On the Diophantine Equation \((p + 4n)x + py = z^2\)

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Abstract. In this paper, we study the Diophantine equation \((p + 4n)x + py = z^2\), where \(n\) is a non-negative integer and \(p, p + 4n\) are prime numbers such that \(p \equiv 7 \pmod{12}\). We show that the non-negative integer solutions of such equation are \((x, y, z) \in \{(0, 1, \sqrt{p + 1}) \cup (1, 0, 2\sqrt{n + \frac{p + 1}{4}})\}\), where \(\sqrt{p + 1}\) and \(\sqrt{n + \frac{p + 1}{4}}\) are integers.

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1. Introduction

A problem related to the Diophantine equation has been investigated by many researchers. It is considered one of the significant problems in elementary number theory. The proving method mainly uses a property in the integer system and algebraic number theory. Some of which appear in a higher system of the integer called the ring of integers. In 2011, Suvarnamani [10] considered a Diophantine equation \(2^x + py = z^2\) when \(p > 2\) and \(p\) is a prime number. The result showed that \((x, y, z) = (3, 0, 3)\) is a solution of the equation for all prime \(p > 2\). If \(p = 3\), then \((x, y, z) = (4, 2, 5)\) is also a solution of the equation. If \(p = 1 + 2k + 1\) for some non-negative integer \(k\), then \((x, y, z) = (2k, 1, 1 + 2k)\).

In 2012, the Diophantine equation \(4^x + py = z^2\), where \(x, y\) and \(z\) are non-negative integers and \(p\) is a positive prime number was studied by Chotchaisthit [2]. The study revealed that the equation has no non-negative integer solution. In 2014, Suvarnamani [11] proved that the equation \(p^x + (p + 1)^y = z^2\) has a unique non-negative integer solution \((x, y, z) = (3, 1, 0, 2)\) when \(p\) is an odd prime number. In 2016, Hoque [6] proved that there are exactly two solutions to \((Mpq)^x + (Mpq + 1)^y = z^2\), where \(p, q \in \mathbb{Z}\) such that \(p > 0, q > 1\) and \(Mpq = p^y - 1\). In 2018, Kumar et al. [7] showed that the non-linear diophantine equation \(p^x + (p + 6)^y = z^2\) has no solution. Moreover, Fernando [4] showed

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that a Diophantine equation $p^x + (p + 8)^y = z^2$ has no positive-integer solution, when $p, p + 8$ are primes such that $p > 3$. In 2019, Kumar et al. [8] proved that the solution of an exponential Diophantine equation $p^x + (p + 12)^y = z^2$ has no non-negative integer solution, when $p$ and $p + 12$ are prime numbers such that $p$ is in the form of $6n + 1$. In 2020, Burshtein [1] proved that a Diophantine equation $p^x + (p + 12)^y = z^2$ has no positive integer solution $(x, y, z)$, when $p$ is a prime number such that $p + 5 = 2^{2a}$. In 2021, Dokchan and Pakapongpun [3] studied a Diophantine equation $p^x + (p + 20)^y = z^2$, when $p$ and $p + 20$ are primes and showed that the equation has no positive integer solution $(x, y, z)$. In the same year, Gayo and Bacani [5] solved the Diophantine equation $M_p^x + (M_q + 1)^y = z^2$ when $M_p$ and $M_q$ are Mersenne primes.

In this work, we give solutions of the Diophantine equations $1 + b^y = z^2, 1 + (d + 4t)^x = z^2$ where $b, t, d$ are positive integers. Then, we extend to the solutions of the Diophantine equation $(p + 4n)^x + p^y = z^2$ where $p, p + 4n$ are prime numbers such that $p \equiv 7 \pmod{12}$ and $n$ is a positive integer such that $n \equiv 0, 1 \pmod{3}$.

2. Main results

**Proposition 1.** (Catalan’s conjecture) $(a, b, x, y) = (3, 2, 2, 3)$ is the unique solution of the Diophantine equation $a^x - b^y = 1$, where $a, b, x$ and $y$ are integers such that $\min\{a, b, x, y\} > 1$.

This proposition was proved in 2004 by Mihailescu [9].

**Lemma 1.** Let $b$ be a positive integer. The non-negative integer solutions to the Diophantine equation $1 + b^y = z^2$ is $(y, z) = (1, \sqrt{b + 1})$ if $\sqrt{b + 1}$ is a positive integer.

**Proof.** Let $b$ be a positive integer. We have $z^2 - b^y = 1$. By proposition 1, it is sufficient to consider the case $b = 1, z \leq 1$ or $y \leq 1$. Hence, it remains to consider the following cases of $b, y$ and $z$. If $b = 1$, then we have $z^2 = 2$, which is impossible. If $z = 0$ or $z = 1$, then there is no solution. If $y = 0$, then we have $z^2 = 2$ which is impossible. If $y = 1$, then we have $z^2 = b + 1$ or $z = \sqrt{b + 1}$. Thus, we have $(y, z) = (1, \sqrt{b + 1})$.

**Corollary 1.** Let $p$ be a prime number such that $p \equiv 7 \pmod{12}$. The non-negative integer solutions to the Diophantine equation $1 + p^y = z^2$ is $(y, z) = (1, \sqrt{p + 1})$ if $\sqrt{p + 1}$ is a positive integer.

**Lemma 2.** Let $t$ and $d$ be positive integers. The non-negative integer solutions of the Diophantine equation $1 + (d + 4t)^x = z^2$ is $(x, z) = \left(1, 2\sqrt{t + \frac{d + 1}{4}}\right)$ if $\sqrt{t + \frac{d + 1}{4}}$ is a positive integer.

**Proof.** Let $t, d$ be positive integers such that $\sqrt{t + \frac{d + 1}{4}}$ is a positive integer. We have $z^2 - (d + 4t)^x = 1$. By proposition 1, it is sufficient to consider only the case that $z \leq 1$ or $x \leq 1$. Hence, we consider the following cases of $z$ and $x$. For $z = 0$ and $z = 1$, there is no solution. If $x = 0$, then we have $z^2 = 2$, which is impossible. If $x = 1$, then we have
$z^2 = 4t + d + 1$. Thus $z = 2\sqrt{t + \frac{d+1}{4}}$ where $\sqrt{t + \frac{d+1}{4}}$ is a positive integer. Therefore, $(x, z) = (1, 2\sqrt{t + \frac{d+1}{4}})$.

**Corollary 2.** Let $n$ be a positive integer and $p, (p+4n)$ be prime numbers such that $n \equiv 0, 1 \pmod{3}$ and $p \equiv 7 \pmod{12}$. The non-negative integer solutions of the Diophantine equation $1 + (p+4n)^x = z^2$ is $(x, z) = \left(1, 2\sqrt{n + \frac{p+1}{4}}\right)$ if $\sqrt{n + \frac{p+1}{4}}$ is a positive integer.

**Theorem 1.** Let $n$ be a positive integer such that $n \equiv 0, 1 \pmod{3}, p \equiv 7 \pmod{12}$. If $\sqrt{p+1}$ and $\sqrt{n + \frac{p+1}{4}}$ are also integers, then all of the non-negative integer solutions to the Diophantine equation $(p + 4n)^x + p^y = z^2$ are given by $(x, y, z) \in \{(0, 1, \sqrt{p+1})\} \cup \{(1, 0, 2\sqrt{n + \frac{p+1}{4}})\}$, where $p$ and $p+4n$ are prime number.

**Proof.** Since $p$ is a prime number such that $p \equiv 7 \pmod{12}$, it is clear that $p \equiv 3 \pmod{4}$ and $p \equiv 1 \pmod{3}$. Let $(x, y, z)$ be a non-negative integer solution of $(p + 4n)^x + p^y = z^2$. If $x = 0$ or $y = 0$, then $(x, y, z) = (0, 1, \sqrt{p+1})$ or $(x, y, z) = \left(1, 0, 2\sqrt{n + \frac{p+1}{4}}\right)$.

Suppose $x > 0$ and $y > 0$. We consider the following cases.

**Case 1.** $x$ and $y$ are even numbers. Since $(p + 4n)^x + p^y = z^2$, it follows that $z$ is even. So $z^2 \equiv 0 \pmod{4}$. Note that $(p + 4n)^x \equiv 1 \pmod{4}$ and $p^y \equiv 1 \pmod{4}$. Thus $(p + 4n)^x + p^y \equiv 2 \pmod{4}$ which contradicts with $z^2 \equiv 0 \pmod{4}$.

**Case 2.** $x$ and $y$ are odd numbers. Since $(p + 4n)^x \equiv 3 \pmod{4}$ and $p^y \equiv 3 \pmod{4}$, it follows that $(p + 4n)^x + p^y \equiv 2 \pmod{4}$ which contradicts with $z^2 \equiv 0 \pmod{4}$.

**Case 3.** $x$ is an even number and $y$ is an odd number. Let $x = 2k, k \geq 1$ and $y = 2s+1, s \geq 0$. We have $(p + 4n)^{2k} + p^{2s+1} = z^2$, or equivalently $p^{2s+1} = z^2 - (p + 4n)^{2k} = [z + (p + 4n)^k][z - (p + 4n)^k]$. Thus, there exist non-negative integers $\alpha, \beta$ such that $\alpha^2 = z + (p + 4n)^k$ and $\beta^2 = z - (p + 4n)^k$, where $\alpha > \beta$ and $\alpha + \beta = 2s + 1$. Then, we have $2(p + 4n)^k = \beta^2(p^{2s-\beta} - 1)$. This implies that $\beta = 0$. We have $2(p + 4n)^k = (p^{2s+1} - 1)$, which is impossible because $2(p + 4n)^k \equiv 1, 2 \pmod{3}$ but $(p^{2s+1} - 1) \equiv 0 \pmod{3}$.

**Case 4.** $x$ is an odd number and $y$ is an even number. Let $x = 2k + 1, k \geq 0$ and $y = 2s, s \geq 1$. We have $(p + 4n)^{2k+1} + p^{2s} = z^2$, or equivalently $(p + 4n)^{2k+1} = z^2 - p^{2s} = (z + p^s)(z - p^s)$. Thus, there exist non-negative integer $\alpha, \beta$ such that $(p + 4n)^\alpha = z + p^s$ and $(p + 4n)^\beta = z - p^s$ where $\alpha > \beta$ and $\alpha + \beta = 2k + 1$. Then, we have $2(p)^s = (p + 4n)^\beta((p + 4n)^\alpha - \beta - 1)$. This implies that $\beta = 0$. We have $2(p)^s = (p + 4n)^{2k+1} - 1$, which is impossible because $2(p)^s \equiv 2 \pmod{3}$ but $(p + 4n)^{2k+1} - 1 \equiv 0, 1 \pmod{3}$.

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References


