Restrained Strong Resolving Hop Domination in Graphs

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Abstract. A set $S \subseteq V(G)$ is a restrained strong resolving hop dominating set in G if for every $v \in V(G) \backslash S$, there exists $w \in S$ such that $d_G(v,w) = 2$ and $S = V(G)$ or $V(G) \backslash S$ has no isolated vertex. The smallest cardinality of such a set, denoted by $\gamma_{rsRh}(G)$, is called the restrained strong resolving hop domination number of G. In this paper, we obtained the corresponding parameter in graphs resulting from the join, corona and lexicographic product of two graphs. Specifically, we characterize the restrained strong resolving hop dominating sets in these types of graphs and determine the bounds or exact values of their restrained strong resolving hop domination numbers.

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1. Introduction

The study of domination can be traced way back 1960. Since then numerous authors contribute several interesting domination parameters to nurture the growth of this research area. In 1977, E.J Cockayne and S.T Hedetniemi introduced the notation $\gamma(G)$ for the domination number of graph G. Until the initiation of the concept of 2-step domination number by Chartrand et al [3] in 1995, which is closely related to hop domination number. Subsequently, Natarajan and Ayyaswamy (2015) introduced the Hop Domination concept. Some variation of domination can be seen in these papers [2], [5], [4].

In this study, the researcher defines and establishes a new concept of hop domination called a restrained strong resolving hop domination and generates some characterizations

of restrained strong resolving hop domination in graphs. For an application, in [8] Haynes and Henning considered a factory with large number of employees and a need to implement a quality assurance checking system of their workers. The factory manager decides to designate an internal committee to do this, i.e, the manager will select a subset of the workers to form a quality assurance team to inspect the work of their co-workers. The manager desires to keep this team as small as possible in order to minimize costs (inspectors' extra pay) and to protect privacy (keeping the identity of inspector secret).To avoid bias, an inspector should neither be close friends nor enemies with any of the workers he/she is responsible for inspecting. To model this situation, a social network graph can be constructed, where each worker is represented by a vertex and an edge between two workers represent possible bias, i.e if the two workers are either close friend or enemies. Ideally, an inspector should not be adjacent to any worker under his inspection. In hop domination, every worker will be inspected by the nearest non-biased inspector, that is, an inspector who is a close friend (or enemy) of the worker's close friend (or enemy). This is to save time and effort locating a particular worker. If we desire a situation where every worker including the inspector has his/her work inspected, then restrained strong resolving hop domination numbers gives us the minimum number of inspectors needed.

In this study, we only consider graphs that are finite, simple, undirected and connected. Readers are referred to [6] for elementary Graph Theory concepts.

Let G be a connected graph. A set $S \subseteq V(G)$ is a *hop dominating set* of G if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A set $C \subseteq V(G)$ is called a *superclique* in G if $\langle C \rangle$ is a clique and for every pair of distinct vertices $u, v \in C$, there exists $w \in V(G) \setminus C$ such that $w \in N_G(u) \setminus N_G(v)$ or $w \in N_G(v) \setminus N_G(u)$. A superclique C is *maximum* in G if $|C| \geq |C^*|$ for all supercliques C^* in G. The superclique number of G, denoted by $\omega_S(G)$, is the cardinality of a maximum superclique in G.

A superclique C in G is called a *hop dominated superclique* if for every $v \in C$ there exists $u \in V(G) \backslash C$ such that $d_G(u, v) = 2$. A hop dominated superclique C is *maximum* in G if $|C| \geq |C^*|$ for all hop dominated supercliques C^* in G. The *hop dominated superclique number* denoted by $\omega_{hS}(G)$, of G is the cardinality of a maximum hop dominated superclique in G.

A superclique $C \subseteq V(G)$ is called a *point-wise non-dominated superclique* of G if for every $x \in C$ there exists $y \in V(G) \setminus C$ such that $y \notin N_G(x)$. A maximum cardinality of a point-wise non-dominated superclique in G is denoted by $\omega_{pndS}(G)$.

A vertex x of a connected graph G is said to *resolve vertices* u and v of G if $d_G(x, u) \neq d_G(x, v)$. For an ordered set $W = \{x_1, \ldots, x_k\} \subseteq V(G)$ and a vertex v in G, the *k-vector*

$$r_G(v/W) = (d_G(v, x_1), d_G(v, x_2), \ldots d_G(v, x_k))$$

is called the *representation* of v with respect to W. The set W is a *resolving set* for G if and only if no two vertices of G have the same representation with respect to W. The

metric dimension of G, denoted by $dim(G)$, is the minimum cardinality over all resolving sets of G. A resolving set of cardinality $dim(G)$ is called a *basis*.

For two vertices $u, v \in V(G)$, the interval $I_G[u,v]$ between u and v is the collection of all vertices that belong to some shortest u-v path. A vertex w *strongly resolves* two vertices u and v if $v \in I_G[u,w]$ or if $u \in I_G[v,w]$. A set W of vertices in G is a *strong resolving set* of G if every two vertices of G are strongly resolved by some vertex of W. The smallest cardinality of a strong resolving set of G is called the *strong metric dimension* of G and is denoted by $sdim(G)$. A strong resolving set of cardinality $sdim(G)$ is called a *strong metric basis* of G.

A subset $S \subseteq V(G)$ is a *strong resolving hop dominating set* of G if S is both a strong resolving set and a hop dominating set. The minimum cardinality of a strong resolving hop dominating set of G, denoted by $\gamma_{sRh}(G)$, is called the *strong resolving hop domination number* of G. Any resolving hop dominating set with cardinality equal to $\gamma_{sRh}(G)$ is called a γ_{sRh}-*set*.

A set $S \subseteq V(G)$ is a *restrained strong resolving hop dominating set* on G if S is a strong resolving hop dominating set in G and $S = V(G)$ or $\langle V(G) \setminus S \rangle$ has no isolated vertex. The *restrained strong resolving hop domination number* of G, denoted by $\gamma_{rsRh}(G)$, is the smallest cardinality of a restrained strong resolving dominating set in G. A restrained strong resolving hop dominating set of cardinality $\gamma_{rsRh}(G)$ is then referred to as γ_{rsRh}-*set* of G.

2. Preliminary Results

Lemma 1. *[7] Let G be a nontrivial connected graph with $diam(G) \leq 2$. Then $S = V(G) \setminus C$ is a strong resolving set of G if and only if $C = \varnothing$ or C is a superclique in G. In particular, $sdim(G) = |V(G)| - \omega_S(G)$.*

Proposition 1. *Let G be a connected graph of order n and $A = \{x \in G : deg_G(x) = n - 1\}$. If $A \neq \varnothing$ and C is a hop dominated superclique in G, then $C \cap A = \varnothing$.*

Theorem 1. *[7] Let G be a nontrivial connected graph of order n with $\gamma(G) \neq 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a strong resolving set of $K_1 + G$ if and only if $S = V(G)$, $S = V(G) \setminus C$, or $S = V(K_1 + G) \setminus C$ where C is a superclique in G.*

Theorem 2. *[7] Let G and H be nontrivial connected graphs of orders m and n, respectively. A proper subset S of $V(G + H)$ is a strong resolving set of $G + H$ if and only if at least one of the following is satisfied:*

(i) $S = V(G + H) \setminus C_G$ where C_G is a superclique in G.

(ii) $S = V(G + H) \setminus C_H$ where C_H is a superclique in H.

(iii) If $\gamma(G) \neq 1$ or $\gamma(H) \neq 1$,

$$S = V(G + H) \setminus (C_G \cup C_H) = (V(G) \setminus C_G) \cup (V(H) \setminus C_H),$$

where C_G and C_H are supercliques in G and H, respectively.

Lemma 2. *[1] Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) \neq 1$. Then $A \times C \subseteq V(G[H])$ is a superclique in $G[H]$ if and only if A is a nonempty subset of $V(G)$ and C is a superclique in H.*

Theorem 3. *[1] Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) \neq 1$. A subset S of $V(G[H])$ is a strong resolving set of $G[H]$ if and only $S = V(G[H]) \backslash (A \times C)$, where A is a subset of $V(G)$ and $C = \varnothing$ or C is a superclique in H.*

Lemma 3. *[1] Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) = 1$. Then $A \times C \subseteq V(G[H])$ is a superclique in $G[H]$ if and only if A is a nonempty subset of $V(G)$ and C is a superclique in H such that $|A| = 1$ whenever $C \cap C^* \neq \varnothing$ for some γ-set C^* of H.*

Theorem 4. *[7] Let G be a nontrivial connected graph and H a connected graph. A proper subset S of $V(G \circ H)$ is a strong resolving set of $G \circ H$ if and only if one of the following holds:*

(i) $S = A \cup \left(\underset{u \in V(G)}{\cup} V(H^u) \right)$ *where $A \subseteq V(G)$.*

(ii) $S = \cup \left(\underset{u \in V(G) \backslash \{v\}}{\cup} V(H^u) \right) \cup B_v$ *for a unique v in $V(G)$,*
where $A \subseteq V(G)$ and B^v is a strong resolving set of H^v if $\gamma(H) = 1$ or B_v is a resolving set of $\{v\} + H^v$ if $\gamma(H) \neq 1$.

Remark 1. *Every restrained strong resolving hop dominating set of a connected graph G is a strong resolving set. Hence, $sdim(G) \leq \gamma_{rsRh}(G)$. Also, every restrained strong resolving hop dominating set of G is a hop dominating set. Thus, $\gamma_h(G) \leq \gamma_{rsRh}(G)$.*

Remark 2. *For any connected graph G of order n, $1 \leq \gamma_{rsRh}(G) \leq n$. Moreover, $\gamma_{rsRh}(G) = 1$ if G is a trivial graph and $\gamma_{rsRh}(K_n) = n$ for $n \geq 1$.*

The next result follows immediately from Lemma 1.

Proposition 2. *Let G be a nontrivial connected graph with $diam(G) \leq 2$. Then $S \subseteq V(G)$ is a restrained strong resolving hop dominating set of G if and only if $S = V(G) \backslash C$ where $C = \varnothing$ or C is a nonsingleton hop dominated superclique in G. In particular, $\gamma_{rsRh}(G) = |V(G)| - \omega_{hS}(G)$.*

3. Join of Graphs

Definition 1. [6] The *join $G + H$* of graphs G and H, is the graph with vertex-set $V(G+H) = V(G) \,\dot\cup\, V(H)$ and edge-set $E(G+H) = E(G) \,\dot\cup\, E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 5. *Let G be a nontrivial connected graph of order n with $\gamma(G) \neq 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a restrained strong resolving hop dominating set of $K_1 + G$ if and only if $S = V(K_1 + G) \setminus C$ where $C = \varnothing$ or C is a hop dominated superclique of G.*

Proof: Let S be a restrained strong resolving set of $K_1 + G$. Since S is strong resolving, by Theorem 1, $S = V(G)$ or $S = V(G) \setminus C^*$ or $S = V(K_1 + G) \setminus C^*$ where C^* is a superclique in G. Since S is restrained hop dominating set in $K_1 + G$, so $S = V(G + K_1)$ or $\langle V(G + K_1) \setminus S \rangle$ has no isolated vertex and $v \in S$. Hence , $S \neq V(G)$ and $S = V(G) \setminus C^*$. Hence, $S = V(K_1 + G) \setminus C^*$ where $C^* = \varnothing$ or C^* is a nonsingleton hop dominated superclique of G.

The converse follows immediately from Theorem 1. □

Theorem 6. *Let G be a nontrivial connected graph of order n with $\gamma(G) = 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a restrained strong resolving hop dominating set of $K_1 + G$ if and only if $S = (V(K_1 + G) \setminus C) \cup \{x \in C : deg_G(x) = n - 1\}$ where $C = \varnothing$ or C is a hop dominated superclique in G.*

Proof: Let S be a restrained strong resolving hop dominating set of $K_1 + G$. Then by Theorem 1,

$$S = V(G) \text{ or } S = V(K_1 + G) \setminus C^* \text{ or } S = (V(G) \setminus C^*) \cup \{x \in C^* : deg_G(x) = n - 1\}$$

where C^* is a superclique of G. Since S is a restrained hop dominating set, $S = V(K_1 + G)$ or $\langle V(K_1 + G) \setminus S \rangle$ has no isolated vertex and $v, x \in S$ where $deg_G(x) = n - 1$. Thus, $S = (V(G) \setminus C^*) \cup \{x \in C^* : deg_G(x) = n - 1\}$ where $C^* = \varnothing$ or C^* is nonsingleton hop dominated superclique of G.

The converse follows immediately from Theorem 1. □

Corollary 1. *Let G be a nontrivial connected graph of order n. Then*

$$\gamma_{rsRh}(K_1 + G) = n - \omega_{hS}(G) + 1.$$

Corollary 2. *Let $P_n = [v_1, v_2, \ldots, v_n]$ and $C_m = [c_1, c_2, \ldots c_m, c_1]$ where $n, m \geq 4$.*

(i) The set $[\, V(P_n) \cup \{v\} \,] \setminus \{v_k, v_{k+1}\}$ for $k = 1, 2, \ldots, n - 1$ are the restrained strong resolving hop dominating sets of $\langle v \rangle + P_n$.

(ii) The sets $[(V(C_m) \cup \{v\}) \setminus \{c_i, c_{i+1}\}]$ and $(V(C_m) \cup \{v\} \setminus \{c_1, c_m\})$ for $i = 1, 2, \ldots, m - 1$, are the restrained strong resolving hop dominating sets of $\langle v \rangle + C_n$.

Theorem 7. *Let G be a disconnected graph whose components are G_i for $i = 1, 2, \ldots, n$. A subset S of $V(K_1 + G)$ is a restrained strong resolving hop dominating set of $K_1 + G$ if and only if $S = V(K_1 + G) \setminus C_i$ where $C_i = \varnothing$ or C_i is a nonsingleton superclique of G_i.*

Proof: Let S be a restrained strong resolving hop dominating set of $K_1 + G$. Then by Theorem 1, $S = V(G)$ or $S = V(G) \setminus C_i^*$ or $S = V(K_1 + G) \setminus C_i^*$ where C_i^*

is a superclique of G_i and $C_i^* = \varnothing$ or C_i^* is a nonsingleton superclique in G_i. Since S is a restrained and hop dominating, $S = V(K_1 + G)$ or $\langle V(K_1 + G) \setminus S \rangle$ has no isolated vertex. Hence, $S \neq V(G)$ and $S \neq V(G) \setminus C_i^*$ where $C_i^* = \varnothing$ or C_i^* is a nonsingleton superclique in G_i.

The converse follows immediately from Theorem 1. $\square$

Corollary 3. *Let G_i be connected graphs of order n_i and G be a disconnected graph whose components are G_i for $i = 1, 2, \ldots, m$. Then,*

$$\gamma_{rsRh}(K_1 + G) = \sum_{i=1}^{m} n_i - max\{\omega_S(G_i) : i = 1, \cdots, m\}.$$

In the join of two graphs G and H, the results of Theorem 5 and Theorem 6 have already considered the case when G or H is trivial. Hence, the next result considers the characterizations of the restrained strong resolving hop dominating sets of nontrivial connected graphs G and H.

The next result follows from Theorem 2.

Theorem 8. *Let G and H be nontrivial connected graphs of orders m and n, respectively. A subset S of $V(G + H)$ is a restrained strong resolving hop dominating set of $G + H$ if and only if at least one of the following is satisfied:*

(i) $S = V(G + H) \setminus C_G$ where C_G is a nonsingleton hop dominated superclique of G.

(ii) $S = V(G + H) \setminus C_H$ where C_H is a nonsingleton hop dominated superclique of H.

(iii) If $\gamma(G) = 1$ and $\gamma(H) = 1$,
$S = [V(G+H) \setminus (C_G \cup C_H)] \cup \{z \in C_G : deg_G(z) = m-1\} \cup \{w \in C_H : deg_H(w) = n-1\}$
where C_G and C_H are hop dominated supercliques in G and H, respectively.

(iv) If $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$,

$$S = [V(G + H) \setminus (C_G \cup C_H)] = (V(G) \setminus C_G) \cup (V(H) \setminus C_H)$$

where C_G and C_H are hop dominated supercliques in G and H, respectively.

Corollary 4. *Let G and H be nontrivial connected graphs of orders m and n, respectively. Then*

$$\gamma_{srRh}(G + H) = \begin{cases} (m - \omega_{hS}(G)) + (n - \omega_{hS}(H)) + 1, & \text{if } \gamma(G) = 1 \text{ or } \gamma(H) = 1 \\ (m - \omega_{hS}(G)) + (n - \omega_{hS}(H)), & \text{if } \gamma(G) \neq 1 \text{ and } \gamma(H) \neq 1. \end{cases}$$

Example 1. Consider the graphs $\langle w \rangle + C_4$. Then $\gamma_{rsRh}(\langle w \rangle + C_6) = 3$.

4.   Corona of Graphs

Definition 2.   [8] The *corona* $G \circ H$ of graphs G and H, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. *For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$, $v \in V(G)$.*

Theorem 9. *Let G be a nontrivial connected graph and H a connected graph. A proper subset $S \subseteq V(G \circ H)$ is a restrained strong resolving hop dominating set of $G \circ H$ if and only if one of the following holds:*

(i) $S = A \cup \left(\bigcup\limits_{u \in V(G)} V(H^u) \right)$ where $A \subseteq V(G)$ and $\langle V(G) \setminus A \rangle$ has no isolated vertex.

(ii) $S = A \cup \left(\bigcup\limits_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup B_v$ for a unique vertex v in G, where $A = V(G) \setminus \{v\}$
or $\langle V(G) \setminus (A \cup \{v\}) \rangle$ has no isolated vertex and B_v is a strong resolving hop dominating set of $H^v + \langle v \rangle$ if $N_G(v) \cap A = \varnothing$ and B_v is a strong resolving set where $\langle V(H^v) \setminus B_v \rangle$ has no isolated vertex if $v \in A$ and $N_G(v) \cap A \neq \varnothing$.

Proof: Suppose S is a restrained resolving hop dominating set of $G \circ H$. Then S is a strong resolving hop dominating set and by Theorem 4, one of the following holds:

(a) $S = A \cup \left(\bigcup\limits_{u \in V(G)} V(H^u) \right)$, where $A \subseteq V(G)$;

(b) $S = A \cup \left(\bigcup\limits_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup B_v$ for a unique vertex v in G, where $A \subseteq V(G) \setminus \{v\}$ and
B_v is a strong resolving set of H^v if $\gamma(H) = 1$ or B_v is a strong resolving set of $\langle v \rangle + H^v$ if $\gamma(H) \neq 1$.

Suppose (a) holds. Since S is a proper restrained hop dominating subset of $G \circ H$, $\langle V(G \circ H) \setminus S = V(G) \setminus A \rangle$ has no isolated vertex. Thus, (i) holds.

On the other hand, suppose (b) holds. Since S is a restrained hop dominating set of $G \circ H$, $S = A \cup \left(\bigcup\limits_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup B_v$ for a unique vertex $v \in V(G)$, where $A = V(G) \setminus \{v\}$ or $\langle V(G) \setminus (A \cup \{v\}) \rangle$ has no isolated vertex and B_v is a strong resolving hop dominating set of $H^v + \langle v \rangle$ if $N_G(v) \cap A = \varnothing$ and B_v is a strong resolving set, where $\langle V(H^v) \setminus B_v \rangle$ has no isolated vertex if $v \in A$ and $N_G(v) \cap A \neq \varnothing$. Since $v \in S$, $\langle V(H^v) \setminus B_v \rangle$ has no isolated vertex and B_v is a strong hop dominating set of $H^v + \langle v \rangle$, if $v \in S$ and B_v is strong resolving set of H^v and $\langle V(H^v) \setminus B_v \rangle$ has no isolated vertex if $v \in S$. Thus (ii) holds.

Conversely, suppose (i) and (ii) hold. By Theorem 4, S is a strong resolving set of $G \circ H$. If (i) holds, then $\langle V(G \circ H) \setminus S = V(G) \setminus A \rangle$ has no isolated vertex. If (ii) holds

then
$$V(G \circ H) \setminus S = (V(G) \setminus A) \cup (V(H^v + \langle v \rangle) \setminus B_v).$$

Since $A = V(G) \setminus \{v\}$ or $\langle V(G) \setminus (A \cup \{v\}) \rangle$ has no isolated vertex, $\langle V(G + H) \setminus S \rangle$ has no isolated vertex. In either case, $\langle V(G \circ H) \setminus S \rangle$ has no isolated vertex. Therefore, S is a restrained strong resolving hop dominating set of $G \circ H$. $\square$

Corollary 5. Let G and H be nontrivial connected graphs of orders m and n, respectively. Then, $\gamma_{rsRh}(G \circ H) = (m-1)n + \gamma_{sR}(H + K_1)$.

Proof: Let S be a γ_{rsRh}-set of $G \circ H$. Then by Theorem 9 (ii),

$$S = A \bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \bigcup B_v$$

for a unique vertex v in G and B_v is a strong resolving hop dominating set of H^v. Hence,

$$\gamma_{rsRh}(G \circ H) = |S| = |V(H)||V(G) \setminus \{v\}| + |B_v|$$
$$\geq (m-1)n + \gamma_{sRh}(H).$$

Let C_v be a minimum strong resolving hop dominating set of $K_1 + H^v$. For a unique vertex $v \in V(G)$, let $\langle B_v \rangle \cong \langle C_v \rangle$. Then by Theorem 9, $S = A \bigcup \left(\bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right) \bigcup B_v$ is a restrained strong resolving hop dominating set of $G \circ H$. Thus,

$$\gamma_{rsRh}(G \circ H) \leq |S|$$
$$= \left| \bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right| + |B_v|$$
$$= (m-1)(n) + |C_v|$$
$$= (m-1)(n) + \gamma_{sRh}(K_1 + H).$$

Therefore, $\gamma_{rsRh}(G \circ H) = (m-1)n + \gamma_{sRh}(K_1 + H)$. $\square$

Example 2. Consider the graph $P_3 \circ P_3$. Then the minimum restrained strong resolving hop dominating set is $\gamma_{rsRh}(P_3 \circ P_3) = 8$.

5. Lexicographic of Graphs

Definition 3. [6] The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ if and only if either $u_1 u_2 \in E(G)$ or $u_1 = u_2$ and $v_1 v_2 \in E(H)$.

Lemma 4. *Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) \neq 1$. Then $A \times C \subseteq V(G[H])$ is a hop dominated superclique in $G[H]$ if and only if A is a nonempty subset of $V(G)$ and C is a superclique in H.*

Proof: Suppose that $A \times C \subseteq V(G[H])$ is a hop dominated superclique in $G[H]$. By Lemma 2 and Lemma 3, A is a nonempty subset of $V(G)$ and C is a superclique in H. Let $x \in C$. Then $(a, x) \in A \times C$ for any $a \in A$. Since $A \times C$ is hop dominated superclique, there exists $(b, y) \in [V(G[H]) \setminus (A \times C)] \cap N_{G[H]}((a, x), 2)$. Suppose $\gamma(H) = 1$. Since $G = K_n$ for $n > 1$, $a = b$ and $y \in [(V(H) \setminus C) \cap N_H(x, 2)]$. If $\gamma(H) = 1$, then by Proposition 1, $C \cap C^* = \varnothing$ for all γ-sets C^* of H. Thus, $x \in C \setminus C^*$ and $y \in N_H(x, 2)$ exists. Hence, C is a hop dominated superclique in H.

For the converse, suppose that A is a nonempty subset of $V(G)$ and C is a hop dominated superclique in H. By Lemma 2, Lemma 3 and Proposition 1, $A \times C$ is a superclique in $G[H]$.

Let $(a, x) \in A \times C$ and $\gamma(H) \neq 1$. Since C is a hop dominated superclique in H, there exists $y \in [\langle (V(H) \setminus C) \rangle \cap N_H(x, 2)]$. Hence, $(a, y) \in [V(G[H]) \setminus (A \times C)] \cap N_{G[H]}((a, x), 2)$. Suppose $\gamma(H) = 1$. Then by Proposition 1, $C \cap C^* = \varnothing$ for all γ-sets C^* of H. Thus, $x \in C \setminus C^*$. This implies that a vertex $z \in N_H(x, 2)$ exists. Since C is a superclique, $z \in \langle V(H) \setminus C \rangle$. Hence, $(a, z) \in [V(G[H]) \setminus (A \times C) \cap N_G[H]((a, x), 2)]$.

Therefore, $A \times C$ is a hop dominated superclique in $G[H]$. $\qquad\qquad\square$

Theorem 10. *Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) \neq 1$. A subset S of $V(G[H])$ is a restrained strong resolving dominating set of $G[H]$ if and only if $S = V(G[H]) \setminus (A \times C)$ and one of the following is satisfied:*

(i) $A \subseteq V(G)$ and $C = \varnothing$.

(ii) A is a singleton subset of $V(G)$ and C is a nonsingleton superclique in H.

(iii) A is a nonempty nonsingleton subset of $V(G)$ and C is a hop dominated superclique in H.

Proof: Let S be a restrained strong resolving hop dominating set of $G[H]$. By Theorem 3, $S = V(G[H]) \setminus (A \times C)$ where A is a subset of $V(G)$ and $C = \varnothing$ or C is a superclique in H. Since S is a restrained strong resolving set, $S = V(G[H])$ or $\langle V(G[H]) \setminus S \rangle$ has no isolated vertex. If $S = V(G[H])$ then $A \times C = \varnothing$, showing that $A \subseteq V(G)$ and $C = \varnothing$. Thus, (i) holds. If $\langle V(G[H]) \setminus S \rangle$ has no isolated vertex, then $A \times C$ is a nonsingleton hop dominated superclique in $G[H]$. This implies that A is a singleton subset of $V(G)$ and C is a nonsingleton superclique in H or A is nonempty nonsingleton subset of $V(G)$ and C is hop dominated superclique in H. Hence (ii) or (iii) holds.

For the converse, suppose $S = V(G[H]) \setminus (A \times C)$, where A and C satisfy (i),(ii) or (iii). Then, either $A \times C = \varnothing$ or by Lemma 4, $A \times C$ is a nonsingleton hop dominated superclique in $G[H]$. By Theorem 3, S is a strong resolving set in $G[H]$. Since $A \times C$ is hop dominated superclique, S is a strong resolving hop dominating set of $G[H]$. If (i) is true, then $A \subseteq V(G)$ and $C = \varnothing$, that is, $A \times C = \varnothing$ and $S = V(G[H])$. If (ii) or (iii) is satisfied, then $\langle V(G[H]) \setminus S \rangle$ has no isolated vertex. Therefore, S is a restrained strong resolving hop dominating set $G[H]$. $\qquad\square$

Example 3. Consider the graph of $K_3[P_5]$ then the $\gamma_{rsRh}(K_3[P_5]) = 13$.

Acknowledgments

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