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Grundy Hop Domination in Graphs

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Abstract. Let G be an undirected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of G and let $\hat{S} = \{v_1, v_2, \ldots, v_k\}$. Then S is a legal closed hop neighborhood sequence of G if $N_G^2[v_i] \setminus \cup_{j=1}^{i-1} N_G^2[v_j] \neq \varnothing$ for each $i \in \{2, \cdots, k\}$. If, in addition, $\hat{S}$ is a hop dominating set of G, then S is called a Grundy hop dominating sequence. The maximum length of a Grundy hop dominating sequence in a graph G, denoted by $\gamma_{gr}^h(G)$, is called the Grundy hop domination number of G. In this paper, we determine some (extreme) values for the Grundy hop domination number. It is pointed out that the Grundy hop domination number is at least equal to the hop domination. Bounds for the Grundy hop domination numbers of some graphs resulting from some binary operations of two graphs are also obtained.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Hop domination, hop domination number, closed hop neighborhood sequence, Grundy hop dominating sequence, Grundy hop domination number

1. Introduction

One of the several considered variations of the standard domination concept is hop domination. This concept was introduced and initially studied by Natarajan and Ayyaswamy in [15]. Just like domination, hop domination and its variations find plenty of applications in various fields and networks. In fact, some real-life problems (including protection strategies and facility location) that can be modeled by the concept of domination can be slighty modified for the concept of hop domination. Domination, hop domination, and some of their variations are also studied in [1], [2], [9], [10], [11], [12], [13], and [16].

In 2014, the concept of Grundy domination in graphs was introduced by Bresar et al. [6]. The newly defined parameter has subsequently attracted other researchers in the area who generated more interesting results (see [3], [4], [5], and [7]). Grundy domination was further studied in [5], where exact formulas for Grundy domination numbers of Sierpinski

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graphs were proven and a linear algorithm for determining these numbers in arbitrary interval graphs was given. Grundy domination number was also studied in Kneser graphs [7] and graph products in [3] and [14].

In this study, the concept of Grundy hop domination in a graph will be introduced and initially investigated. In particular, bounds for the parameter will be given for the join, corona, and lexicographic product of graphs.

2. Terminology and Notation

Two vertices u, v of a graph G are *adjacent*, or *neighbors*, if uv is an edge of G. Moreover, an edge uv of G is *incident* to two vertices u, v of G. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u in G. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{u\in X} N_G(u)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

Let G be a graph. A set $D \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of a graph G, and let $\hat{S} = \{v_1, v_2, \cdots, v_k\}$. Then S is a *legal closed neighborhood sequence* if $N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \varnothing$ for every $i \in \{2, \cdots, k\}$. If, in addition, $\hat{S}$ is a dominating set of G, then S is called a *Grundy dominating sequence*. The maximum length of a Grundy dominating sequence in a graph G is called the *Grundy domination number* of G, and is denoted by $\gamma_{gr}(G)$.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N^2_G(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N^2_G[u] = N^2_G(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N^2_G(X) = \bigcup_{u\in X} N^2_G(u)$. The *closed hop neighborhood* of X in G is the set $N^2_G[X] = N^2_G(X) \cup X$.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if $N^2_G[S] = V(G)$, that is, for every $v \in V(G)\setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of G and let $\hat{S} = \{v_1, \cdots, v_k\}$. Then S is a *legal closed hop neighborhood sequence* of G if $N^2_G[v_i]\setminus \cup_{j=1}^{i-1} N^2_G[v_j] \neq \varnothing$ for each $i \in \{2, \cdots, k\}$. If, in addition, $\hat{S}$ is a hop dominating set of G, then S is called a *Grundy hop dominating sequence*. The maximum length of a Grundy hop dominating sequence in a graph G, denoted by $\gamma^h_{gr}(G)$, is called the *Grundy hop domination number* of G. We say that vertex v_i *hop-footprints* the vertices from $N^2_G[v_i]\setminus \cup_{j=1}^{i} N^2_G[v_j]$, and that v_i is their *hop-footprinter*. A legal closed hop neighborhood sequence $S = (v_1, v_2, \cdots, v_k)$ with maximum length, i.e., $k = \max\{p \in \mathbb{N} : \exists$ a legal closed hop neighborhood sequence $(x_1, \cdots, x_p)$ of $G\}$, will be referred to as a *maximum legal closed hop neighborhood sequence*.

Let $S_1 = (v_1, \cdots, v_n)$ and $S_2 = (u_1, \cdots, u_m)$, $n, m \geq 1$ be two sequences of distinct vertices of G. The *concatenation* of S_1 and S_2, denoted by $S_1 \oplus S_2$, is the sequence given by

$$S_1 \oplus S_2 = (v_1, \cdots, v_n, u_1, \cdots, u_m).$$

A sequence $S = (v_1, v_2, \cdots, v_k)$ of distinct vertices of a graph G is a *co-legal closed neighborhood sequence* in G if

$$[V(G) \setminus N_G(v_i)] \setminus \cup_{j=1}^{i-1} [V(G) \setminus N_G(v_j)] \neq \varnothing$$

for each $i \in \{2, \ldots, k\}$. A co-legal sequence $S = (v_1, v_2, \ldots, v_k)$ is a *co-Grundy dominating sequence* if $V(G) = \cup_{i=1}^{k} [V(G) \setminus N_G(v_i)]$. The maximum length of a co-Grundy dominating sequence in a graph G is called the *co-Grundy domination number* of G, and is denoted by $\gamma_{cogr}(G)$.

A set $D \subseteq V(G)$ is *hop independent* if for every pair of distinct vertices $v, w \in D$, we have $d_G(v, w) \neq 2$. This concept was introduced and studied in [8]. Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of a graph G and let $\hat{S} = \{v_1, v_2, \ldots, v_k\}$. Then S is a *legal closed hop independent neighborhood sequence* in G if it is a legal closed hop neighborhood sequence and $\hat{S}$ is a hop independent set. A legal closed hop independent neighborhood sequence $S = (v_1, v_2, \ldots, v_k)$ is a *Grundy hop independent hop dominating sequence* if $\hat{S}$ is a hop independent hop dominating set of G. The maximum length of a Grundy hop independent hop dominating sequence in a graph G is called the *Grundy hop independent hop domination number* of G, and is denoted by $\gamma_{gr}^{hih}(G)$.

Let G and H be any two graphs. The *join* of G and H, denoted by $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* G and H, denoted by $G \circ H$, the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} + H^v \rangle$. The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(v, a)(u, b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$. We note that any non-empty set $C \subseteq V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Specifically, $T_x = \{a \in V(H) : (x, a) \in C\}$ for each $x \in S$.

3. Results

Remark 1. *The vertex set of a graph need not form a legal closed hop neighborhood sequence (a Grundy hop dominating sequence).*

To see this, consider the graph $G = C_5$ in Figure 1. Let $S = (v_1, v_2, v_3, v_4, v_5)$. Notice that

$$N_G^2[v_5] = \{v_2, v_3, v_5\} \subseteq N_G^2[v_1] \cup N_G^2[v_2] = \{v_1, v_2, v_3, v_4, v_5\}.$$

Hence, $N^2_G[v_5] \setminus \cup_{j=1}^4 N^2_G[v_j] = \varnothing$. Thus, S is not a legal closed hop neighborhood sequence (hence, not a Grundy hop dominating sequence). It is routine to show that any rearrangement of the terms of S is not legal closed hop neighborhood sequence of G.

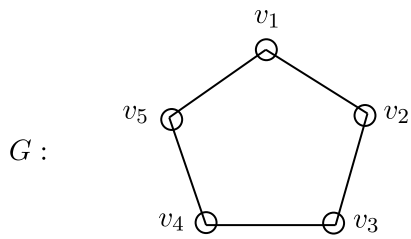

Figure 1: A graph G such that $V(G)$ does not form a legal closed hop neighborhood sequence

Remark 2. *A proper hop dominating set need not form a legal closed hop neighborhood sequence (a Grundy hop dominating sequence).*

Consider the graph G in Figure 2. Let $S = (v_1, v_2, v_3)$. Clearly, $\hat{S}$ is a proper hop dominating set of G. Observe that $N^2_G[v_3] = \{v_2, v_3, v_5\} = N^2_G[v_2]$.
Hence,

$$N^2_G[v_3] \setminus \cup_{j=1}^2 N^2_G[v_j] = \varnothing.$$

Thus, S is not a legal closed hop neighborhood sequence (not a Grundy hop dominating sequence).

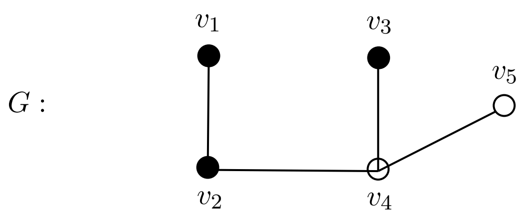

Figure 2: A graph G with a proper hop dominating set which does not form a legal closed hop neighborhood sequence

Our first result shows that every graph G admits a Grundy hop dominating sequence.

Theorem 1. *Let G be any graph on n vertices. Then the following statements hold.*

(i) If $\gamma_h(G) = k$ and $D = \{v_1, v_2, \ldots, v_k\}$ is a minimum hop dominating set of G, then $S = (v_1, v_2, \cdots, v_k)$ is a Grundy hop dominating sequence. In particular, $\gamma_h(G) \leq \gamma^h_{gr}(G)$.

(ii) If $S = (v_1, v_2, \cdots, v_m)$ is a minimum Grundy hop dominating sequence, then $\gamma_h(G) = |\hat{S}|$.

Proof. (i) Suppose there exists $i \in \{2,3,\ldots,k\}$ such that $N_G^2[v_i] \setminus \cup_{j=1}^{i-1} N_G^2[v_j] = \varnothing$. Then $N_G^2[v_i] \subseteq \cup_{j=1}^{i-1} N_G^2[v_j]$. It follows that $D \setminus \{v_i\}$ is a hop dominating set of G, contradicting the minimality of D. Therefore, $N_G^2[v_i] \setminus \cup_{j=1}^{i-1} N_G^2[v_j] \neq \varnothing$ for each $i \in \{2,3,\ldots,k\}$. Consequently, $\gamma_h(G) \leq \gamma_{gr}^h(G)$.

(ii) Note that from (i), every γ_h-set of G forms a Grundy hop dominating sequence. Since S is a minimum Grundy hop dominating sequence, it follows that $|\hat{S}| \leq \gamma_h(G)$. On the other hand, since every Grundy hop dominating sequence forms a hop dominating set (by definition), it follows that $\gamma_h(G) \leq |\hat{S}|$. This establishes the desired equality. □

Theorem 2. *Let G be any graph. Then $S = (v_1, v_2, \cdots, v_k)$ is a maximum legal closed hop neighborhood sequence of G if and only if S is a Grundy hop dominating sequence of G and $\gamma_{gr}^h(G) = k$.*

Proof. Let $S = (v_1, \cdots, v_k)$ be a maximum legal closed hop neighborhood sequence of G. Suppose $\hat{S}$ is not a hop dominating set of G. Then there exists $v \in V(G) \setminus N_G^2[\hat{S}]$. This implies that $v \notin N_G^2[u]$ for every $u \in \hat{S}$. Let $S' = (v_1, \cdots, v_k, v)$. Since S is a legal closed hop neighborhood sequence, $N_G^2[v_i] \setminus \cup_{j=1}^{i-1} N_G^2[v_j] \neq \varnothing$ for each $i \in \{2,3,\ldots,k\}$. Also, since $v \in N_G^2[v]$ and $v \notin N_G^2[u]$ for every $u \in \hat{S}$, it follows that $N_G^2[v] \setminus \cup_{j=1}^{k} N_G^2[v_i] \neq \varnothing$. Hence, S' is a legal closed hop neighborhood sequence of G, contradicting the maximality of S. Therefore, $\hat{S}$ is a hop dominating set of G. Since S is a maximum legal closed hop neighborhood sequence of G, it is a Grundy hop dominating sequence and $\gamma_{gr}^h(G) = k$.

For the converse, suppose that S is a Grundy hop dominating sequence and $\gamma_{gr}^h(G) = k$. Then S is a maximum legal closed hop neighborhood sequence of G. □

Corollary 1. *Let G be a graph and let $D = (x_1, \cdots, x_t)$ be a legal closed hop neighborhood sequence of G. Then $|\hat{D}| = t \leq \gamma_{gr}^h(G)$.*

Proof. Let k be the length of a maximum legal closed hop neighborhood sequence of G. Then $t \leq k$. By Theorem 2, $|\hat{D}| = t \leq \gamma_{gr}^h(G)$. □

Throughout, $[n] = \{1, 2, \ldots, n\}$ for each positive integer n.

Theorem 3. *Let G be any graph on $n \geq 2$ vertices. Then $2 \leq \gamma_{gr}^h(G) \leq n$. Moreover, each of the following statements holds.*

(i) $\gamma_{gr}^h(G) = 2$ if and only if for each pair of distinct vertices $x, y \in V(G)$ such that $N_G^2[x] \neq N_G^2[y]$, we have $V(G) = N_G^2[x] \cup N_G^2[y]$ (i.e., $\{x, y\}$ is a hop dominating set of G).

(ii) $\gamma_{gr}^h(G) = n$ if and only if every component C of G is complete.

Proof. Clearly, $2 \leq \gamma_{gr}^h(G) \leq n$.

(i) Suppose $\gamma_{gr}^h(G) = 2$. Then by Theorem 2, the maximum length of a legal closed hop neighborhood sequence of G is 2. Let x and y be distinct vertices of G such that $N_G^2[x] \neq N_G^2[y]$. We may assume that $N_G^2[y] \setminus N_G^2[x] \neq \varnothing$. Then (x, y) is a legal closed hop neighborhood sequence of G. Suppose there exists $z \in V(G) \setminus (N_G^2[x] \cup N_G^2[y])$. Since $z \in N_G^2[z]$, it follows that $N_G^2[z] \setminus (N_G^2[x] \cup N_G^2[y]) \neq \varnothing$. This implies that (x, y, z) is a legal closed hop neighborhood sequence of G, contrary to the assumption that $\gamma_{gr}^h(G) = 2$. Hence, $V(G) = N_G^2[x] \cup N_G^2[y]$.

For the converse, suppose that for each pair of distinct vertices $x, y \in V(G)$ such that $N_G^2[x] \neq N_G^2[y]$, we have $V(G) = N_G^2[x] \cup N_G^2[y]$. Since G is a non-trivial graph, the assumption implies that $\gamma_h(G) = 2$. Hence, $\gamma_{gr}^h(G) = k \geq 2$. Let $(v_1, v_2, \cdots, v_k)$ be a Grundy hop dominating sequence of G. Because $N_G^2[v_2] \setminus N_G^2[v_1] \neq \varnothing$, $V(G) = N_G^2[v_1] \cup N_G^2[v_2]$, by assumption. Therefore, $\gamma_{gr}^h(G) = k = 2$.

(ii) Suppose $\gamma_{gr}^h(G) = n$ and let $S = (v_1, v_2, \cdots, v_n)$ be a Grundy hop dominating sequence of G. Note that since $N_G^2[v_n] \setminus \cup_{j=1}^{n-1} N_G^2[v_j] \neq \varnothing$ and $v_j \in N_G^2[v_j]$ for each $j \in [n]$, $N_G^2[v_n] \setminus \cup_{j=1}^{n-1} N_G^2[v_j] = \{v_n\}$. This implies that $v_n \notin N_G^2[v_j]$ for all $j \in [n-1]$, i.e., $N_G^2[v_n] = \{v_n\}$. This would imply that $N_G^2[v_{n-1}] \setminus \cup_{j=1}^{n-2} N_G^2[v_j] = \{v_{n-1}\}$. Using the same argument as earlier, $N_G^2[v_{n-1}] = \{v_{n-1}\}$. Continuing in this fashion, we find that $N_G^2[v_i] = \{v_i\}$ for each $i \in [n]$, i.e., $deg_G(v_i) = 0$ or v_i is adjacent to every other vertex in the component it belongs. Therefore, every component of G is complete.

Conversely, if every component of G is complete, then $N_G^2[v] = \{v\}$ for each $v \in V(G)$. Hence, if $V(G) = \{v_1, v_2, \ldots, v_n\}$, then

$$N_G^2[v_i] \setminus \cup_{j=1}^{i-1} N_G^2[v_j] = \{v_i\} \setminus \{v_j : j \neq i\} = \{v_i\} \neq \varnothing.$$

for each $i \in \{2, 3, \ldots, n\}$. This shows that $(v_1, v_2, \cdots, v_n)$ is a Grundy hop dominating sequence of G. Therefore $\gamma_{gr}^h(G) = n$. □

The next result is immediate from the Theorem 3(ii).

Corollary 2. *Let G be a connected graph on n vertices. Then each of the following statements holds.*

(i) *$\gamma_{gr}^h(G) = n$ if and only if $G = K_n$.*

(ii) *If G is non-complete, then $\gamma_{gr}^h(G) \leq n - 1$.*

Theorem 4. *Let G be a graph on n vertices.*

(i) *If G is complete, then $\gamma_{gr}^h(G) + \gamma_{gr}^h(\overline{G}) = 2n$.*

(ii) *If G is non-complete, then*

(a) *$4 \leq \gamma_{gr}^h(G) + \gamma_{gr}^h(\overline{G}) \leq 2n - 1$, and*

(b) *$4 \leq \gamma_{gr}^h(G) \cdot \gamma_{gr}^h(\overline{G}) \leq n^2 - n$.*

Proof. (i) The equality follows from Theorem 3(ii).

(ii) By Corollary 2(ii), $\gamma^h_{gr}(G) \leq n-1$ and by Theorem 3, $\gamma^h_{gr}(\overline{G}) \leq n$. These imply that $\gamma^h_{gr}(G) + \gamma^h_{gr}(\overline{G}) \leq n-1+n = 2n-1$, and $\gamma^h_{gr}(G) \cdot \gamma^h_{gr}(\overline{G}) \leq (n-1)n = n^2-n$. The left inequalities follow from Theorem 3. □

The bounds in Theorem 4(ii) are tight. Indeed, one can easily verify that

$$\gamma^h_{gr}(P_4) + \gamma^h_{gr}(\overline{P}_4) = \gamma^h_{gr}(P_4) \cdot \gamma^h_{gr}(\overline{P}_4) = 4,$$

$$\gamma^h_{gr}(P_3) + \gamma^h_{gr}(\overline{P}_3) = 5 = 2(3) - 1, \text{ and}$$

$$\gamma^h_{gr}(P_3) \cdot \gamma^h_{gr}(\overline{P}_3) = 6 = 3^2 - 3.$$

Proposition 1. *For any positive integer $n \geq 2$,*

$$\gamma^h_{gr}(P_n) = \begin{cases} 2 & if\ n = 2,3 \\ n-2 & if\ n \geq 4. \end{cases}$$

Proof. Let $G = P_n = [v_1, v_2, \cdots, v_n]$. Clearly, $\gamma^h_{gr}(P_n) = 2$ for $n = 2,3$. So suppose that $n \geq 4$. Let $S' = (v_1, \cdots, v_{n-2})$. Clearly, S' is a Grundy hop dominating sequence in G. Since G is not a complete graph, $\gamma^h_{gr}(G) \leq n-1$ by Corollary 2(ii). Thus, $n-2 \leq \gamma^h_{gr}(G) \leq n-1$. Suppose $\gamma^h_{gr}(G) = n-1$, say, $S = (w_1, \cdots, w_{n-1})$ is a Grundy hop dominating sequence. Then $N^2_G[w_{n-1}] \setminus \bigcup_{j=1}^{n-2} N^2_G[w_j] \neq \varnothing$. Notice that $N^2_G[w_{n-1}] \setminus \bigcup_{j=1}^{n-2} N^2_G[w_j] \subseteq \{w_{n-1}, v_r\}$ for some $r \in \{1, \ldots, n\}$, $v_r \neq w_{n-1}$. Consider the following two cases:

Case 1: $N^2_G[w_{n-1}] \setminus \bigcup_{j=1}^{n-2} N^2_G[w_j] = \{w_{n-1}, v_r\}$ for some $v_r \in V(G) \setminus \{w_1, \ldots, w_{n-1}\}$.

Let $w_{n-1} = v_q$. Then v_q and v_r are not hop dominated by each w_j, where $j \in \{1, 2, \ldots, n-2\}$. If $q < r$, then v_q is v_1 or v_2. If $v_q = v_1$, then $v_r = v_3$ and $v_n = v_4$. This is not possible because $N^2_G[v_2] = N^2_G[v_4] = \{v_2, v_4\}$ where $v_2, v_4 \in \hat{S}$. If $v_q = v_2$, then $v_r = v_4$ and $n = 4$ or 5. Again, this is not possible. A similar situation happens when $q > r$.

Case 2: $N^2_G[w_{n-1}] \setminus \bigcup_{j=1}^{n-2} N^2_G[w_j] = \{w_{n-1}\}$.

Then w_{n-1} is not hop dominated by any of the vertices $w_1, \cdots, w_{n-2}$. Moreover, since $n \geq 4$, $d_G(w_{n-1}, v_t) = 2$ for some vertex $v_t \in V(G) \setminus \hat{S}$. Now, because $N^2_G[w_{n-1}] \setminus \bigcup_{j=1}^{n-2} N^2_G[w_j] = \{w_{n-1}\}$, v_t must be hop-dominated by some vertex w_j where $1 \leq j \leq n-2$. This is not possible when $n = 4$, and so $n \geq 5$. This would imply that w_{n-1} is either v_1, v_2, v_{n-1}, or v_n. It is routine to show that any of these vertices will contradict the assumption that S is a legal closed hop neighborhood sequence.

Therefore, $\gamma^h_{gr}(P_n) = n-2$ when $n \geq 4$. □

Lemma 1. *Let G be a graph on n vertices. If $|N^2_G[v]| = 3$ for every $v \in V(G)$, then $\gamma^h_{gr}(G) \leq n-2$.*

Proof. Suppose that $|N^2_G[v]| = 3$ for every $v \in V(G)$. Then $G \neq K_n$. Thus, $\gamma^h_{gr}(G) \leq n-1$. Suppose that $\gamma^h_{gr}(G) = n-1$, say, $S = (v_1, \cdots, v_{n-1})$ is a Grundy hop dominating sequence of G. Then $N^2_G[v_i] \setminus \bigcup_{j=1}^{i-1} N^2_G[v_j] \neq \varnothing$ for each $i \in \{2, \ldots, n-1\}$. Let $p, q \in N^2_G(v_{n-1})$. Since $N^2_G[v_{n-1}] \setminus \bigcup_{j=1}^{n-2} N^2_G[v_j] \neq \varnothing$, $p \notin S$ or $q \notin S$, say $p \notin S$. Then $q = v_k \in \hat{S}$ for some $k \neq n-1$ and $v_k, v_{n-1} \in N^2_G[v_k]$. Now, since $|N^2_G[p]| = 3$, there exists $j \neq n-1$ such that $p \in N^2(v_j)$. It follows that $v_k, p, v_{n-1} \in N^2_G[v_k] \cup N^2_G[v_j]$, a contradiction. Therefore, $\gamma^h_{gr}(G) \leq n-2$. □

The next result shows that the bound in Lemma 1 is tight.

Proposition 2. *For any positive integer $n \geq 3$,*

$$\gamma^h_{gr}(C_n) = \begin{cases} 3 & if\ n = 3 \\ 2 & if\ n = 4 \\ n-4 & if\ n \geq 6 \text{ and even} \\ n-2 & if\ n \geq 5 \text{ and odd.} \end{cases}$$

Proof. Let $G = C_n = [v_1, v_2, \cdots, v_n, v_1]$. Clearly, $\gamma^h_{gr}(C_3) = 3$ and $\gamma^h_{gr}(C_4) = 2$. Let $n \geq 6$. Let $S_0 = (v_1, v_2, \cdots, v_{n-4})$. Then $N^2_G[v_2] \setminus N^2_G[v_1] = \{v_2, v_4, v_n\} \neq \varnothing$ and $v_{i+2} \in N^2_G[v_i] \setminus \cup_{j=1}^{i-1} N^2_G[v_j]$ for all $i \in \{3, 4, \cdots, n-4\}$. It follows that S_0 is a Grundy hop dominating sequence and $\gamma^h_{gr}(C_n) \geq |\hat{S_0}| = n-4$. Suppose that n is even and suppose that S is a Grundy hop dominating sequence of C_n with $\gamma^h_{gr}(C_n) = |\hat{S}|$. Observe that if i is even and j is odd, then $N^2_G[v_i] \cap N^2_G[v_j] = \varnothing$. Hence, we may express S as concatenation $S_1 \oplus S_2$ where the subscripts of the terms of S_1 and S_2 are even and odd, respectively. Now, since $N^2_G[v_1] = \{v_1, v_3, v_{n-1}\}$, $N^2_G[v_2] = \{v_2, v_4, v_n\}$, $N^2_G[v_n] = \{v_2, v_{n-2}, v_n\}$, $N^2_G[v_{n-1}] = \{v_1, v_{n-3}, v_{n-1}\}$, and $N^2_G[v_i] = \{v_{i-2}, v_i, v_{i+2}\}$ for $i \in \{3, 4, \cdots, n-4\}$, it follows that each of S_1 and S_2 can only have at most $\frac{n}{2} - 2$ terms. Thus, $\gamma^h_{gr}(C_n) = n-4$.

Next, suppose that $n \geq 5$ and is odd. Clearly, $\gamma^h_{gr}(C_n) = 3$ if $n = 5$. For $n \geq 7$, the sequence $S = (v_1, v_3, \cdots, v_n, v_2, v_4, \cdots, v_{n-5}) = (v_1, v_3, \cdots, v_n) \oplus (v_2, v_4, \cdots, v_{n-5})$ can be verified to be a Grundy hop dominating sequence of C_n. This and Lemma 1 would imply that $\gamma^h_{gr}(G) = n-2$. □

Lemma 2. *Let G be a graph. A sequence S is a co-legal closed neighborhood sequence in G if and only if S is a legal closed neighborhood sequence in $\overline{G}$. Moreover, S is a co-Grundy dominating sequence in G if and only if it is a Grundy dominating sequence in $\overline{G}$. In particular, $\gamma_{cogr}(G) = \gamma_{gr}(\overline{G})$.*

Proof. Let $S = (v_1, \cdots, v_k)$ be a sequence in G. Since $V(G) \setminus N_G(v_i) = N_{\overline{G}}[v_i]$ for each $i \in [k]$, it follows that

$$[V(G) \setminus N_G(v_i)] \setminus \cup_{j=1}^{i-1} [V(G) \setminus N_G(v_j)] = N_{\overline{G}}[v_i] \setminus \cup_{j=1}^{i-1} N_{\overline{G}}[v_j].$$

Hence, S is a co-legal closed neighborhood sequence in G if and only if it is a legal closed neighborhood sequence in $\overline{G}$. Clearly, a co-legal closed neighborhood sequence in G is a co-Grundy dominating sequence if and only if it is a Grundy dominating sequence in $\overline{G}$. Hence, $\gamma_{cogr}(G) = \gamma_{gr}(\overline{G})$. □

Theorem 5. *Let G and H be any two graphs. A sequence S of distinct vertices of $G+H$ is a legal closed hop neighborhood sequence if and only if one of the following holds:*

(i) S is a co-legal closed neighborhood sequence in G (legal closed neighborhood sequence in $\overline{G}$).

(ii) S is a co-legal closed neighborhood sequence in H (legal closed neighborhood sequence in $\overline{H}$).

(iii) S is a concatenation $S_G \oplus S_H$, where S_G and S_H are co-legal closed neighborhood sequences in G and H, respectively.

Proof. Suppose that $S = (w_1, \cdots, w_k)$ is a legal closed hop neighborhood sequence in $G+H$ and let $\hat{S} = \{w_1, \ldots, w_k\}$. Suppose $\hat{S} \subseteq V(G)$. By the legality condition in S, we have

$$N^2_{G+H}[w_i] \setminus \cup_{j=1}^{i-1} N^2_{G+H}[w_j] \neq \varnothing \text{ for all } \ i \in \{2, 3, \ldots, k\}.$$

Since $N^2_{G+H}[w_i] = V(G) \setminus N_G(w_i)$ for each $i \in [k]$, it follows that

$$[V(G) \setminus N_G(w_i)] \setminus \cup_{j=1}^{i-1} [V(G) \setminus N_G(w_j)] \neq \varnothing \text{ for all } \ i \in \{2, 3, \ldots, k\}.$$

Therefore, S is a co-legal closed neighborhood sequence in G, showing that (i) holds. Similarly, (ii) holds if $\hat{S} \subseteq V(H)$.

Next, suppose that $\hat{S_G} = \hat{S} \cap V(G) \neq \varnothing$ and $\hat{S_H} = \hat{S} \cap V(H) \neq \varnothing$. Since $N^2_{G+H}[w_j] \subseteq V(G)$ for all $w_j \in \hat{S_G}$ and $N^2_{G+H}[w_s] \subseteq V(H)$ for all $w_s \in \hat{S_H}$, we may assume that $\hat{S_G} = \{w_1, w_2, \ldots, w_m\}$ and $\hat{S_H} = \{w_{m+1}, w_{m+1}, \ldots, w_k\}$. Then $S = S_G \oplus S_H$. Since S is a legal closed hop neighborhood sequence,

$$[V(G) \setminus N_G(w_i)] \setminus \cup_{j=1}^{i-1} [V(G) \setminus N_G(w_j)] = N^2_{G+H}[w_i] \setminus \cup_{j=1}^{i-1} N^2_{G+H}[w_j] \neq \varnothing$$

for all $i \in \{2, 3, \ldots, m\}$, showing that S_G is a co-legal closed neighborhood sequence in G. Similarly, S_H is a co-legal closed neighborhood sequence in H. This shows that (iii) holds.

The converse is clear. □

Corollary 3. *Let G and H be any two graphs. A sequence S of distinct vertices of $G+H$ is a Grundy hop dominating sequence in $G+H$ if and only if $S = S_G \oplus S_H$, where S_H and S_H are co-Grundy dominating sequences in G and H, respectively (Grundy dominating sequences in $\overline{G}$ and $\overline{H}$, respectively). Moreover,*

$$\gamma^h_{gr}(G+H) = \gamma_{cogr}(G) + \gamma_{cogr}(H) = \gamma_{gr}(\overline{G}) + \gamma_{gr}(\overline{H}).$$

In particular, each of the following holds.

(i) $\gamma^h_{gr}(K_1 + G) = 1 + \gamma_{cogr}(G) = 1 + \gamma_{gr}(\overline{G})$.

(ii) $\gamma^h_{gr}(K_{m,n}) = 2$ *for* $m, n \geq 1$.

(iii) $\gamma^h_{gr}(W_n) = 1 + \gamma_{cogr}(C_n) = 1 + \gamma_{gr}(\overline{C}_n)$ *for all* $n \geq 3$.

(iv) $\gamma^h_{gr}(F_n) = 1 + \gamma_{cogr}(P_n) = 1 + \gamma_{gr}(\overline{P}_n)$ *for all* $n \geq 1$.

Theorem 6. *Let G be a non-trivial connected graph on m vertices and let H be any graph. Then $\gamma^h_{gr}(G \circ H) \geq m \cdot \gamma_{cogr}(H) = m \cdot \gamma_{gr}(\overline{H})$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_m\}$ and let $S_{v_i} = (w^1_{v_i}, w^2_{v_i}, \cdots, w^k_{v_i})$ be a co-Grundy dominating sequence in H^{v_i} for each $i \in [m] = \{1, 2, \ldots, m\}$, where $k = \gamma_{cogr}(H)$. Let $S = S_{v_1} \oplus S_{v_2} \oplus \cdots \oplus S_{v_m}$. Let $x \in V(G \circ H) \setminus \hat{S}$ and let $v_t \in V(G)$ such that $x \in V(v_t + H^{v_t})$. Suppose first that $x = v_t$. Let $v_s \in N_G(v_t)$ and pick any $w^j_{v_s} \in \hat{S}_{v_s}$. Then $w^j_{v_s} \in \hat{S} \cap N^2_{G\circ H}(v_t)$. Suppose $x \neq v_t$. Then $x \in V(H^{v_t}) \setminus \hat{S}_{v_t}$. Since $\hat{S}_{v_t}$ is a co-Grundy dominating set in H^{v_t}, it follows that there exists $w^l_{v_t} \in \hat{S}_{v_t} \subseteq \hat{S}$ such that $d_{H^{v_t}}(x, w^l_{v_t}) \neq 1$. It follows that $d_{G\circ H}(x, w^l_{v_t}) = 2$. Thus, $\hat{S}$ is a hop dominating set in $G \circ H$.

We relabel the terms in S, say $S = (x_1, x_2, \cdots, x_k, \cdots, x_{mk})$. Next, let $i \in [mk] \setminus \{1\}$ and let $x_i = w^t_{v_r}$, where $r \in [m]$ and $t \in [k]$. Then

$$\begin{aligned} N^2_{G\circ H}[x_i] \setminus \cup^{i-1}_{j=1} N^2_{G\circ H}[x_j] &= N^2_{G\circ H}[w^t_{v_r}] \setminus [(\cup^{t-1}_{s=1} N^2_{G\circ H}[w^s_{v_r}]) \cup \\ &\quad (\cup\{N^2_{G\circ H}[w^p_{v_q}] : p \in [k] \text{ and } 1 \leq q \leq r-1\})]. \end{aligned}$$

If $t = 1$, then $N^2_{G\circ H}[w^t_{v_r}] \setminus (\cup^{t-1}_{s=1} N^2_{G\circ H}[w^s_{v_r}]) = N^2_{G\circ H}[w^t_{v_r}]$. Clearly,

$$w^t_{v_r} \in N^2_{G\circ H}[w^t_{v_r}] \setminus [\cup\{N^2_{G\circ H}[w^p_{v_q}] : p \in [k] \text{ and } 1 \leq q \leq r-1\}].$$

Suppose $t \neq 1$. Then

$$\begin{aligned} N^2_{G\circ H}[w^t_{v_r}] \setminus (\cup^{t-1}_{s=1} N^2_{G\circ H}[w^s_{v_r}]) &= [V(H^{v_r}) \setminus N_{H^{v_r}}(w^t_{v_r})] \setminus \\ &\quad [\cup^{t-1}_{s=1}(V(H^{v_r}) \setminus N_{H^{v_r}}(w^s_{v_r}))] \\ &\neq \varnothing \end{aligned}$$

because S_{v_r} is a co-legal neighborhood sequence in H^{v_r}. Since

$$N^2_{G\circ H}[w^t_{v_r}] \setminus (\cup^{t-1}_{s=1} N^2_{G\circ H}[w^s_{v_r}]) \cap [\cup\{N^2_{G\circ H}[w^p_{v_q}] : p \in [k] \text{ and } 1 \leq q \leq r-1\}] = \varnothing,$$

$N^2_{G\circ H}[x_i] \setminus \cup^{i-1}_{j=1} N^2_{G\circ H}[x_j] \neq \varnothing$. Therefore, S is a Grundy hop dominating sequence in $G \circ H$. Accordingly,

$$\gamma^h_{gr}(G \circ H) \geq |\hat{S}| = \sum^m_{i=1} |\hat{S}_{v_i}| = m \cdot \gamma_{cogr}(H) = m \cdot \gamma_{gr}(\overline{H}).$$

This proves the assertion. □

Remark 3. *The bound given in Theorem 6 is tight. Moreover, strict inequality can also be attained.*

To see this, consider $G = K_3$ and $H = P_3$. Then $\gamma_{cogr}(H) = 2$ and $\gamma^h_{gr}(G \circ H) = 3\gamma_{cogr}(H) = 6$.

For the strict inequality, consider the graphs G and $G \circ K_1$ in Figure 3. Then $\gamma_{cogr}(K_1) = 1$ and $S = (7, 5, 1, 4, 3)$ is a Grundy hop dominating sequence in $G \circ K_1$ with

$$\gamma^h_{gr}(G \circ K_1) = |S| = 5 > 4 = 4\gamma_{cogr}(K_1).$$

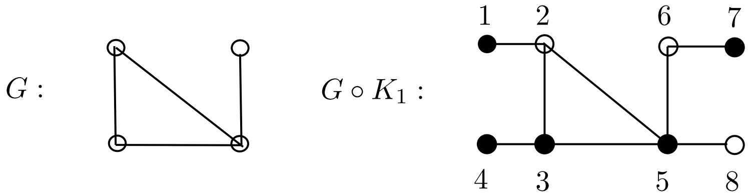

Figure 3: The corona $G \circ K_1$ with $\gamma^h_{gr}(G \circ K_1) = 5$

Theorem 7. *Let G and H be non-trivial connected graphs. Let $S_G = (v_1, v_2, \cdots, v_k)$ be a legal closed hop independent neighborhood sequence of G and let $S_H = (a_1, a_2, \cdots, a_t)$ be a co-legal neighborhood sequence of H. Then*

$$S = ((v_1, a_1), (v_1, a_2), \cdots, (v_1, a_t), \cdots, (v_k, a_1), (v_k, a_2), \cdots, (v_k, a_t))$$

is a legal closed hop neighborhood sequence of $G[H]$.

Proof. Suppose S_G is a legal closed hop independent neighborhood sequence of G and let S_H be a co-legal neighborhood sequence of H. Let $i \in [k]$ and $j \in [t]$. Then

$$N^2_{G[H]}[(v_i, a_j)] \setminus \cup_{l=1}^{j-1} N^2_{G[H]}[(v_i, a_l)] = (\{v_i\} \times V(H) \setminus N_H(a_j)) \setminus (\{v_i\} \times \cup_{l=1}^{j-1} [V(H) \setminus N_H(a_l)]).$$

Now, since S_H is a co-legal sequence in H,

$$V(H) \setminus N_H(a_j)) \setminus \cup_{l=1}^{j-1} [V(H) \setminus N_H(a_l)] \neq \varnothing.$$

Equality implies that

$$N^2_{G[H]}[(v_i, a_j)] \setminus \cup_{l=1}^{j-1} N^2_{G[H]}[(v_i, a_l)] \neq \varnothing.$$

The assumption that S_G is a legal closed hop independent neighborhood sequence would imply that $(\{v_i\} \times V(H) \setminus N_H(a_j)) \setminus (\{v_i\} \times \cup_{l=1}^{j-1} [V(H) \setminus N_H(a_l)])$ and $\cup\{N^2_{G[H]}[(v_r, a_l)] :$

$r \in [i-1]$ and $l \in [t]\}$ are disjoint. Thus,

$$N^2_{G[H]}[(v_i, a_j)] \setminus \cup\{N^2_{G[H]}[(v_r, a_l)] : r \in [i], l \in [t] \text{ with } (v_i, a_j) \neq (v_r, a_l)\} \neq \varnothing.$$

Therefore, S is a legal closed hop neighborhood sequence in $G[H]$. □

Theorem 8. *Let G and H be non-trivial connected graphs. If $S_G = (v_1, v_2, \cdots, v_k)$ is a Grundy hop independent hop dominating sequence in G and $S_H = (a_1, a_2, \cdots, a_t)$ is a co-Grundy dominating sequence in H. Then*

$$S = ((v_1, a_1), (v_1, a_2), \cdots, (v_1, a_t), \cdots, (v_k, a_1), (v_k, a_2), \cdots, (v_k, a_t))$$

is a Grundy hop dominating sequence of $G[H]$. In particular, $\gamma^h_{gr}(G[H]) \geq \gamma^{hih}_{gr}(G)\gamma_{cogr}(H)$.

Proof. By Theorem 7, S is a legal closed hop neighborhood sequence in $G[H]$. Let $(v, a) \in V(G[H]) \setminus \hat{S}$. Suppose that $v \in V(G) \setminus S_G$. Since S_G is a hop dominating set, there exists $v_j \in S_G$ such that $v \in N^2_G(v_j)$. It follows that $(v_j, a_1) \in \hat{S}$ and $(v, a) \in N^2_{G[H]}(v_j, a_1)$. Suppose $v = v_i$ for some $i \in [k]$. Then $a \in V(H) \setminus S_H$. Since S_H is a co-Grundy dominating sequence, there exists $a_s \in S_H$ such that $a \notin N_H(a_s)$. Clearly, $(v_i, a_s) \in \hat{S}$ and $(v, a) \in N^2_{G[H]}(v_i, a_s)$. Thus, $\hat{S}$ is a hop dominating set, showing that S is a Grundy hop dominating sequence in $G[H]$. Therefore, $\gamma^h_{gr}(G[H]) \geq \gamma^{hih}_{gr}(G)\gamma_{cogr}(H)$. □

Remark 4. *The bound given in Theorem 8 is tight.*
To see this, consider $G = H = P_3$. Then $\gamma^{hih}_{gr}(G) = 2$ and $\gamma_{cogr}(H) = 2$. It can easily be verified that $\gamma^h_{gr}(G[H]) = \gamma^{hih}_{gr}(G)\gamma_{cogr}(H) = 4$.

4. Conclusion

This study did introduce the concept of Grundy hop domination and make an initial investigation of the concept. It was pointed out and proved that every graph admits a Grundy hop dominating sequence. Extremal values of the Grundy hop domination number were given. Moreover, exact value or tight lower bound for each of the Grundy hop domination numbers of the join, corona, and lexicographic product of two graphs was determined. Bounds for this newly defined parameter in terms of other parameters (e.g. minimum degree, maximum degree, diameter, etc.) may be obtained. The parameter can be investigated further for trees and graphs under other binary operations.

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References

[1] S. Ayyaswamy, B. Krishnakumari, B. Natarjan, and Y. Venkatakrishnan. Bounds on the hop domination number of a tree. *Proceedings-Mathematical Sciences.*, 125(4):449–455, 2015.

[2] S. Ayyaswamy, C. Natarajan, and G. Sathiamoorphy. A note on hop domination number of some special families of graphs. *International Journal of Pure and Applied Mathematics.*, 119(12):11465–14171, 2018.

[3] B. Bresar, Cs. Bujtas, T. Gologranc, S. Klavzar, G. Kosmrlj, B. Patkos, Zs. Tuza, and M. Vizer. Dominating sequence in grid-like and toroidal graphs. *Electron. J. Combin.*, (23):1–17, 2016.

[4] B. Bresar, Cs. Bujtas, T. Gologranc, S. Klavzar, G. Kosmrlj, B. Patkos, Zs. Tuza, and M. Vizer. Grundy dominating sequence and zero forcing sets. *Discrete Optim.*, (26):66–77, 2017.

[5] B. Bresar, T. Gologranc, and T. Kos. Dominating sequences under atomic changes with applications in sierpinski and interval graphs. *Appl. Anal. Discrete Math.*, (10):518–531, 2016.

[6] B. Bresar, T. Gologranc, M. Milanic, D. Rall, and R. Rizzi. Dominating sequence in graphs. *Discrete Math.*, (336):22–36, 2014.

[7] B. Bresar, T. Kos, and P. Torres. Grundy domination and zero forcing in kneser graphs. *Ars Math. Contemp.*, (17):419–430, 2019.

[8] J. Hassan, S. Canoy Jr., and A. Aradais. Hop independent sets in graphs. *Eur. J. Pure Appl. Math.*, 15(2):467–477, 2022.

[9] M. Henning and N. Rad. On 2-step and hop dominating sets in graphs. *Graphs and Combinatorics.*, 33(4):913–927, 2017.

[10] S. Canoy Jr. and G. Malacas. Determining the intruder's location in a given network: Locating-dominating sets in a graph. *NRCP Research Journal.*, 13(1):1–8, 2013.

[11] S. Canoy Jr. and G. Malacas. Differentiating-dominating sets in graphs under binary operations. *Tamkang J. Math.*, 46(1):51–60, 2015.

[12] S. Canoy Jr., R. Mollejon, and J. G. Canoy. Hop dominating sets in graphs under binary operations. *Eur. J. Pure Appl. Math.*, 12(4):1455–1463, 2019.

[13] S. Canoy Jr. and G. Salasalan. Locating-hop domination in graphs. *Kyungpook Mathematical Journal.*, 62:193–204, 2022.

[14] G. Nasini and P. Torres. Grundy dominating sequences on x-join product. *Discrete Applied Mathematics.*, (284):138–149, 2020.

[15] C. Natarajan and S. Ayyaswamy. Hop domination in graphs ii. *Versita*, 23(2):187–199, 2015.

[16] G. Salasalan and S. Canoy Jr. Global hop domination numbers of graphs. *Eur. J. Pure Appl. Math.*, 14(1):112–125, 2021.