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Applications of double fuzzy Sumudu Adomian decomposition method for two-dimensional fuzzy Volterra integral equations

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Abstract. In this paper, we combine double Sumudu transform with Adomian decomposition method to solve two dimensional fuzzy convolution Volterra integral equations. We give some definition for fuzzy number, fuzzy valued function which are prerequest for combine the double fuzzy Sumudu transform with Adomian decomposition transform method. We describe the method and finally, numerical examples are given to illustrate the efficiency and applicability of our method.

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1. Introduction

An important branch in fuzzy mathematics is the topics of fuzzy integral equations. The concept of fuzzy numbers and arithmetic operations firstly introduced by Zadeh [11]. The topic of integration of fuzzy functions was initially coined out by Dubois and Prade [12]. The study of fuzzy Volterra integral equations begins in Kaleva [20], Seikkala [26] and Mordeson and Newman [24], such integral equations are applied in control mathematical models.

The problems posed in the study of fuzzy integral equations are: existence and uniqueness, boundedness of the solutions [13, 16] and the construction of numerical methods for the approximate solution.

Integral transforms constitute fundamental tools in operational calculus. They are mathematical operators that have been used widely in solving many practical problems in applied mathematics, physics and engineering [25, 27]. There are a number of papers on the theories and applications of integral transforms, some of which are Laplace, Mellin

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and Hankel transforms [23, 28, 29]. Subsequently, the concept of integral transforms was expanded to remove the necessity of finite intervals. Watugala [18, 19] has proposed a new integral transform called the Sumudu transform. One of the most recent methods in handling problems modelled under fuzzy environment is fuzzy Sumudu transform (FST). FST has been used for solving various kinds of fuzzy differential and integral equations [21, 30]. By using FST, the problem is reduced to algebraic problem which is much simpler to be solved. This fall under the topic of operational calculus and under this topic, FST is considered as one of a powerful method alongside with fuzzy Laplace transform. .
The Adomian decompositon method (ADM) has been recently intensively studied by scientists and engineers and used for solving nonlinear diferencial and integral problems. The adomian decompositon method (ADM) intruduced by Adomian [1, 2] for solving different kind of functional equations and has been subject of extensive numerical and analytical studies. Babolian at al.[7], ADM to solve fuzzy Fredholm integral equations, respectively Allahviranloo at al.[8], solved fuzzy system of Fredholm integral equations. Also, Behzadi [5], solving nonlinear i fuzzy Volterra Fredholm integral equations with Adomain decompositon method.
In many papers have combine with Laplace -Adomian transform method and Sumudu -Adomain transform to solve diferential and integral equations. Alidema [3] using Adomian decomposition method for solving two-dimensional nonlinear Volterra fuzzy integral equations.
Morever in the recent years contributed in this field [36–38] and [39–41]. Bushnaq at al. [9] find solution a fuzzy Volterra Abel's integral equation of the second kind with Laplace - ADM. Georgieva [4, 5, 14], gave solution two dimensional fuzzy Volterra-Fredholm with Adomain and use double fuzzy Sumudu Transform to solve partial Volterra Fuzzy Integro-Differential Equations and fuzzy Sawi decompositon for nonlinear differential equations. L. Al-Taee at al.[34, 35], solving system of nonlinear Volterra integral equations with Sumudu-ADM and fuzzy system of Volterra integro-differential equations by using ADM.

The subject of this paper is to apply the double fuzzy Sumudu transform with Adomian decomposition method for solving fuzzy convolution Volterra integral equation in two variables.

We give some preliminaries on fuzzy numbers, fuzzy functions and fuzzy integrals. Then we provide the proposed DFST with ADM, properties with DFST. We introduced the concept of double fuzzy convolution. Later we construct in detail procedure for solving two-dimensional fuzzy convolution Volterra integral equation using DFST with ADM. Numerical example is provided to demonstrate the proposed method.

2. Basic Preliminaries

Definition 1. *[31]A fuzzy number is a function $u : R \to [0, 1]$ satisfying the following properties:*
(i) u is upper semi-continuous on R,

(ii) $u(x) = 0$ *outside of some interval* $[c, d]$,
(iii) there are the real numbers a *and* b *with* $c \le a \le b \le d$, *such that* u *is increasing on* $[c, a]$, *decreasing on* $[b, d]$ *and* $u(x) = 1$ *for each* $x \in [a, b]$,
(iv) u *is fuzzy convex set i. e. that is* $u(\lambda x + (1-\lambda)y) \ge \min\{u(x), u(y)\}$ *for all* $x, y \in R$ *and* $\lambda \in [0, 1]$.

The set of all fuzzy numbers is denoted by E^1. Any real number $a \in R$ can be interpreted as a fuzzy number $\tilde{a} = \chi_{[a]}$ and therefore $R \subset E^1$.

According to [32] for any $0 < r \le 1$ we denote the $r-$level set $[u]^r = \{x \in R : u(x) \ge r\}$ that is a closed interval and $[u]^r = [u_-^r, u_+^r]$ for all $r \in [0, 1]$. These lead to the usual parametric representation of a fuzzy number, by an ordered pair of functions (u_-^r, u_+^r), which satisfies the following properties: u_-^r is bounded left continuous non-decreasing function over $[0, 1]$, u_+^r is bounded left continuous non-increasing function over $[0, 1]$ and $u_-^r < u_+^r$.

For $u, v \in E^1$, $k \in R$, the addition and the scalar multiplication are defined by $[u \oplus v]^r = [u]^r + [v]^r = [u_-^r + v_-^r, u_+^r + v_+^r]$ and $[k \odot u]^r = k.[u]^r = \begin{cases} [ku_-^r, ku_+^r], & \text{if } k \ge 0 \\ [ku_+^r, ku_-^r], & \text{if } k < 0 \end{cases}$ for all $r \in [0, 1]$.

The neutral element respect to $\oplus$ in E^1, denoted by $\tilde{0} = \chi_{[0]}$. The algebraic properties of addition and scalar multiplication of fuzzy numbers are given in [32].

As a distance between fuzzy numbers we use the Hausdorff metric [32] defined by $D(u, v) = \sup\limits_{r \in [0,1]} max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}$ for any $u, v \in E^1$.

The metric space (E^1, D) is complete, separable and local compact.

Lemma 1. *[32] The Hausdorff metric has the following properties:*

(i) $D(u \oplus w, v \oplus w) = D(u, v)$ for all $u, v, w \in E^1$,

(ii) $D(u \oplus v, w \oplus e) \le D(u, w) + D(v, e)$ for all $u, v, w, e \in E^1$,

(iii) $D(u \oplus v, \tilde{0}) \le D(u, \tilde{0}) + D(v, \tilde{0})$ for all $u, v \in E^1$,

(iv) $D(\lambda \odot u, \lambda \odot v) = |k| D(u, v)$ for all $u, v \in E^1$, for all $\lambda \in R$,

(vi) $D(\lambda_1 \odot u, \lambda_2 \odot u) = |\lambda_1 - \lambda_2| D(u, \tilde{0})$ for all $\lambda_1, \lambda_2 \in E^1$, *with* $\lambda_1, \lambda_2 \ge 0$ *and* for all $u \in E^1$.

For any two-variable fuzzy-valued function $f : R_+^2 \to E^1$ we can define the functions $\underline{f}(.,.,r)$, $\overline{f}(.,.,r) : R_+^2 \to R$. These functions are called the left and right $r-$level functions of f.

Definition 2. *[33] A fuzzy-number-valued function* $f: R\times R \to E^1$ *is said to be continuous at* $(s_0,t_0)\in R\times R$ *if for each* $\varepsilon>0$ *there is* $\delta>0$ *such that* $D(f(s,t),f(s_0,t_0))<\varepsilon$ *whenever* $((s-s_0)^2+(t-t_0)^2)^{\frac{1}{2}}<\delta$*. If* f *be continuous for each* $(s,t)\in R\times R$ *then we say that* f *is continuous on* $R\times R$*. A fuzzy number* $u\in E^1$ *is upper bound for a fuzzy-number-valued function* $f: R\times R\to E^1$ *if* $\underline{f}(s,t,r)^r\le u_-^r$ *and* $\overline{f}(s,t,r)\le u_+^r$ *for all* $(s,t)\in R\times R$*,* $r\in[0,1]$*. A fuzzy number* $u\in E^1$ *is lower bound for a fuzzy number-valued function* $f: R\times R\to E^1$ *if* $u_+^r\le\overline{f}(s,t,r)$ *and* $u_-^r\le\underline{f}(s,t,r)$ *for all* $(s,t)\in R\times R$*,* $r\in[0,1]$*. A fuzzy-number-valued function* $f: A\to E^1$ *is said to be bounded it has a lower bound and an upper bound.*

For any two-variable fuzzy-valued function $f: R_+^2\to E^1$ we can define the functions $\underline{f}(.,.,r),\ \overline{f}(.,.,r): R_+^2\to R$. These functions are called the left and right $r-$level functions of f.

Definition 3. *[31] A fuzzy-valued function* $f: R^2\to E^1$ *is said to be continuous at* $(s_0,t_0)\in R^2$ *if for each* $\varepsilon>0$ *there is* $\delta>0$ *such that* $D(f(s,t),f(s_0,t_0))<\varepsilon$ *whenever* $|s-s_0|+|t-t_0|<\delta$*. We say that* f *is continuous on* R^2 *if* f *is continuous at each* $(s,t)\in R^2$*.*

Theorem 1 ([4])**.** *Let the fuzzy-valued function of two-variable* $f: R_+^2\to E^1$ *represented by* $(\underline{f}(x,y,r),\overline{f}(x,y,r))$*. For any* $r\in[0,1]$ *assume that the functions* $\underline{f}(x,y,r)$ *and* $\overline{f}(x,y,r)$ *are Riemann-integrable on* $[0,X]\times[0,Y]$ *and there are positive constants* $\underline{N}(r)$ *and* $\overline{N}(r)$*, such that*

$$\int_0^Y\int_0^X |\underline{f}(x,y,r)|dxdy\le\underline{N}(r),\quad \int_0^Y\int_0^X |\overline{f}(x,y,r)|dxdy\le\overline{N}(r)$$

for every $X,\ Y>0$.
Then the function $f(x,y)$ *is improper fuzzy Riemann-integrable on* R_+^2 *and*

$$(FR)\int_0^\infty(FR)\int_0^\infty f(x,y)dxdy=\left(\int_0^\infty\int_0^\infty\underline{f}(x,y,r)dxdy,\int_0^\infty\int_0^\infty\overline{f}(x,y,r)dxdy\right)$$

Theorem 2. *[4] Let* $f: R_+^2\to E^1$ *be fuzzy-valued function of two-variable. Suppose that for each* $x\in[0,\infty)$*, the fuzzy integral* $(FR)\int_0^\infty f(x,y)dy$ *is convergent and moreover the fuzzy integral* $(FR)\int_0^\infty f(x,y)dx$ *as a function of* y *is convergent on* $[0,\infty)$*. Then*

$$(FR)\int_0^\infty(FR)\int_0^\infty f(x,y)dydx=(FR)\int_0^\infty(FR)\int_0^\infty f(x,y)dxdy.$$

Theorem 3. *[6] Let f and g be fuzzy-valued functions. Let $F(u,v)$ and $G(u,v)$ be double fuzzy Sumudu transforms for f and g, respectively. Then the DFST of the double fuzzy convolution f and g,*

$$(f**g)(x,y) = \int_0^y \int_0^x f(\alpha,\beta)g(x-\alpha,y-\beta)d\alpha d\beta$$

is given by

$$S[(f*g)(x,y)] = uvF(u,v)G(u,v).$$

Definition 4. *[6] Let $f : A \to E^1$, for $\Delta_x^n : a = x_0 < x_1 < ... < x_n = b$ and $\Delta_y^n : c = y_0 < y_1 < ... < y_n = d$, be two partitions of the intervals $[a,b]$ and $[c,d]$, respectively. Let one consider the intermediates points $\xi_i \in [x_{i-1}, x_i]$ and $\eta_j \in [y_{j-1}, y_j]$, $i = 1,...,n$; $j = 1,...,n$, and $\delta : [a,b] \to R_+$ and $\sigma : [c,d] \to R_+$. The divisions $P_x = ([x_{i-1}, x_i]; \xi_i)$, $i = 1,...,n$, and $P_y = ([y_{j-1}, y_i]; \eta_j)$, $j = 1,...,n$ are said to be δ-fine and σ-fine, respectively, if $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $[y_{j-1}, y_j] \subseteq (\eta_j - \sigma(\eta_j), \eta_j + \sigma(\eta_j))$.*

The function f is said to be two-dimensional Henstock integrable to $I \in E^1$ if for every $\varepsilon > 0$ there are functions $\delta : [a,b] \to R_+$ and $\sigma : [c,d] \to R_+$ such that for any δ-fine and σ-fine divisions we have $D(\sum_{j=1}^{n}\sum_{i=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j), I) < \varepsilon$, where $\sum$ denotes the fuzzy summation. Then, I is called the two-dimensional Henstock integral of f and is denoted by $I(f) = (FH)\int_c^d (FH)\int_a^b f(s,t)dsdt$. If the above δ and σ are constant functions, then one recaptures the concept of Riemann integral. In this case, $I \in E^1$ will be called two-dimensional integral of f on A and will be denoted by $(FR)\int_c^d (FR)\int_a^b f(s,t)dsdt$.

3. Double Fuzzy Sumudu Transform

Definition 5. *[6] Let $f : R \times R \to E^1$ be a continuous fuzzy-valued function . Suppose that $e^{-x-y} \odot f(ux, vy)$ is improper fuzzy Riemann-integrable on $R_+ \times R_+$, then $(FR)\int_0^\infty (FR)\int_0^\infty e^{-x-y} \odot f(ux,vy)dxdy$ is called the double fuzzy Sumudu transform and is denoted by*

$$F(u,v) = S[f(x,y)] = (FR)\int_0^\infty (FR)\int_0^\infty e^{-x-y} \odot f(ux,vy)dxdy, \quad for\ u \in [-\tau_1, \tau_2]\ and\ v \in [-\sigma_1, \sigma_2], \tag{1}$$

where the variables u, v are used to factor the variables x, y in the argument of the fuzzy-valued function and $\tau_1,\ \tau_2,\ \sigma_1,\ \sigma_2 > 0$.

DFST also can be presented parametrically as follows

$$S[f(x,y)] = (s[\underline{f}(x,y,r)], s[\overline{f}(x,y,r)]) = (\underline{F}(u,v,r), \overline{F}(u,v,r)) = F(u,v).$$

From the classical double Sumudu transform, we have

$$s[\underline{f}(x,y,r)] = \int_0^\infty \int_0^\infty e^{-x-y}\underline{f}(ux,vy,r)dxdy \ \ and \ \ s[\overline{f}(x,y,r)] = \int_0^\infty \int_0^\infty e^{-x-y}\overline{f}(ux,vy,r)dxdy$$

Definition 6. *[6] Double fuzzy inverse Sumudu transform can be written as the formula*

$$S^{-1}\left[F(u,v)\right] = f(x,y) = \left(s^{-1}[\underline{F}(u,v,r)], s^{-1}[\overline{F}(u,v,r)]\right),$$

where

$$s^{-1}[\underline{F}(u,v,r)] = \frac{1}{2\pi\imath}\int_{\gamma-\imath\infty}^{\gamma+\imath\infty} e^{\frac{x}{u}}du\frac{1}{2\pi\imath}\int_{\delta-\imath\infty}^{\delta+\imath\infty} e^{\frac{y}{v}}\underline{F}(u,v,r)dv,$$

$$s^{-1}[\overline{F}(u,v,r)] = \frac{1}{2\pi\imath}\int_{\gamma-\imath\infty}^{\gamma+\imath\infty} e^{\frac{x}{u}}du\frac{1}{2\pi\imath}\int_{\delta-\imath\infty}^{\delta+\imath\infty} e^{\frac{y}{v}}\overline{F}(u,v,r)dv.$$

For all $r \in [0,1]$ the functions $\underline{F}(u,v,r)$ and $\overline{F}(u,v,r)$ must be analytic functions for all u and v in the region defined by the inequalities $Reu \geq \gamma$ and $Rev \geq \delta$, where γ and δ are real constants to be chosen suitably.

Some properties of Sumudu transform and double Sumudu transform in [9, 10, 22].

The convolution theorem is provided below.

Definition 7. *[6] If $f(x,y)$ and $g(x,y)$ are fuzzy Riemann integrable functions, then double fuzzy convolution of $f(x,y)$ and $g(x,y)$ is given by*

$$(f**g)(x,y) = \int_0^y \int_0^x f(\alpha,\beta)g(x-\alpha,y-\beta)d\alpha d\beta$$

and the symbol $**$ denotes the double convolution respect to x and y.

From the definition 6, it can easily be verified that the following properties of convolution hold :

(i) $(f**g)(x,y) = (g**f)(x,y)$, (commutative);

(ii) $c(f**g)(x,y) = [(cf)**g](x,y)$, c is constant;

(iii) $[f**(g**h)](x,y) = [(f**g)**h](x,y)$, (associative).

4. Double fuzzy Sumudu ADM for solving 2D-VIEs

This section contains Sumudu decomposition method for solving two dimensional fuzzy convolution Volterra integral equations.

The general form of this integral equation is given by

$$f(x,y) = g(x,y) \oplus (FR) \int_0^t \int_0^s k(x-\alpha, y-\beta) \odot G(f(\alpha,\beta)) d\alpha d\beta, \quad (x,y) \in [0,b] \times [0,b], \quad (2)$$

where $k(x-\alpha, y-\beta)$ are arbitrary given convolution kernel functions and g is a continuous fuzzy-valued function.

5. Main results

We combine DFST with ADM to solve two dimensional fuzzy convolution VIEs. Operating Sumudu transform method in both sides of equation (2). After that using the property and convulotion theorem of Sumudu transform.

By using convolution, we have,

$$S[f(x,y)] = S[g(x,y)] \oplus uvS[K(x,y)] \odot S[Gf(x,y)], \quad (3)$$

Operating inverse fuzzy Sumudu transformation on both sides.

$$f(x,y) = g(x,y) \oplus uvS^{-1}(K(x,y)) \odot S(Gf(x,y)), \quad (4)$$

$$\sum_{i=0}^{\infty} f_i(x,y,r) = g(x,y,r) + S^{-1}[uv[k(x,y)]S[\sum_{i=0}^{\infty} \underline{f}(x,y,r)]]$$

In the case $k(x-\alpha, y-\beta) = (x-\alpha)(y-\beta) > 0$, for $0 \leq \alpha \leq x \leq 1$ and $0 \leq \beta \leq y \leq 1$. $0 \leq r \leq 1$ The parametric form of equations is

$$\underline{f}_{i+1}(x,y,r) = \underline{g}(x,y,r) + S^{-1}[uv[k(x,y)]S[\underline{f}_i(x,y,r)]], \quad i \geq 0.$$

$$\overline{f}_{i+1}(x,y,r) = \overline{g}(x,y,r) + S^{-1}[uv[k(x,y)]S[\overline{f}_i(x,y,r)]], \quad i \geq 0.$$

In the case $k(x-\alpha, y-\beta) = (x-\alpha)(y-\beta) < 0$, for $0 \leq \alpha \leq x \leq 1$ and $0 \leq \beta \leq y \leq 1$. $0 \leq r \leq 1$ The parametric form of equations is

$$\underline{f}_{i+1}(x,y,r) = \underline{g}(x,y,r) + S^{-1}[uv[k(x,y)]S[\overline{f}_i(x,y,r)]], \quad i \geq 0.$$

$$\overline{f}_{i+1}(x,y,r) = \overline{g}(x,y,r) + S^{-1}[uv[k(x,y)]S[\underline{f}_i(x,y,r)]], \quad i \geq 0.$$

The ADM assume an infinite series solution for the functions $(\underline{f}(x,y,r))$ and $(\overline{f}(x,y,r))$

$$\underline{f}(x,y,r) = \sum_{i=0}^{\infty} \underline{f_i}(x,y,r), \quad \overline{f}(x,y,r) = \sum_{i=0}^{\infty} \overline{f_i}(x,y,r) \quad (5)$$

The parametric form of the equations is [3]

$$\sum_{i=0}^{\infty} \underline{f}_i(x,y,r) = \underline{g}(x,y,r) + S^{-1}[uv[k(x,y)]S[\sum_{i=0}^{\infty} \underline{f}_i(x,y,r)]] \tag{6}$$

$$\sum_{i=0}^{\infty} \overline{f}_i(x,y,r) = \overline{g}(x,y,r) + S^{-1}[uv[k(x,y)]S[\sum_{i=0}^{\infty} \overline{f}_i(x,y,r)]] \tag{7}$$

The operator $G(\underline{f}(x,y,r))$ and $G(\overline{f}(x,y,r))$ into an infinite series of polynomials given by

$$G(\underline{f}(x,y,r)) = \sum_{i=0}^{\infty} \underline{A}_i(\underline{f}_0(x,y,r), \underline{f}_1(x,y,r), ..., \underline{f}_i(x,y,r)),$$

$$G(\overline{f}(x,y,r)) = \sum_{i=0}^{\infty} \overline{A_i}(\overline{f}_0(x,y,r), \overline{f}_1(x,y,r), ..., \overline{f}_i(x,y,r)),$$

where the $A_i = (\underline{A}_i, \overline{A}_i)$, $i \geq 0$ are the so-called Adomian polynomials defined by

The components $\underline{f}(x,y,r) and\ \overline{f}(x,y,r)$, $i \geq 0.$, computed using the following recursive relations

$$\underline{f}_0(x,y,r) = \underline{g}(x,y,r), \tag{8}$$

$$\underline{f}_0(x,y,r) = \underline{g}(x,y,r),\ \ \underline{f_1}(x,y,r) = S^{-1}(u^2v^2S(\underline{A}_0)), \underline{f_2}(x,y,r) = S^{-1}(u^2v^2S(\underline{A}_1)), ...,$$

$$\underline{f}_{i+1}(x,y,r) = S^{-1}[uv[k(x,y)]S(\underline{A}_i),\ \ i \geq 0.$$

and,

$$\overline{f}_0(x,y,r) = \overline{g}(x,y,r),$$

$$\overline{f_1}(x,y,r) = S^{-1}(u^2v^2S(\overline{A}_0)), \overline{f_2}(x,y,r) = S^{-1}(u^2v^2S(\overline{A}_1)), \overline{f_3}(x,y,r) = S^{-1}(u^2v^2S(\overline{A}_2)), ..., \tag{9}$$

$$\overline{f}_{i+1}(x,y,r) = S^{-1}[uv[k(x,y)]S[\overline{A}_i(x,y,r)]],\ \ i \geq 0.$$

Using relation (4.5) respectively (4.6), we find the values of $\underline{f}_0(x,y,r)$, $\underline{f}_1(x,y,r)$, $\underline{f}_2(x,y,r)$, $\underline{f}_3(x,y,r)$, $\overline{f}_0(x,y,r)$, $\overline{f}_1(x,y,r)$, $\overline{f}_2(x,y,r)$, $\overline{f}_3(x,y,r)$...easily. After finding these values, we use for required solution.

Then, we find approximate solution for $f(s,t,r)$

6. Numerical Examples

Consider the following 2D-VFIEs

$$f(x,y) = g(x,y) \oplus \int_0^y \int_0^x (x-\alpha)(y-\beta) \odot f^2(\alpha,\beta) d\alpha d\beta,\ \ (x,y) \in [0,1]\times[0,1]\ \ r \in [0,1] \tag{10}$$

where,

$$g(x,y,r) = ((xy - \frac{1}{144}x^4y^4)(1+r),(xy - \frac{1}{144}x^4y^4)(3-r))$$

and $k(x-\alpha, y-\beta) = (x-\alpha)(y-\beta) > 0$. The exact solution is $f_{exact}(x,y,r) = (xy(1+r), xy(3-r))$.

Using double fuzzy Sumudu-Adomain, we have and the general form of the equations is

$$S\underline{f}(x,y) = S\underline{g}(x,y) + \int_0^y \int_0^x (x-\alpha)(y-\beta)\underline{f}^2(\alpha,\beta)d\alpha d\beta,$$

$$\underline{f}(x,y) = g(x,y) + S^{-1}(uvS(xy)S(f^2(\alpha,\beta))d\alpha d\beta,$$

$$\underline{f}(x,y) = g(x,y) + S^{-1}(u^2v^2Sf^2(\alpha,\beta)d\alpha d\beta,$$

$$\sum_{i=0}^{\infty}\underline{f}_i(x,y) = g(x,y) + S^{-1}(u^2v^2S(\sum_{i=0}^{\infty}A_i))$$

$$\underline{f}_0(x,y,r) = g(x,y,r)$$

$$\underline{f}_0(x,y,r) = (xy - \frac{1}{144}x^4y^4)(1+r)$$

Using of Sumudu transform for, $S(x^my^n) = (m!)(n!)u^mv^n$

$$x^my^n = (m!)(n!)S^{-1}(u^mv^n)$$

$$S^{-1}(u^mv^n) = \frac{1}{(m!)(n!)}(x^my^n)$$

$$\underline{f}_1(x,y,r) = S^{-1}(u^2v^2S(\underline{A}_0)) = S^{-1}(u^2v^2S(f_0)^2) =$$

$$S^{-1}[u^2v^2S(xy - \frac{1}{144}x^4y^4)^2](1+r)^2 = S^{-1}[u^2v^2S(x^2y^2 - \frac{x^5y^5}{72} + \frac{x^8y^8}{20736})](1+r)^2 =$$

Using Sumudu properties,

$$\underline{f}_1(x,y,r) = (\frac{x^4y^4}{24} - \frac{x^7y^7}{(42)^2 72} + \frac{x^{10}y^{10}}{(90)^2 20736})(1+r)^2$$

$$\underline{f}_2(x,y,r) = S^{-1}(u^2v^2S(\underline{A}_1)) = S^{-1}(u^2v^2S(2f_0f_1)) =$$

$$S^{-1}(u^2v^2S[2((xy) - \frac{1}{144}x^4y^4))(\frac{x^4y^4}{24} - \frac{x^7y^7}{((42)^2 * 72)} + \frac{x^{10}y^{10}}{((90)^2 * 20736)})(1+r)^3$$

$$\underline{f}_2(x,y,r) = S^{-1}(u^2v^2S(\frac{x^5y^5}{12} - \frac{151x^8y^8}{254016} + \frac{499x^{11}y^{11}}{4115059200} - \frac{x^{14}y^{14}}{12093235200})(1+r)^3$$

Using Sumudu properties,

$$\underline{f}_2(x,y,r) = (\frac{x^7y^7}{18144} - \frac{151x^{10}y^{10}}{2057529600} + \frac{499x^{13}y^{13}}{10014408069120} - \frac{x^{16}y^{16}}{792542262067200})(1+r)^3$$

$$\underline{f}_3(x,y,r) = S^{-1}(u^2v^2S(\underline{A}_2)) = S^{-1}(u^2v^2S2(f_1^2 + 2u_0u_2))$$

$$\underline{f}_4(x,y,r) = S^{-1}(u^2v^2S(\underline{A}_3)) = S^{-1}(u^2v^2S2(f_1f_2 + 2u_0u_3))$$

...

Using relation (4.7), we find the values of $\underline{f}_3(x,y,r)$, $\underline{f}_4 = (x,y,r)$, $\underline{f}_5(x,y,r)$, $\underline{f}_6(x,y,r)$,...easily. After finding these values, we use (14) for required solution.

Analogously, we can obtain the solution of equation

$$\overline{f}_0(x,y,r) = (xy - \frac{1}{144}x^4y^4)(3-r)$$

$$\overline{f}_1(x,y,r) = (\frac{x^4y^4}{24} - \frac{x^7y^7}{(42)^2 72} + \frac{x^{10}y^{10}}{(90)^2 20736})(3-r)^2$$

$$\overline{f}_2(x,y,r) = (\frac{x^7y^7}{18144} - \frac{151x^{10}y^{10}}{2057529600} + \frac{499x^{13}y^{13}}{10014408069120} - \frac{x^{16}y^{16}}{792542262067200})(3-r)^3$$

Using relation (4.8), we find the values of $\overline{f}_3(x,y,r)$, $\overline{f}_4 = (x,y,r)$, $\overline{f}_5(x,y,r)$, $\overline{f}_6(x,y,r)$,...easily

After finding these values, we use for required solution of the approximate solution is $f(x,y,r) = (\underline{f}(x,y,r), \overline{f}(x,y,r))$.

Conlusion

In this article, we successfully combine the double fuzzy Sumudu transform and Adomian decomposition method to solve two dimensional fuzzy Volterra integral equations. The results demonstrate the method are efficient and reliable. Moreover, the proposed method was evaluated by solving numerical examples.

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