + K_{1}) \circ H) = \min\{(n + 1)\gamma(H), \gamma(G)(1 + m - \gamma(H)) + n\gamma(H)\}
\]

Proof. Put \( K = (G + K_{1}) \circ H \), and \( \alpha = \min\{2n + 2, n + m + 1\} \). By Corollary 3, with \( \gamma(G + K_{1}) = 1 \), we have \( \gamma_{2}(K) \leq \alpha \).

Now, let \( S \subseteq V(K) \) be an outer-connected semitotal dominating set of \( K \). Since \( S \) is a dominating set of \( K \), \( S \cap V(H^x + x) \neq \emptyset \) for all \( x \in V(G + K_{1}) \). First, suppose that \( S \cap V(G + K_{1}) = \emptyset \). By Theorem 4(ii), \( S \cap V(H^x) \) is a nonsingleton dominating set of \( H^x + x \) for all \( x \in V(G + K_{1}) \). This means that \( |S| \geq 2(n + 1) = 2n + 2 \geq \alpha \). Next, suppose that \( S \cap V(G + K_{1}) \neq \emptyset \). Clearly, if \( V(G + K_{1}) \subseteq S \), then \( |S| \geq 2n + 2 \). Assume that \( V(G + K_{1}) \setminus S \neq \emptyset \). Let \( w \in S \cap V(G + K_{1}) \). Since \( S \) is an outer-connected semitotal dominating set of \( K \) and \( V(G + K_{1}) \setminus S \subseteq V(K) \setminus S \), \( V(H^w) \subseteq S \). This means that
\[
|S| \geq |V(H^w + w)| + \sum_{x \in V(G + K_{1}) \setminus \{w\}} |S \cap V(H^x + x)| \geq m + 1 + n \geq \alpha.
\]

Since \( S \) is arbitrary, \( \gamma_{2}(K) \geq \alpha \).

Similar arguments will prove (ii). \( \square \)

It is worth noting that the wheel graphs and the fan graphs are among the graphs represented by \( G + K_{1} \) in Corollary 4.

**Theorem 5.** [1] Let \( G \) and \( H \) be nontrivial connected graphs, and let \( C = \cup_{x \in S}(\{x\} \times T_{x}) \subseteq V(G[H]) \). Then \( C \) is a semitotal dominating set in \( G[H] \) if and only if one of the following holds:

(i) \( S \) is a total dominating set in \( G \);

(ii) \( S \) is a semitotal dominating set in \( G \) and for each \( x \in S \setminus N_{G}(S), T_{x} \) is a dominating set in \( H \);

(iii) \( S \) is a dominating set in \( G \), such that, \( T_{x} \) is a dominating set in \( H \) for each \( x \in S \setminus N_{G}(S) \), and \( |T_{x}| \geq 2 \) for each \( x \in S \setminus N_{G}^2(S) \).

For \( C \subseteq V(G[H]) \), define \( \overline{C}_{G} = \{x \in V(G) : (x, y) \notin C \text{ for some } y \in V(H)\} \).

**Theorem 6.** Let \( G \) be a nontrivial connected graph and \( n \geq 2 \), and \( C = \cup_{x \in S}(\{x\} \times T_{x}) \neq V(G[K_{n}]) \). Then \( C \) is an outer-connected semitotal dominating set of \( G[K_{n}] \) if and only if each of the following holds:

(i) One of the following holds:

(a) \( S \) is a semitotal dominating set in \( G \).
Case 1: 

(ii) Exactly one of the following holds:

(a) \( C_G = \{x\} \) for some \( x \in V(G) \).

(b) \( |C_G| \geq 2 \) and for each distinct \( u, v \in C_G \), \( G \) has a u-v geodesic \( P \) for which either \( S \cap V(P) = \emptyset \) or \( |T_x| < n \) for each \( x \in S \cap V(P) \).

Proof. Assume that \( C \) is an outer-connected semitotal dominating set of \( G[K_n] \). Since \( C \) is a semitotal dominating set of \( G[K_n] \) and every nonempty subset of \( V(K_n) \) is a dominating set of \( K_n \), (i) holds by Theorem 5. To show (ii), if \( |C_G| = 1 \), then (ii)(a) holds. Suppose that \( |C_G| \geq 2 \), and let \( u, v \in C_G \) with \( u \neq v \). Pick \( z, w \in V(K_n) \) such that \( (u, z), (v, w) \notin C \). Since \( \langle V(G[K_n]) \setminus C \rangle \) is connected, there exists a \((u, z)-(v, w)\) geodesic \([u, z) = (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) = (v, w)\] in \( G[K_n] \) such that \((x_k, y_k) \notin C\) for all \( k = 1, 2, \ldots, n \). It implies that for some \( k_1 < k_2 < \cdots < k_r \in \{1, 2, \ldots, n\} \), \( P = [u = x_{k_1}, x_{k_2}, \ldots, x_{k_r} = v] \) is a u-v geodesic in \( G \). If \( V(P) \cap \emptyset \neq 0 \), then we are done. Suppose that \( V(P) \cap \emptyset = \emptyset \), and let \( x_{k_j} \in S \cap V(P) \). Necessarily, \( y_{k_j} \notin T_{x_{k_j}} \). Thus, \( T_{x_{k_j}} \neq V(K_n) \). This shows that (ii)(b) holds.

Conversely, condition (i) implies that \( C \) is a semitotal dominating set of \( G[K_n] \) by Theorem 5. If condition (ii)(a) holds, then

\[
V(G[K_n]) \setminus C = \begin{cases} 
\{x\} \times V(K_n), & \text{if } x \notin S \\
\{x\} \times (V(K_n) \setminus T_x), & \text{if } x \in S,
\end{cases}
\]

and \( C \) is an outer-connected semitotal dominating set of \( G[H] \). Now, suppose that condition (ii)(b) holds. Let \((u, z), (v, w) \in V(G[K_n]) \setminus C \) be distinct.

Case 1: \( u \neq v \)

Since \( u, v \in C_G \), \( G \) has a u-v geodesic \( P = [u = x_1, x_2, \ldots, x_n = v] \) as being described in (ii)(b). If \( S \cap V(P) = \emptyset \), then for any \( y \in V(K_n) \), \((u, z), (x_2, y), (x_3, y), \ldots, (x_n-1, y), (v, w)\) is a \((u, z)-(v, w)\) path in \( \langle V(G[K_n]) \setminus C \rangle \). Suppose that \( S \cap V(P) \neq \emptyset \). Put \( y_1 = z \) and \( w = y_n \). For each \( k \in \{2, \ldots, n-1\} \), pick any \( y_k \in V(K_n) \) whenever \( x_k \notin S \); otherwise, pick \( y_k \in V(K_n) \setminus T_{x_k} \). Then \([u, z) = (x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n) = (v, w)\] is a \((u, z)-(v, w)\) path in \( \langle V(G[K_n]) \setminus C \rangle \).

Case 2: \( u = v \)

Pick \( x \in C_G \setminus \{u\} \). Let \( P = [x = x_1, x_2, \ldots, x_{n-1}, x_n = u] \) be a \( u-u \) geodesic in \( G \) as described in condition (ii)(b). In particular, \( x_{n-1} \in C_G \). Pick \( y \in V(K_n) \) such that \((x_{n-1}, y) \notin C\). Then \([u, z), (x_{n-1}, y), (v, w)\] is a \((u, z)-(v, w)\) path in \( \langle V(G[K_n]) \setminus C \rangle \).

The above cases imply that \( \langle V(G[K_n]) \setminus C \rangle \) is connected. Therefore, \( C \) is an outer-connected semitotal dominating set of \( G[K_n] \).

Now, we provide proof for the following lemma, which is very useful to get the desired result in this section. The lemma is given without proof in [1].
Lemma 1.  [1] If $G$ is a nontrivial connected graph and $S \subseteq V(G)$ is a dominating set in $G$, then

$$\gamma_{t_2}(G) \leq 2|S \setminus N^2_G(S)| + |S \cap N^2_G(S)|.$$  

Proof. Let $S \subseteq V(G)$ be a dominating set of $G$. For each $x \in S \setminus N^2_G(S)$, pick $u_x \in V(G)$ such that $xu_x \in E(G)$. Then $S^* = S \cup \{u_x : x \in S \setminus N^2_G(S)\}$ is a semitotal dominating set of $G$. Thus,

$$\gamma_{t_2}(G) \leq |S^*| = |S \cap N^2_G(S)| + 2|S \setminus N^2_G(S)|.$$ 

Corollary 5. Let $G$ be a nontrivial connected graph and $n \geq 2$. Then

$$\tilde{\gamma}_{t_2}(G[K_n]) = \gamma_{t_2}(G).$$ 

Proof. Let $S \subseteq V(G)$ be a $\gamma_{t_2}$-set of $G$. Choose $v \in V(K_n)$ and define $C = S \times \{v\}$. Since conditions (i)(a) and (ii)(b) of Theorem 6 hold for $C$, $C$ is an outer-connected semitotal dominating set of $G[K_n]$. Consequently, $\tilde{\gamma}_{t_2}(G[K_n]) \leq |S| = \gamma_{t_2}(G).$

Let $C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[K_n])$ be an outer-connected semitotal dominating set of $G[K_n]$. By Theorem 6, $S$ is a dominating set of $G$. If $S$ is a semitotal dominating set of $G$, then

$$\gamma_{t_2}(G) \leq |S| \leq \sum_{x \in S} |T_x| = |C|.$$ 

Suppose that $S$ is not a semitotal dominating set in $G$. Let $S_1 = S \setminus N^2_G(S)$ and $S_2 = S \cap N^2_G(S)$. By Theorem 6,

$$C = \left(\bigcup_{x \in S_1} (\{x\} \times T_x)\right) \cup \left(\bigcup_{x \in S_2} (\{x\} \times T_x)\right),$$

where $|T_x| \geq 2$ for all $x \in S_1$. Thus,

$$|C| = \sum_{x \in S_1} |T_x| + \sum_{x \in S_2} |T_x| \geq 2|S_1| + |S_2| = 2|S \setminus N^2_G(S)| + |S \cap N^2_G(S)|.$$ 

By Lemma 1, $\gamma_{t_2}(G) \leq |C|$. Since $C$ is arbitrary, $\gamma_{t_2}(G) \leq \gamma_{t_2}(G[K_n])$. 

$\square$
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References


