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On Salem formal power series

Oussama Dammak[1,2], Saber Mansour[1,3,*]

[1] *Department of Mathematics, Umm Al-Qura University, College of First Common year, P.O. Box 14035, Holly Makkah 21955, Saudi Arabia*
[2] *University of Gabes, Science Faculty of Gabes, Omar Ibn AlKhattab-Zrig Road, 6029 Gabes, Tunisia*
[3] *Sfax University, Department of Mathematics, Faculty of Sciences of Sfax, Soukra Road, 3018 Sfax, BP 802, Tunisia*

Abstract. In this paper, we will take a look at Salem elements dealing with formal power series on F_q, where F_q is a finite field . Our main result, presents a criteria for an element to be the smallest Salem element (SSE) via an order extending a given order in F_q. Moreover, we provide the CFE of the (SSE) for each n.

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1. Introduction

Let α_1 be an algebraic integer of degree n with Galois conjugates $\alpha_2, \alpha_3, \cdots, \alpha_n$. If

$$\begin{cases} & |\alpha_1| > 1 \\ and & |\alpha_i| < 1, \ \forall 2 \leq i \leq n, \end{cases}$$

α_1 is called to be a Pisot number. If

$$\begin{cases} & |\alpha_1| > 1 \\ and & |\alpha_i| = 1, \ \text{for } 2 \leq i \leq n, \\ \text{and} & |\alpha_j| \leq 1, \ \text{for } 2 \leq j \leq n, j \neq i, \end{cases}$$

α_1 is called to be a Salem number.

The set of the so called, Pisot numbers, is usually denoted by S, it is denoted by T, the set of Salem numbers. Thus, Pisot numbers are commonly referred to as $S-$numbers,

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Email addresses: omdammak@uqu.edu.sa (O. Dammak), samansour@uqu.edu.sa (S. Mansour)

while Salem numbers are referred to as $T-$numbers. The sets S and T appears in a variety of algebraic number theory problems, Diophantine approximation, Fourier analysis, distribution, the so-called β-expansions, etc. The set S was defined approximately simultaneously and separately by C. Pisot [9] and Vihayaraghavan [18, 19]. R. Salem [11] a few years later, defined the set T. However, some of the first research in this approach and related to uniform distributions were published earlier by Thue [17].

In 1919, Hardy showed that if α is an algebraic integer such that $\alpha^n \to 0(\mod 1)$ as $n \to \infty$, then α is a Pisot number.

Later, this concept, was investigated by Salem, who proved that the only algebraic numbers that have this property of being badly distributed modulo 1 are S-numbers [12]. In brief, this occurs because for all algebraic number α and $n \in \mathbb{N}^*$, the sum of the nth powers of α and its conjugates is an integer.

when α is an $S-$number, the nth powers of the conjugates of α tends to 0 as n tends to ∞, because they all have modulus strictly less than 1. This fundamental property of S-numbers raises the following significant unanswered question about the characterization of the set S. Asuume $\alpha > 1$ is a real number with $\alpha^n \to 0(\mod 1)$ as $n \to \infty$.

Can we then conclude, under no other assumptions, that θ is an algebraic integer in the set S ? This is possibly one of the oldest unsolved problems involving S-numbers, as it appears in [12].

The sets S and T have been widely investigated, and a large number of results are known about them. Here are a handful of the more notable results. Because they exclusively include algebraic numbers, both S and T are clearly countable sets. Furthermore, S contains infinitely many limit points, because it is possible to consider each integer $a \geq 2$ as a limit of a sequence of elements in S. Then, S contains an infinity of limit points.

The derived set of S is denoted by S' and contains all of S's limits points. Equally, the derived set of T, is the set of all T limits points, and it is denoted by T'.

While the product of two algebraic integers is also an algebraic integer, Pisot numbers are not. For instance, We have just shown, that both 2 and $\dfrac{1+\sqrt{5}}{2}$ are Pisot; however, their product $1+\sqrt{5}$ with conjugate equals to $1-\sqrt{5}$ clearly is not a Pisot number.

If the two Pisot numbers are equal, their product is a Pisot number as well.

The sets S and T are tightly connected and contain fascinating linkages, as one might expect given their comparable definitions. In [10] it was proved, by Salem, that $S' \subset S$ is closed. So, it must have a smallest element because it is bounded below. According to Siegel [14], the smallest known Pisot number is equal to the largest root of $x^3 = x + 1$, which is around 1.3247179.

The smallest Pisot number of degree $n \geq 3$ was identified by Dufresnoy and Pisot [5].

They arrived to the following theorem :

Theorem 1. *Let a_n be the smallest Pisot number of degree $n \geq 3$, then the following assertions holds*
i) $P_n(z) = z^n - z^{n-1} - z^{n-2} + z^2 - 1$ *is the minimal polynomial of* a_n
ii) the sequence $(a_n)_{n\geq 1}$ *is increasing and eventually converges to* $\dfrac{1+\sqrt{5}}{2} \approx 1.61803$, *a*

root of $x^2 = x + 1$.

It's worth noting that similar claims concerning the set T have yet to be discovered. There are, indeed, two of the most well-known unanswered questions about S and T-numbers. The first question concerns T's limit point. The set S is known to be contained in T', or, to put it another way, any point of S is a limit point of T, on both sides. Salem [11] was the first to demonstrate this astounding fact by creating polynomial sequences that provided the needed $T-$numbers. Although it is known that $S \subseteq T'$, There's no way of knowing if the set T has any limit points other than the ones identified in S; This is an inquiry that is addressed in [12], but it is yet unsolved.

Small $T-$numbers are the subject of the second open question. We can't assume the existence of a lowest Salem number because T isn't closed. While we can't assume that T contains a smallest element, there is a possibility. It is conjectured that 1.1762808..., a root of the 10^{th} degree polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1,$$

discovered by Lehmer [7] in 1933, is the smallest Salem number.

There hasn't been a smaller Salem number in almost eighty years. All Salem numbers < 1.3 of degree 20 over [3], are included in the list of 39 Salem numbers given in [2]; it will enough for the applications below. There are currently around 47 Salem numbers < 1.3 are known and the list is believed to be comprehensive up to degree 44..

Although this has not been proven, it is widely assumed that Lehmer's Salem number 1.1762808... is an isolated point of T.

It was stated and proven some basic results about Salem numbers in [15], and then a survey of the literature about them was conducted. Chris Smyth's intention was to supplement rather than duplicate other general treatises on these numbers. This is especially true of Bertin and her coauthors' work [1], as well as Ghate and Hironaka's application-rich Salem number survey [6]. He did, however, cite some findings from Salem's classic monograph [13].

Moreover, the concept of the Mahler measure of a matrix arose from the investigation of Salem numbers, which appeared as Mahler measures of graphs. It was demonstrated that certain limit points of these Salem numbers are Pisot numbers, providing yet another example of a more general result. it was focused on an interlacing construction for Salem numbers, which evolved historically from the graph construction but is far more general (For more details one can see [8]).

Chandoul et al. [4], established that the minimal polynomial of the so-called, smallest Pisot element (SPE) of degree n in in $F_q((X^{-1}))$ is $P(Y) = Y^n - aXY^{n-1} - a^n$, where a is the least element of the finite field $\mathbb{F}_q \setminus \{0\}$ (as a finite total ordered set).

It was shown that the sequence of SPEs in the case of degree n is decreasing. Moreover it converges to aX.

But why is finding the smallest element of a set so important?

The answer is summed up in the possibility of discovering a total order and a point that reduces all the properties of the set which is the smallest according to this order.

The smallest Salem element (SSE) of a given degree n in $Fq((X-1))$ is presented in this work.

The following is how the paper is structured: In Section 2, we define the lexicographic order on in $F_q((X^{-1}))$ and provide some early definitions. We offer the (SSE) of degree n in in $F_q((X^{-1}))$ in Section 3. Section 4 investigates the (SSE)'s CFE over in in $F_q((X^{-1}))$.

2. Formal power series

Let $\mathbb{F}_q$ be a field with q elements of characteristic p, $\mathbb{F}_q[X]$ the set of polynomials of coefficients in $\mathbb{F}_q$ and $\mathbb{F}_q(X)$ its field of fractions. The set $\mathbb{F}_q((X^{-1}))$ of formal power series over $\mathbb{F}_q$ is defined as follows

$$\mathbb{F}_q((X^{-1})) = \{\sum_{j=s}^{+\infty} a_j X^{-j} \ : \ a_j \in \mathbb{F}_q, \ a_s \neq 0 \text{ with } s \in \mathbb{Z}\}.$$

Let $\omega = \sum_{j=s}^{+\infty} a_j X^{-j} \in \mathbb{F}_q((X^{-1}))$. The polynomial part of ω is denoted by $[\omega]$ and its fractional part is denoted by $\{\omega\}$. We remark that $\omega = [\omega] + \{\omega\}$. As in Sprindžuk [16] a non archimedean absolute value on $\mathbb{F}_q((X^{-1}))$ is definied by $\mid \omega \mid = e^{-s}$. Clearly, we have, $\mid P \mid = e^{\deg P}$, for all $P \in \mathbb{F}_q[X]$, and, $\mid \dfrac{P}{Q} \mid = e^{\deg P - \deg Q}$, for all $Q \in \mathbb{F}_q[X]$, such that $Q \neq 0$.

It is well known that $\mathbb{F}_q((X^{-1}))$ is complete. In terms of the metric provided by this absolute value, $\mathbb{F}_q((X^{-1}))$ is locally compact.

An algebraic closure of $\mathbb{F}_q((X^{-1}))$ is denoted by $\overline{\mathbb{F}}_q((X^{-1}))$. It is worth noting that the absolute value has a distinct extension to $\overline{\mathbb{F}}_q((X^{-1}))$. We will use the same symbol $|\cdot|$ for the two absolute values, slightly abusing the notations.

As $\mathbb{F}_q$ is a finite total order set, we denote by $\preceq$ a totally order on $\mathbb{F}_q$.

Now, we extend $\preceq$ to the field of formal power series as follows :

Let $w = \sum_{i=m}^{+\infty} w_i X^{-i}$ and $v = \sum_{i=k}^{+\infty} v_i X^{-i}$ with $w_m v_k \neq 0$, then, $w \preceq v$ if and only if $m > k$ or $w = v$ or $m = k$ and there exists $j \geq m$, such that $w_i = v_i$, for $i < j$ and $w_j \preceq v_j$.

Let $w \in_q ((X^{-1}))$, with $|w| > 1$ is called to be a Salem element if it is algebraic over $_q[X]$ whose conjugates w_i in $\overline{\mathbb{F}}_q((X^{-1}))$ have modulus $\mid w_i \mid \leq 1$, with at least one case of equality.

Theorem 2. *Let w such that $|w| > 1$, be an element of $_q((X^{-1}))$, These two affirmations are equivalent:*
1) w is a Salem element.

2) the minimal polynomial P of w can be written as $P(Y) = Y^s + A_{s-1}Y^{s-1} + \cdots + A_0$, with $A_i \in_q [X] \setminus \{0\}$, $A_i \in_q [X]$ for $i = 0, \ldots, s-1$, and $|A_{s-1}| = \max\limits_{i \neq s-1} |A_i|$.

Now, we will go over the main results.

3. Main result

Theorem 3. *Let $\mathcal{S}(n) = \{$Salem elements of degree $n\}$, $n \geq 2$ and a is the least element of $\mathbb{F}_q \backslash \{0\}$, then $w_n = \inf \mathcal{S}(n)$ is a Salem element of minimal polynomial*

$$P_n(Y) = Y^n - aXY^{n-1} - aY + aX - a$$

In addition, the sequence $(w_n)_{n \geq 1}$ of $w_n = \inf \mathcal{S}(n)$ is decreasing one and converges to aX.

The following lemmas are needed to prove this theorem.

Lemma 1. *Let $P(Y) = A_dY^d + \cdots + A_0$ with $A_i \in_q [X]$, $A_d \neq 0$ and $|A_{n-1}| > |A_i|$, for all $i \neq n-1$. Then P has exactly one root $w \in_q ((X^{-1}))$ such that $|w| > 1$. Furthermore $[w] = -\left[\dfrac{A_{n-1}}{A_n}\right]$.*

Lemma 2. *Let $H(Y) = Y^d - AY^{d-1} - B$, $A, B \in \mathbb{F}_q[X] \setminus \{0\}$, $\deg A \geq \deg B$. Then H is irreducible over $\mathbb{F}_q[X]$.*

Proof. According to Lemma 1, H has exactly one root w such that $\mid w \mid > 1$ and $[w] = A$. Let w_i be the other roots of H, $2 \leq i \leq d$ and $w = w_1$.

Because H is a monic polynomial, then, we have $\sum\limits_{k=1}^{d} w_i^k \in \mathbb{F}_q[X]$, for every $k \in \mathbb{N}$, which implies $\lim\limits_{m \to +\infty} \{w^m\} = 0$.

Let $P(Y) = Y^n + A_{n-1}Y^{n-1} + \cdots + A_0$ be the minimal polynomial of w, it is obvious that $A_{n-1} = -A$, since $[w] = A$. From Theorem 2, the polynomial P satisfies $\deg A_{n-1} \geq \max\limits_{i \neq n-1} \deg A_i$.

Let now $H(Y) = P(Y)Q(Y)$, with $Q(Y) = Y^m + B_{m-1}Y^{m-1} + \cdots + B_0$. Suppose that $m \geq 1$, then

$$B_{m-1} + A_{n-1} = -A \text{ and } A_0B_0 = -B \tag{1}$$

$$\sum_{\substack{i+j=s \\ 0 \leq i \leq n \\ 0 \leq j \leq m}} A_iB_j = 0 \ ; \ s \in \{1, 2, \cdots, d-2\}. \tag{2}$$

Since $A_{n-1} = -A$, then from (1) $B_{m-1} = 0$. Let $i_0 \in \{0, 1, \cdots, m\}$ such that $\deg B_{i_0} = \max\limits_{0 \leq i \leq m} \deg B_i$.

If $B_{i_0} \neq 0$, then $\deg(A_{n-1}B_{i_0}) > \deg(A_iB_j),\ (i,j) \neq (n-1, i_0)$. Consequently

$$\deg\left(\sum_{\substack{i+j=n+i_0-1\\0\leq i\leq n\\0\leq j\leq m}} A_iB_j\right) = \deg(A_{n-1}B_{i_0}),$$

there is a contradiction with (2).

Finally, we arrive to $H(Y) = Y^mP(Y)$, if $m \geq 1$, which leads to a contradiction (due to the fact that $B \neq 0$) and arising $H(Y) = P(Y)$.

Proof. **of Theorem 3.** Consider the polynomial $P_n(Y) = Y^n - aXY^{n-1} - aY + aX - a$, such that a is the least element of $\mathbb{F}_q\backslash\{0\}$,. It follows from lemma 2 that P is irreducible. Moreover, P has exactly one root w_n satisfying $\mid w_n \mid > 1$ and $[w_n] = aX$, all of whose other conjugates w_i satisfy $\mid w_i \mid \leq 1$ with at least one case of equality. Because P is a monic polynomial, w_n is a Salem element of degree n. Additionally, if $w_n = aX + \dfrac{1}{h}$, then h is an algebraic formal power series satisfying

$$a^nh^n - (aX)^{n-1}h^{n-1} - \sum_{k=0}^{n-2}\binom{n-1}{k}(aX)^kh^k = 0$$

and using lemma 1, we have

$$[h] = \frac{X^{n-1}}{a}. \tag{3}$$

Now, we consider an other Salem element $v_n \neq w_n$ of degree n such that $[v_n] = aX$, then from Theorem 2 the minimal polynomial of v_n can be written as $F(Y) = Y^n - aXY^{n-1} - \sum_{i=0}^{n-2} A_iY^i$ where $\deg A_i \leq 1$, with at least one case of equality. Let $v_n = aX + \dfrac{1}{g}$, then

$$\begin{aligned}
F\left(aX + \frac{1}{g}\right) &= \left(aX + \frac{1}{g}\right)^n - \left(aX + \frac{1}{g}\right)^{n-1} \\
&- \sum_{j=0}^{n-2} A_j\left(aX + \frac{1}{g}\right)^j \\
&= \sum_{j=0}^{n} A_j \sum_{k=0}^{j}\binom{j}{k}(aX)^{j-k}g^{-k} \\
&;\quad A_n = 1,\ A_{n-1} = aX \\
&= \sum_{k=0}^{n}\left(\sum_{j=n-k}^{n} A_j\binom{j}{n-k}(aX)^{j+k-n}\right)g^k = 0.
\end{aligned}$$

Let

$$B_k = \sum_{j=n-k}^{n} A_j \begin{pmatrix} j \\ n-k \end{pmatrix} (aX)^{j+k-n}. \tag{4}$$

Then

$$\sum_{k=0}^{n} B_k g^k = 0.$$

We have, by using lemma 1 and the equation (4)

$$[g] = \left[\frac{(aX)^{n-1} - \sum_{i=1}^{n-2} iA_i (aX)^{i-1}}{\sum_{i=0}^{n-2} A_i (aX)^i} \right],$$

• if $(A_1, \dots, A_{n-2}) \neq (0, \dots, 0)$, we have $\deg[g] < n-1 = \deg[h]$ then $\frac{1}{h} \preceq \frac{1}{g}$,

• if $(A_1, \dots, A_{n-2}) = (0, \dots, 0)$, we have $[g] = \frac{a^{n-1}}{A_0} X^{n-1}$ and from (3), we obtain $\frac{1}{h} = aX^{-(n-1)} + \cdots \preceq \frac{A_0}{a^{n-1}} X^{-(n-1)} + \cdots = \frac{1}{g}$ ($A_0 \neq a^n$ if not v_n is not a Salem element.

Hence, we get in the two cases $\frac{1}{h} \preceq \frac{1}{g}$, which implies that $w_n \preceq f_n$, and consequently w_n is the (SSE) of degree n.

Since $w_n = aX + \frac{1}{h}$, then from (3) $|w_n - aX| = |\frac{1}{h}| = |\frac{a}{X^{n-1}}| = e^{-(n-1)}$, consequently $\lim_{n \to +\infty} w_n = aX$.

4. CFE of the SSE

Let $\mathcal{J} = \{f \in_q ((X^{-1}))/ \mid f \mid < 1\}$ and let $T : \mathcal{J} \to \mathcal{J}$ be the map given by

$$T(\omega) := \frac{1}{\omega} - \left[\frac{1}{\omega}\right], \ \omega \neq 0, \ T(0) = 0,$$

recall that the map T generate the continued fraction expansion of ω of the form

$$\omega = A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \ddots + \cfrac{1}{A_n + \ddots}}}}, \tag{5}$$

where $A_n = \left[\frac{1}{T^{n-1}(w)}\right]$. When used as a shorthand for (5) we can write

$$\omega = [A_0; A_1, A_2, \ldots].$$

Let ω the (SSE) of degree $q^n + 1$, with $n \in \mathbb{N}$, then $\omega = [A_0, A_1, \ldots, A_s, \ldots]$, where $A_0 = aX$, $A_1 = X^{q^n-1}$

$$A_{2s} = aXX^{q^{(2s)n}-q^{(2s-1)n}} \text{ and } A_{2s+1} = X^{q^n-1}X^{q^{(2s+1)n}-q^{(2s)n}}.$$

Proof. Let $P(Y) = Y^{q^n+1} - aXY^{q^n} - aX$ the minimal polynomial of (SPE) $w = w_{q^n+1}$. Let $z_0 = \omega,\ A_0 = [z_0] = aX,\ U_0 = 1,\ V_0 = -aX,\ R_0 = 0,\ T_0 = -aX$ and $z_{s+1} = \dfrac{1}{z_s - [z_s]}$, then from lemma 1 and lemma 2 we know that z_s satisfies the equation

$$U_s z_s^{q^n+1} + V_s z_s^{q^n} + R_s z_s + T_s = 0$$

with $\deg V_s > \max(\deg U_s, \deg R_s, \deg T_s)$ for all $s \geq 1$ and

$$\begin{aligned}
&U_{s+1} = U_s A_s^{q^n+1} + V_s A_s^{q^n} + R_s A_s + T_s,\\
&V_{s+1} = U_s A_s^{q^n},\\
&R_{s+1} = V_s + A_s U_s,\\
&T_{s+1} = U_s,\\
&A_{s+1} = -\left[\frac{V_{s+1}}{U_{s+1}}\right]
\end{aligned}$$

Now one shows, using a simple recurrence on s , that

$$\begin{aligned}
&U_{2s} = 1,\ U_{2s+1} = -aX,\\
&V_{2s} = -(aX)X^{q^n-1}X^{q^{(2s-1)n}-q^{(2s-2)n}q^n},\\
&V_{2s+1} = a(XX^{q^{(2s)n}-q^{(2s-1)n}})^{q^n},\\
&R_s = 0,\ T_{2s} = -aX,\ T_{2s+1} = 1,\\
&A_{2s} = aXX^{q^{(2s)n}-q^{(2s-1)n}}\\
&\text{and } A_{2s+1} = X^{q^n-1}X^{q^{(2s+1)n}-q^{(2s)n}}.
\end{aligned}$$

Example 1. *The minimal polynomial of the (SSE) w of degree 2 over $_2((X^{-1}))$ is $P(Y) = Y^2 - XY - X$, so $w = \sum_{i=-1}^{\infty} w_i X^{-i}$ is defined by*

$$\begin{cases} w_{-1} = w_0 = w_1 = 1, \\ w_{2n} = 0 \\ w_{2n+1} = w_n \end{cases}, \text{ for all } n \geq 0.$$

The continued fraction of w is $w = [X, X^3, X^{11}, X^{48}, \cdots]$.

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