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Supercliques in a Graph

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Abstract. A set $S \subseteq V(G)$ of a (simple) undirected graph G is a superclique in G if it is a clique and for every pair of distinct vertices $v, w \in S$, there exists $u \in V(G) \setminus S$ such that $u \in N_G(v) \setminus N_G(w)$ or $u \in N_G(w) \setminus N_G(v)$. The maximum cardinality among the supercliques in G, denoted by $\omega_s(G)$, is called the superclique number of G. In this paper, we determine the superclique numbers of some graphs including those resulting from some binary operations of graphs. We will also show that the difference of the clique number and the superclique number can be made arbitrarily large.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Clique, clique number, superclique, superclique number

1. Introduction

Clique is one of the basic concepts in graph theory. This concept was used in many mathematical problems and constructions on graphs. The term clique was introduced by Luce and Perry [12]. In a study in [8], Gaquing and Canoy characterized the cliques in the lexicographic and Cartesian products of graphs. From the characterizations, the corresponding clique numbers of these graphs have been subsequently determined. Some studies involving cliques can be found in [5], [6], [14], and [17].

With the objective of identifying the exact location of an intruder in a network, Slater [19] introduced the concepts of resolving set and metric dimension. These concepts were independently considered by Harary and Melter in [10]. Later, Chartrand et al. (see [4]) also studied resolving set and metric dimension of a graph. Oellermann and Fransen [15] considered the metric dimension of Cartesian products of graphs. It is known that the problem of finding the metric dimension of a graph is NP-hard (see [9]). In 2003, Brigham et al. [2] combined the concepts of resolving and domination. In their article, they defined

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resolving dominating set as a set that is both resolving and dominating. Along with the concept, they also studied the parameter called resolving domination number of a graph.

A more restrictive concept of resolving set, called strong resolving set, and its associated invariant (the strong metric dimension) were introduced in [18]. The invariant was revisited and studied by Oellermann and Fransen in [16] for graphs and digraphs. Recently, Acal, Sumaoy, and Rara in [1], [13], and [20] investigated the concepts of strong connected resolving domination and restrained resolving domination of graphs under some binary operations. In these studies, they specifically introduced the concept of superclique and determined the values of the invariants for the join and corona, and lexicographic product of two graphs in terms of the superclique number of a graph. This work is therefore motivated by the recent studies on these variations of strong resolving set and strong metric dimension that utilize the concepts of superclique and superclique number. Other works on graph-theoretic parameters that involved some binary operations can be found in [3], [7], and [11].

2. Terminologies and Notations

Let $G = (V(G), E(G))$ be a simple undirected graph. The distance between two vertices u and v of G, denoted by $d_G(u, v)$, is equal to the length of a shortest path connecting u and v. Any path connecting u and v of length $d_G(u, v)$ is called a *u-v geodesic*. The *open neighbourhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighbourhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighbourhood* of a subset S of $V(G)$ is the set $N_G(S) = \cup_{v\in S} N_G(v)$ and its *closed neighbourhood* is the set $N_G[S] = N_G(S) \cup S$. The *degree* of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$.

A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The smallest cardinality of a dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G. A dominating set of G with with cardinality $\gamma(G)$ is called a *γ-set* of G.

A set $S \subseteq V(G)$ is a *clique* in a graph G if the graph $G[S]$ induced by S is a complete subgraph of G. A clique C in G is called a *superclique* if for every pair of distinct vertices $u, v \in C$, there exists $w \in V(G) \setminus C$ such that $w \in N_G(u) \setminus N_G(v)$ or $w \in N_G(v) \setminus N_G(u)$. A superclique C in G is called a *point-wise non-dominated superclique* if for every $u \in C$, there exists $v \in V(G) \setminus C$ such that $uv \notin E(G)$. A superclique (resp. point-wise non-dominated superclique) C is *maximum* in G if $|C| \geq |C'|$ for all supercliques (resp. pointwise non-dominated supercliques) C' in G. The *superclique number* (resp. *pointwise non-dominated superclique number*), denoted by $\omega_S(G)$ (resp. $\omega_{pnds}(G)$) of G is the cardinality of a maximum superclique (resp. maximum pointwise non-dominated superclique) in G.

The *shadow graph* $S(G)$ of a graph G is the graph obtained by taking two copies of G, say G_1 and G_2, and joining each vertex $u \in V(G_1)$ to the neighbors of the corresponding vertex $u' \in V(G_2)$. For a graph G, the *complementary prism*, denoted by $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex corresponding to v in $\overline{G}$. In simple terms, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every vertex $v \in V(G)$.

Let G and H be graphs. The *join* of G and H, denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of G and H, denoted by $G \circ H$, is the graph obtained from G by taking a copy H^v of H and forming the join $\langle v \rangle + H^v = v + H^v$ for each $v \in V(G)$. The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ such that $(v, a)(u, b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$. The *Cartesian product* of G and H, denoted by $G \square H$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ such that $(v, a)(u, b) \in E(G[H])$ if and only if either $uv \in E(G)$ and $a = b$ or $u = v$ and $ab \in E(H)$. We note that every non-empty subset C of $V(G) \times V(H)$ can be expressed as $C = \cup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x = \{a \in V(H) : (x, a) \in C\}$ for each $x \in S$.

3. Results

Two adjacent vertices v and w of a graph G are *true twins* if $N_G[v] = N_G[w]$.

Theorem 1. *Let G be any graph. Then each of the following statements holds:*

(*i*) *G admits a superclique and $1 \leq \omega_s(G) \leq \omega(G)$.*

(*ii*) *$\omega_s(G) = 1$ if and only if every component of G is complete.*

(*iii*) *$\omega_s(G) = \omega(G)$ if and only if G has a maximum clique containing no true twin vertices.*

Proof. G admits a superclique because every singleton subset of $V(G)$ is a superclique in G. Moreover, since every superclique is a clique, (i) holds. Suppose that $\omega_s(G) = 1$. Suppose further that G has a component H which is not complete. Then there exist vertices $x, y \in V(H)$ such that $d_H(x, y) = d_G(x, y) = 2$. Let $z \in N_G(x) \cap N_G(y)$. Then $S = \{x, z\}$ is a superclique in G. Hence, $\omega_s(G) \geq |S| = 2 > 1$, contrary to our assumption that $\omega_s(G) = 1$. Thus, every component of G is complete. The converse is easy.

Next, suppose that $\omega_s(G) = \omega(G)$. Let S be a superclique in G such that $|S| = \omega_s(G)$. The assumption $\omega_s(G) = \omega(G)$ would imply that S is a maximum clique of G. Suppose S contains true twin vertices, say v and w. Then

$$(V(G) \setminus S) \cap (N_G(v) \setminus N_G(w)) = (V(G) \setminus S) \cap (N_G(w) \setminus N_G(v)) = \varnothing,$$

contrary to the assumption that S is a superclique in G. Thus, S does not contain true twin vertices.

Conversely, suppose that G has a maximum clique S containing no true twin vertices. Then clearly, S is a superclique in G. This implies that $\omega_s(G) = \omega(G)$. $\square$

Remark 1. *Each of the following statements holds:*

(*i*) *For any positive integer n, $\omega_s(K_n) = 1$.*

(*ii*) *For any positive integer* $n \geq 2$,

$$\omega_s(P_n) = \begin{cases} 1 & if\ n = 2 \\ 2 & if\ n \geq 3. \end{cases}$$

(*iii*) *For any positive integer* $n \geq 3$,

$$\omega_s(C_n) = \begin{cases} 1 & if\ n = 3 \\ 2 & if\ n \geq 4. \end{cases}$$

Theorem 2. *Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $\omega_s(G) = a$ and $\omega(G) = b$.*

Proof. Consider the following cases:

Case 1. $a = b$.

Consider the graph G in Figure 1 obtained from the complete graph K_a by adding the a pendant vertices. Then clearly, $\omega_s(G) = \omega(G) = a$.

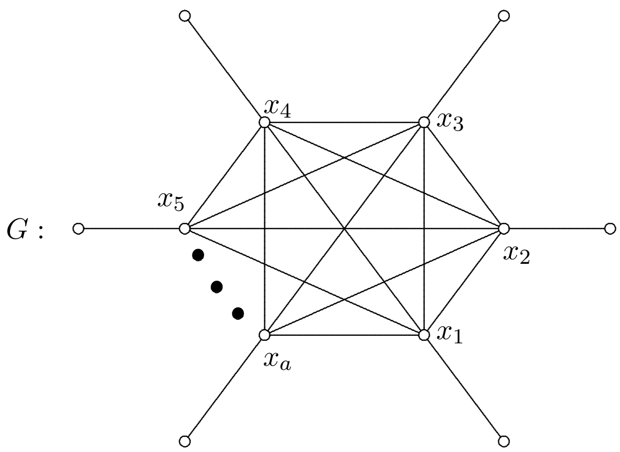

Figure 1: A graph G with $\omega_s(G) = \omega(G)$

Case 2. $a < b$.

Consider the graph G in Figure 2 where $G[\{x_1, x_2, \ldots, x_a\}] = K_a$ and $G[\{y_1, y_2, \ldots, y_b\}] = K_b$. Let $S_1 = \{x_1, x_2, \ldots, x_a\}$ and $S_2 = \{y_1, y_2, \ldots, y_b\}$. Clearly, S_2 is the (only) maximum clique in G and so $\omega(G) = b$. Since S_2 has true twin vertices, it is not a superclique in G (hence, $\omega_s(G) \neq \omega(G)$) by Theorem 1(*iii*). Also, the only subsets of S_2 that are supercliques in G are the singletons and the sets $\{y_1, y_i\}$ where $i \in \{2, 3, \ldots, b\}$. It can easily be verified that S_1 is a maximum superclique in G. Consequently, $\omega_s(G) = a$.

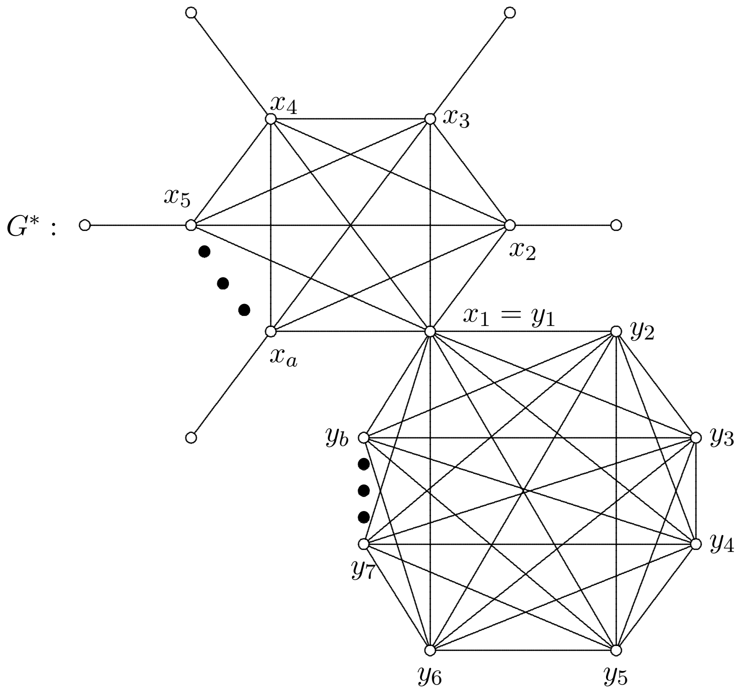

Figure 2: A graph G with $\omega_s(G) < \omega(G)$

Therefore, the assertion holds. □

Corollary 1. *Let n be a positive integer. Then there exists a connected graph G such that $\omega(G) - \omega_s(G) = n$. In other words, the difference $\omega - \omega_s$ can be made arbitrarily large.*

Proof. Let n be a positive integer. By Theorem 2, there exists a connected graph G with $\omega_s(G) = n + 1$ and $\omega(G) = 2n + 1$. Thus, $\omega(G) - \omega_s(G) = n$. □

Theorem 3. *Let G be a non-trivial connected graph and let G_1 and G_2 be two copies of G in the definition of $S(G)$. Then C is a superclique in $S(G)$ if and only if one of the following holds:*

(*i*) *C is a clique in G_1.*

(*ii*) *C is a clique in G_2.*

(*iii*) *$C = C_{G_1} \cup C_{G_2}$, where C_{G_1} and C_{G_2} are cliques in G_1 and G_2, respectively, and satisfy the following conditions:*

  (*a*) *$v' \notin C_{G_2}$ whenever $v \in C_{G_1}$, and*

  (*b*) *$vw \in E(G_1)$ for each $v \in C_{G_1}$ and $w' \in C_{G_2}$.*

Proof. Let C be a superclique in $S(G)$ and set $C_{G_1} = C \cap V(G_1)$ and $C_{G_2} = C \cap V(G_2)$. If $C_{G_2} = \varnothing$ or $C_{G_1} = \varnothing$, then (i) or (ii) holds. So suppose that $C_{G_1} \neq \varnothing$ and $C_{G_2} \neq \varnothing$. Then C_{G_1} and C_{G_2} are cliques in G_1 and G_2, respectively. Let $v \in C_{G_1}$. Then $vv' \notin V(S(G))$ by definition of $S(G)$. Hence, $v' \notin C_{G_2}$, showing that (a) holds. Next, let $v \in C_{G_1}$ and $w' \in C_{G_2}$. Since C is a clique in $S(G)$, $vw' \in E(S(G))$. The definition of $S(G)$ will now imply that $vw \in E(G_1)$ showing that (b) holds. Therefore, (iii) holds.

For the converse, suppose that (i) holds. Let $v, w \in C$ with $v \neq w$. Then $v' \in V(S(G)) \setminus C$ and $v' \in N_{S(G)}(w) \setminus N_{S(G)}(v)$ by the adjacency in $S(G)$. Hence, C is a superclique in $S(G)$. The same conclusion can be made if (ii) holds.

Next, suppose that (iii) holds. Then, by (a) and (b), C is a clique of $S(G)$. Let $x, y \in C$ with $x \neq y$. If $x, y \in C_{G_1}$, then $x' \in V(S(G)) \setminus C$ by (a) and $x' \in N_{S(G)}(y)$ because $xy \in E(G_1)$. It follows that $x' \in N_{S(G)}(y) \setminus N_{S(G)}(x)$. Suppose $x, y \in C_{G_2}$, say $x = v'$ and $y = w'$ where $v, w \in V(G_1)$. Then $v \in V(S(G)) \setminus C$ by (a) and $v \in N_{S(G)}(w')$ because $v'w' \in E(G_2)$. Hence, $v \in N_{S(G)}(w') \setminus N_{S(G)}(v')$. Suppose $x \in C_{G_1}$ and $y \in C_{G_2}$, say $y = z'$, where $z \in V(G_1)$. By (b), $z \in [V(S(G)) \setminus C] \cap N_G(x)$. Thus, $z \in N_{S(G)}(x) \setminus N_{S(G)}(y)$. This shows that C is a superclique in $S(G)$. □

Corollary 2. *Let G be a connected graph. Then*

$$\omega_s(S(G)) = \omega(G).$$

Proof. If $G = K_1$, then $S(G) = \overline{K}_2$ (empty graph). It follows that $\omega_s(S(G)) = \omega(G) = 1$. Next, suppose that G is non-trivial. Let C be a superclique in $S(G)$ of the form given in Theorem 3(iii). Then $C = C_{G_1} \cup C_{G_2}$, where C_{G_1} and C_{G_2} are cliques in G_1 and G_2, respectively, and satisfy conditions (a) and (b). Set $C^*_{G_1} = \{v \in V(G_1) : v' \in C_{G_2}\}$. Then $|C^*_{G_1}| = |C_{G_2}|$. By condition (a), $C_{G_1} \cap C^*_{G_1} = \varnothing$. Let $C^* = C_{G_1} \cup C^*_{G_1}$ and let $x, y \in C^*$ with $x \neq y$. If $x, y \in C_{G_1}$, then $xy \in E(G_1)$. Suppose $x, y \in C^*_{G_1}$. Then $x', y' \in C_{G_2}$. Since C_{G_2} is a clique in G_2, $x'y' \in E(G_2)$. This implies that $xy \in E(G_1)$. Finally, suppose that $x \in C_{G_1}$ and $y \in C^*_{G_1}$. Then $y' \in C_{G_2}$. By condition (b), it follows that $xy \in E(G_1)$. Therefore, C^* is a clique in G_1 and $|C| = |C^*| \leq \omega(G_1) = \omega(G)$. The desired result now follows from Theorem 3. □

Theorem 4. *Let G be a non-trivial graph. Then C is a superclique in $G\overline{G}$ if and only if one of the following holds:*

(i) *C is a clique in G.*

(ii) *C is a clique in $\overline{G}$.*

(iii) *$C = \{v, \overline{v}\}$ for some $v \in V(G)$.*

Proof. Let C be a superclique in $G\overline{G}$ and set $C_G = C \cap V(G)$ and $C_{\overline{G}} = C \cap V(\overline{G})$. If $C_{\overline{G}} = \varnothing$ or $C_G = \varnothing$, then C is a clique in G or $\overline{G}$, showing that (i) or (ii) holds. Suppose that $C_G \neq \varnothing$ and $C_{\overline{G}} \neq \varnothing$. Let $v \in C_G$ and $\overline{w} \in C_{\overline{G}}$. Since C is a clique, $v\overline{w} \in E(G\overline{G})$.

Hence, by definition of $G\overline{G}$, $\overline{w} = \overline{v}$. Now, because $\overline{v} \notin N_{G\overline{G}}(u)$ for all $u \in N_G(v)$, it follows that $C_G = \{v\}$. Similarly, $C_{\overline{G}} = \{\overline{v}\}$. Thus, $C = \{v, \overline{v}\}$, showing that (iii) holds.

For the converse, suppose first that (i) holds. Then C is a clique in $G\overline{G}$. Let $p, q \in C$ with $p \neq q$. Then $\overline{p} \in N_{G\overline{G}}(p) \setminus N_{G\overline{G}}(q)$. This implies that C is a superclique in $G\overline{G}$. The same conclusion holds if (ii) is assumed. Suppose now that (iii) holds, i.e., $C = \{v, \overline{v}\}$ for some $v \in V(G)$. Pick any $w \in V(G) \setminus \{v\}$. If $w \in N_G(v)$, then $w \in N_{G\overline{G}}(v) \setminus N_{G\overline{G}}(\overline{v})$. If $w \notin N_G(v)$, then $\overline{w} \in N_{G\overline{G}}(\overline{v}) \setminus N_{G\overline{G}}(v)$. Thus, C is a superclique in $G\overline{G}$. □

The next results are immediate from Theorem 4.

Corollary 3. *Let G be a non-trivial graph. Then*

$$\omega_s(G\overline{G}) = \max\{2,\ \omega(G),\ \omega(\overline{G})\}.$$

Corollary 4. *Let G be any graph. Then C is a superclique in $G\overline{G}$ if and only if it is a clique in $G\overline{G}$. In particular,*

$$\omega_s(G\overline{G}) = \omega(G\overline{G}).$$

Theorem 5. *Let G and H be any two graphs. Then $C \subseteq V(G+H)$ is a superclique in $G+H$ if and only if one of the following statements holds:*

(i) *C is a superclique in G.*

(ii) *C is a superclique in H.*

(iii) *$C = C_G \cup C_H$, where C_G and C_H are supercliques in G and H, respectively, and at least one of them is pointwise non-dominated.*

Proof. Suppose C is a superclique in $G+H$. Set $C_G = C \cap V(G)$ and $C_H = C \cap V(H)$. If one of C_G and C_H is empty, say $C_H = \varnothing$, then clearly, C_G is a superclique in G. Thus, (i) or (ii) holds. Suppose that $C_G \neq \varnothing$ and $C_H \neq \varnothing$. Since C is a clique, C_G and C_H are cliques in G and H, respectively. Let $v, w \in C_G$ such that $v \neq w$. Since C is a superclique in $G+H$, there exists $z \in V(G+H) \setminus C$ such that $z \in N_{G+H}(v) \setminus N_{G+H}(w)$ or $z \in N_{G+H}(w) \setminus N_{G+H}(v)$. Since $V(H) \setminus C_H \subseteq N_{G+H}(v) \cap N_{G+H}(w)$, it follows that $z \in V(G) \setminus C_G$. Hence, $z \in N_G(v) \setminus N_G(w)$ or $z \in N_G(w) \setminus N_G(v)$, showing that C_G is a superclique in G. Similarly, C_H is a superclique in H. Suppose now that both C_G and C_H are not pointwise non-dominated supercliques in G and H, respectively. Then there exist $a \in C_G$ and $b \in C_H$ such that $V(G) \setminus C_G \subseteq N_G(a)$ and $V(H) \setminus C_H \subseteq N_G(b)$. This would imply that $V(G+H) \setminus C \subseteq N_{G+H}(a) \cap N_{G+H}(b)$, contradicting the assumption that C is a superclique in $G+H$. Therefore, C_G or C_H is a pointwise non-dominated superclique, showing that (iii) holds.

For the converse, suppose that (i) or (ii) holds. Then clearly, C is a superclique in $G+H$. Next, suppose that (iii) holds, i.e., $C = C_G \cup C_H$ and satisfies the given property. Assume that C_H is a pointwise non-dominated superclique in H. Let $v, w \in V(G+H) \setminus C$ such that $v \neq v$. Suppose first that $v, w \in C_G$. Since C_G is a superclique in G, there exists $y \in V(G) \setminus C_G$ such that $y \in N_G(v) \setminus N_G(w)$ or $y \in N_G(w) \setminus N_G(v)$. Hence,

$y \in V(G+H) \setminus C$ and $y \in N_{G+H}(v) \setminus N_{G+H}(w)$ or $y \in N_{G+H}(w) \setminus N_{G+H}(v)$. Similarly, if $v, w \in V(H) \setminus C_H$, then there exists $p \in V(G+H) \setminus C$ and $p \in N_{G+H}(v) \setminus N_{G+H}(w)$ or $p \in N_{G+H}(w) \setminus N_{G+H}(v)$. Suppose now that $v \in C_G$ and $w \in C_H$. Since C_H is pointwise non-dominated, there exists $q \in V(H) \setminus C_H$ such that $q \notin N_H(v)$. It follows that $q \in N_{G+H}(v) \setminus N_{G+H}(w)$. Accordingly, C is a superclique in $G+H$. □

Corollary 5. *Let G be a non-complete graph and let n be a positive integer. Then $C \subseteq V(K_n + G)$ is a superclique in $K_n + G$ if and only if one of the following statements holds:*

(i) C is a superclique in G.

(ii) $C = \{p\}$ for some $p \in V(K_n)$.

(iii) $C = C_G \cup \{p\}$ for some pointwise non-dominated superclique C_G in G and $p \in V(K_n)$.

Proof. Since the only supercliques in K_n are the singleton subsets of $V(K_n)$ and none of these sets is pointwise non-dominated, $C \subseteq V(K_n + G)$ is a superclique in $K_n + G$ if and only if (i), (ii), or (iii) holds by Theorem 5. □

The next result is a consequence of Theorem 5 and Corollary 5.

Corollary 6. *Let G and H be any two graphs and let n be a positive integer. Then*

$$\omega_s(G+H) = \begin{cases} \max\{\omega_s(G) + \omega_{pnds}(H),\ \omega_s(H) + \omega_{pnds}(G)\} & \text{if } G \text{ and } H \text{ are non-complete} \\ \max\{\omega_s(G),\ \omega_{pnds}(G) + 1\} & \text{if } G \text{ is non-complete and } H = K_n. \end{cases}$$

Theorem 6. *Let G be a non-trivial connected graph and let H be any graph. Then $C \subseteq V(G \circ H)$ is a superclique in $G \circ H$ if and only if one of the following statements holds:*

(i) C is a clique in G.

(ii) C is a superclique in H^v for some $v \in V(G)$.

(iii) $C = C_v \cup \{v\}$ for some $v \in V(G)$ and superclique C_v in H^v.

Proof. Suppose C is a superclique in $G \circ H$. If $C \subseteq V(G)$, then C is a clique in G. So suppose that $C \cap V(H^v) \neq \varnothing$ for some $v \in V(G)$. Then C is a clique in $v + H^v$. Suppose first that $C \subseteq V(H^v)$ and let $x, y \in C$ where $x \neq y$. Since C is a superclique in $G \circ H$, there exists $z \in V(G \circ H) \setminus C$ such that $z \in N_{G \circ H}(x) \setminus N_{G \circ H}(y)$ or $z \in N_{G \circ H}(y) \setminus N_{G \circ H}(x)$. Since $v \in N_{G \circ H}(x) \cap N_{G \circ H}(y)$, $z \neq v$. Hence, $z \in V(H^v) \setminus C$ and $z \in N_{H^v}(x) \setminus N_{H^v}(y)$

or $z \in N_{H^v}(y) \setminus N_{H^v}(x)$. This shows that C is a superclique in H^v. Next, suppose that $v \in C$. Then $C = \{v\} \cup C_v$ where $C_v = C \cap V(H^v)$. It is routine to show that C_v is a superclique in H^v.

For the converse, suppose first that C is a clique in G. Let $a, b \in C$ where $a \neq b$. Pick any $c \in V(H^a)$. Then $c \in V(G \circ H) \setminus C$ and $c \in\in N_{G\circ H}(a) \setminus N_{G\circ H}(b)$. Thus, C is a superclique in $G \circ H$. Next, suppose that C is a superclique in H^v for some $v \in V(G)$. Then clearly, C is a superclique in $G \circ H$. Finally, suppose that $C = C_v \cup \{v\}$ for some $v \in V(G)$ and superclique C_v in H^v. Let $p, q \in C$ where $p \neq q$. If $p, q \in C_v$, then there exists $t \in V(H^v) \setminus C_v$ such that $t \in N_{H^v}(p) \setminus N_{H^v}(q)$ or $t \in N_{H^v}(q) \setminus N_{H^v}(p)$ because C_v is a superclique in H^v. It follows that $t \in V(G \circ H) \setminus C$ and $t \in N_{G\circ H}(p) \setminus N_{G\circ H}(q)$ or $t \in N_{G\circ H}(q) \setminus N_{G\circ H}(p)$. Suppose $p = v$. Choose any $w \in N_G(v)$. Then $w \in N_{G\circ H}(p) \setminus N_{G\circ H}(q)$. This proves that C is a superclique in $G \circ H$. □

The next result is a consequence of Theorem 6.

Corollary 7. *Let G be a non-trivial connected graph and let H be any graph. Then*

$$\omega_s(G \circ H) = \max\{\omega(G),\ \omega_s(H) + 1\}.$$

Gaquing and Canoy in [8] obtained the next result.

Theorem 7. *Let G and H be any connected non-trivial graphs. Then $C = \bigcup_{x\in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a clique in $G[H]$ if and only if S a clique in G and T_x is a clique in H for each $x \in S$. In particular, $\omega(G[H]) = \omega(G)\omega(H)$.*

Theorem 8. *Let G and H be any connected non-trivial graphs. Then $C = \bigcup_{x\in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a superclique in $G[H]$ if and only if S a superclique in G and T_x is a superclique in H for each $x \in S$.*

Proof. Suppose C is a superclique in $G[H]$. Then S and each T_x are cliques in G and H, respectively, by Theorem 7. Let $x, y \in S$ with $x \neq y$. Choose any $a \in T_x$ and $b \in T_y$. Then $(x, a), (y, b) \in C$. Since C is a superclique in $G[H]$, there exists $(z, c) \in V(G[H]) \setminus C$ such that $(z, c) \in N_{G[H]}((x, a)) \setminus N_{G[H]}((y, b))$ or $(z, c) \in N_{G[H]}((y, b)) \setminus N_{G[H]}((x, a))$. Since $\{x\} \times (V(H) \setminus T_x) \subseteq N_{G[H]}((y, b))$ and $\{y\} \times (V(H) \setminus T_y) \subseteq N_{G[H]}((x, b))$, it follows that $z \notin \{x, y\}$ and $z \in V(G) \setminus S$. Hence, $z \in N_G(x) \setminus N_G(y)$ or $z \in N_G(y) \setminus N_G(x)$, showing that S is a superclique in G.

Next, let $x \in S$ and let $p, q \in T_x$ with $p \neq q$. Then $(x, p), (x, q) \in C$. Since C is a superclique in $G[H]$, there exists $(w, t) \in V(G[H]) \setminus C$ such that $(w, t) \in N_{G[H]}((x, p)) \setminus N_{G[H]}((x, q))$ or $(w, t) \in N_{G[H]}((x, q)) \setminus N_{G[H]}((x, p))$. This would imply that $w = x$, $t \in V(H) \setminus T_x$, and $t \in N_H(p) \setminus N_H(q)$ or $t \in N_H(q) \setminus N_H(p)$. Therefore, T_x is a superclique in H.

For the converse, suppose that S a superclique in G and T_x is a superclique in H for each $x \in S$. By Theorem 7, C is a clique in $G[H]$. Let $(v, a), (w, b) \in C$ with $(v, a) \neq (w, b)$ and consider the following cases:

Case 1. $v = w$.
Then $a, b \in T_v$ and $a \neq b$. Since T_v is a superclique in H, there exists $d \in V(H) \setminus T_v$ such that $d \in N_H(a) \setminus N_H(b)$ or $d \in N_H(b) \setminus N_H(a)$. It follows that $(v, d) \in V(G[H]) \setminus C$ and $(v, d) \in N_{G[H]}((v, a)) \setminus N_{G[H]}((v, b))$ or $(v, d) \in N_{G[H]}((v, b)) \setminus N_{G[H]}((v, a))$.

Case 2. $v \neq w$.
Since $v, w \in S$ and S is a superclique in G, there exists $u \in V(G) \setminus S$ such that $u \in N_G(v) \setminus N_G(w)$ or $u \in N_G(w) \setminus N_G(v)$. Then $(u, a) \in V(G[H]) \setminus C$ and $(u, a) \in N_{G[H]}((v, a)) \setminus N_{G[H]}((v, b))$ or $(u, a) \in N_{G[H]}((v, b)) \setminus N_{G[H]}((v, a))$.

Accordingly, C is a superclique in $G[H]$. □

Corollary 8. *Let G and H be any connected non-trivial graphs. Then*

$$\omega_s(G[H]) = \omega_s(G)\omega_s(H).$$

Proof. Let S and D be ω_s-sets in G and H, respectively. Then $C_0 = S \times D$ is a superclique in $G[H]$ by Theorem 8. It follows that

$$\omega_s(G[H]) \geq |C_0| = |S||D| = \omega_s(G)\omega_s(H).$$

Now, let C be an ω_s-set in $G[H]$. Then $C = \bigcup_{x \in S} [\{x\} \times T_x]$ and S and each T_x are supercliques in G and H, respectively, by Theorem 8. Hence,

$$\omega_s(G[H]) = |C| = \sum_{x \in S} |T_x| \leq \omega_s(G)\omega_s(H).$$

This establishes the desired equality. □

The next result is also obtained from [8].

Theorem 9. *Let G and H be any connected graphs. Then C is a clique in $G \square H$ if and only if $C = S \times \{a\}$ for some $a \in V(H)$ and clique S in G or $C = \{x\} \times R$ for some $x \in V(G)$ and clique R in H. In particular,*

$$\omega(G \square H) = \max\{\omega(G), \omega(H)\}.$$

Theorem 10. *Let G and H be any connected non-trivial graphs. Then C is a superclique in $G \square H$ if and only if $C = S \times \{a\}$ for some $a \in V(H)$ and clique S in G or $C = \{x\} \times R$ for some $x \in V(G)$ and clique R in H.*

Proof. Suppose C is a superclique in $G[H]$. Since C is a clique, $C = S \times \{a\}$ for some $a \in V(H)$ and clique S in G or $C = \{x\} \times R$ for some $x \in V(G)$ and clique R in H by Theorem 9.

For the converse, suppose that $C = S \times \{a\}$ for some $a \in V(H)$ and clique S in G. Then C is a clique in $G \square H$ by Theorem 9. Let $(v, a), (w, a) \in C$ such that $v \neq w$. Choose any $b \in N_H(a)$. Then $(v,b) \in V(G\Box H) \setminus C$ and $(v,b) \in N_{G\Box H}((v,a)) \setminus N_{G\Box H}((w,a))$. Next, suppose that $C = \{x\} \times R$ for some $x \in V(G)$ and clique R in H. Again, C is a clique in $G\Box H$ by Theorem 9. Let $(x,p),(x,q) \in C$ with $p \neq q$. Choose any $y \in N_G(x)$. Then $(y,p) \in V(G\Box H) \setminus C$ and $(y,p) \in N_{G\Box H}((x,p)) \setminus N_{G\Box H}((x,q))$. In either case, we find that C is a superclique in $G\Box H$. □

The next result follows from Theorem 9 and Theorem 10.

Corollary 9. *Let G and H be non-trivial connected graphs. Then*

$$\omega_s(G\Box H) = \omega(G\Box H) = \max\{\omega(G), \omega(H)\}.$$

4. Conclusion

Any graph admits a superclique and the superclique number of a graph does not exceed the clique number of the graph. It is shown that the difference of the clique number and superclique number can be made arbitrarily large. Supercliques in the join, corona, lexicographic product, and Cartesian product of two graphs have been characterized. From these characterizations, respective superclique numbers have been determined. This new invariant can also be studied for graphs under other binary operations. Moreover, it may be possible that the invariant has relationship with other graph-theoretic parameters apart from the ones involving the strong metric dimension.

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