Forcing Subsets of Connected Co-Independent Hop Domination in the Edge Corona and Lexicographic Product of Graphs

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\begin{abstract}
This study deals with the forcing subsets of a minimum connected co-independent hop dominating sets in graphs. Bounds or exact values of the forcing connected co-independent hop domination numbers of graphs resulting from some binary operations such as edge corona and lexicographic product of graphs are determined. Some main results generated in this study include: (a) characterization of the minimum connected co-independent hop dominating sets; and (b) characterization of the forcing subsets for these types of sets.

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\end{abstract}

\section{1. Introduction}

Beginning with C. Berge [4] in 1958, the study on domination in graphs was developed. There are now a lot of studies involving domination and its variations. One of its variations is the connected co-independent domination number of graphs that was studied in [7].

Years later, a new domination parameter called hop domination in graph was introduced in [12] by Natarajan and Ayyaswamy and was also studied in [3, 13–15]. A study in 2021 by S. Nanding and H. Rara [11] introduced a new concept of hop domination called the connected co-independent hop domination and generated some characterizations of connected co-independent hop domination in graphs.

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On the other hand, the concept of forcing numbers started from the study of molecular resonance structure which was introduced by Klein and Randic [10] in 1987. Harary et. al [16] first used the name “forcing number” and introduced the concept of the forcing of a perfect match in 1991. Chartrand et. al [5] initiated the investigation on the relation between forcing and domination concepts in 1997 and defined the term “forcing domination number”. In 2017, John et. al [9] investigated the forcing connected domination number of a graph, and C. Armada and S. Canoy Jr. [1] investigated the forcing independent domination number of a graph in 2019. Furthermore, in 2018, Canoy et. al [2] investigated the forcing domination number of graphs under some binary operations.

In this study, the forcing subsets of minimum connected co-independent hop dominating sets in graphs are defined and established and some characterizations of forcing subsets of minimum connected co-independent hop dominating sets of graphs resulting from the edge corona and lexicographic product of two graphs are generated. Also, the values or bounds of their corresponding forcing connected co-independent hop domination numbers are determined.

Connected co-independent hop domination in graphs can have real world applications. For an application, in [6], Desormeaux, Haynes, and Henning inspired their research on these concepts through social networking applications. They considered a factory with a large number of employees and needed to implement a quality assurance checking system of their workers. The factory manager decides to designate an internal committee to do this. In other words, the manager will select some workers to form a quality assurance team to inspect the work of their co-workers. The manager wants to keep this team as small as possible to minimize costs (extra costs for inspectors) and protect privacy (keep the inspectors’ identity confidential). To avoid bias, an inspector should neither be close friends nor enemies with any of the workers he/she is responsible for inspecting. To model this situation, a social network graph can be constructed in which each worker is represented by a vertex and an edge between two workers represents possible bias, that is, whether the two workers are close friends or enemies. Ideally, an inspector should not be adjacent to any worker who is being inspected.

In connected co-independent hop domination [11], every worker will be inspected by the nearest non-biased inspector. That is, an inspector who is a close friend (or an enemy) of a close friend (or enemy) of a worker. This is to save time and effort of locating a particular worker. Also, the inspectors should be acquainted with each other and all non-inspector workers are neither friends nor enemies, that is, they are not adjacent or there is no edge between them. The connected co-independent hop domination number will give the minimum number of inspectors needed.

In forcing subsets of connected co-independent hop domination, in each respective group of minimum number of inspectors that will inspect the workers in the designated areas of the factory, the members of that particular group of minimum number of inspectors will be assigned only to that distinct group of minimum number of inspectors, that is, it will strengthen the bond of the respective group of minimum number of non-biased inspectors with each other, since they are uniquely assigned to particular groups, and they will trust each other more doing their duties and will have a much easier time doing
their job regarding with the respective workers that they are assigned to inspect. The forcing connected co-independent hop domination number will determine the minimum number of members from the respective group of minimum number of inspectors that will be assigned only to that particular group of respective minimum number of inspectors.

In this study, we only consider graphs that are finite, simple, undirected and connected. Readers are referred to [8] for elementary Graph Theoretic concepts.

An independent set $S$ in a graph $G$ is a subset of the vertex-set of $G$ such that no two vertices in $S$ are adjacent in $G$. The cardinality of a maximum independent set is called the independence number of $G$ and is denoted by $\beta(G)$. An independent set $S \subseteq V(G)$ with $|S| = \beta(G)$ is called a $\beta$-set of $G$.

A set $S \subseteq V(G)$ is a co-independent set of $G$ if $V(G) \setminus S$ is independent. The minimum cardinality of a co-independent set in $G$, denoted by $\text{coi}(G)$ is called the co-independent number of $G$. A co-independent set $S$ with $|S| = \text{coi}(G)$ is called a coi-set of $G$.

A dominating set $D \subseteq V(G)$ is called a connected co-independent dominating set of $G$ if the subgraph $\langle D \rangle$ is connected and $V(G) \setminus D$ is an independent set. The cardinality of such a minimum set $D$ is called connected co-independent domination number of $G$ denoted by $\gamma_{c,\text{coi}}(G)$. A connected co-independent dominating set $D$ with $|D| = \gamma_{c,\text{coi}}(G)$ is called a $\gamma_{c,\text{coi}}$-set of $G$.

Let $G$ be a connected graph. A set $S \subseteq V(G)$ is a hop dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set of $G$, denoted by $\gamma_h(G)$, is called the hop domination number of $G$. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a $\gamma_h$-set.

A vertex $v$ in $G$ is a hop neighbor of vertex $u$ in $G$ if $d_G(u, v) = 2$. The set $N_G(u, 2) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the open hop neighborhood of $u$. The closed hop neighborhood of $u$ in $G$ is given by $N_G[u, 2] = N_G(u, 2) \cup \{u\}$. The open hop neighborhood of $X \subseteq V(G)$ is the set $N_G(X, 2) = \bigcup_{u \in X} N_G(u, 2)$. The closed hop neighborhood of $X$ in $G$ is the set $N_G[X, 2] = N_G(X, 2) \cup X$.

Let $G$ be a connected graph. A hop dominating set $S \subseteq V(G)$ is a connected co-independent hop dominating set of $G$ if $\langle S \rangle$ is connected and $V(G) \setminus S$ is an independent set. The minimum cardinality of a connected co-independent hop dominating set of $G$, denoted by $\gamma_{ch,\text{coi}}(G)$, is called the connected co-independent hop domination number of $G$. A connected co-independent hop dominating set $S$ with $|S| = \gamma_{ch,\text{coi}}(G)$ is called a $\gamma_{ch,\text{coi}}$-set of $G$.

Let $W$ be a $\gamma_{ch,\text{coi}}$-set of a graph $G$. A subset $S$ of $W$ is said to be a forcing subset for $W$ if $W$ is the unique $\gamma_{ch,\text{coi}}$-set containing $S$. The forcing connected co-independent hop domination number of $W$ is given by $f_{\gamma_{ch,\text{coi}}}(W) = \min\{|S| : S \text{ is a forcing subset for } W\}$. The forcing connected co-independent hop domination number of $G$ is given by

$$f_{\gamma_{ch,\text{coi}}}(G) = \min\{f_{\gamma_{ch,\text{coi}}}(W) : W \text{ is a } \gamma_{ch,\text{coi}}\text{-set of } G\}.$$
of $G$ is given by
\[ f_{\text{coi}}(G) = \min \{ f_{\text{coi}}(W) : W \text{ is a coi-set of } G \} . \]

Let $G$ be a connected graph and let $M$ be a $\beta$-set (maximum independent set) of $G$. A set $D \subseteq M^c$ is a forcing subset for the complement $M^c$ of $M$ if $M$ is the unique $\beta$-set such that $M^c$ contains $D$. The number $f^{c\beta}(M) = \min \{|D| : D \text{ is a forcing subset for } M^c \}$ is defined as the forcing complement of the independence number $M$. The forcing complement of the independence number of $G$ is given by
\[ f^{c\beta}(G) = \min \{ f^{c\beta}(M) : M \text{ is a } \beta\text{-set of } G \} . \]

The edge corona $G \odot H$ of $G$ and $H$ is the graph obtained by taking one copy of $G$ and $|E(G)|$ copies of $H$ and joining each of the end vertices $u$ and $v$ of each edge $uv$ of $G$ to every vertex of the copy $H^{uv}$ of $H$.

The lexicographic product of two graphs $G$ and $H$, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ such that $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

2. Known Results

The following known results are taken from [11].

Remark 1. Let $G$ be a connected graph of order $n$. Then $1 \leq \gamma_{\text{ch,coi}}(G) \leq n$. Moreover, $\gamma_{\text{ch,coi}}(G) = 1$ if and only if $G = K_1$.

Example 1. The equations below give the connected co-independent hop domination number of the path $P_n$ and cycle $C_n$.

\[
\gamma_{\text{ch,coi}}(P_n) = \begin{cases} 
1 & \text{if } n = 1, \\
2 & \text{if } n = 2, 3, \\
n - 2 & \text{if } n \geq 4.
\end{cases}
\]

\[
\gamma_{\text{ch,coi}}(C_n) = \begin{cases} 
3 & \text{if } n = 3, \\
n - 1 & \text{if } n \geq 4.
\end{cases}
\]

Remark 2. If $G$ is a complete graph, then $\gamma_{\text{ch,coi}}(G) = n$.

Theorem 1. Let $G$ and $H$ be nontrivial connected graphs with $|V(G)| = n$. A subset $C = \bigcup_{x \in S} \{ \{x\} \times T_x \}$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ of $V(G[H])$ is a connected co-independent hop dominating set if and only if
(i) $S = V(G)$.
(ii) For every $x \in V(G)$ such that $T_x \neq V(H)$, $V(H) \setminus T_x$ is an independent set and $T_y = V(H)$ for every $y \in N_G(x)$ where $T_x$ is a hop dominating set of $H$ if $\deg_G(x) = n - 1.$
Corollary 1. Let $G$ be any connected noncomplete graph of order $m$ and $H$ be any nontrivial connected graph of order $n$. Then

$$
\gamma_{ch,\text{coi}}(G[H]) = m(n - \beta(H)) + r(G)\beta(H),
$$

where $r(G) = \min\{|D| : V(G) \setminus D \text{ is an independent set}\}$ and $\beta(H)$ is the independence number of $H$.

3. Forcing Connected Co-Independent Hop Domination Number of Some Special Graphs

Remark 3. Let $G$ be a connected graph. Then

(i) $f_{\gamma_{ch,\text{coi}}}(G) = 0$ if and only if $G$ has a unique $\gamma_{ch,\text{coi}}$-set, and

(ii) $f_{\gamma_{ch,\text{coi}}}(G) = 1$ if and only if $G$ has at least two $\gamma_{ch,\text{coi}}$-sets, one of which, say $B$, contains an element which is not found in any $\gamma_{ch,\text{coi}}$-set of $G$.

Theorem 2. Let $G$ be a connected graph. Then $f_{\gamma_{ch,\text{coi}}}(G) = \gamma_{ch,\text{coi}}(G)$ if and only if for all $\gamma_{ch,\text{coi}}$-set $B$ of $G$ and for each $v \in B$, there exists $u_v \in V(G) \setminus B$ such that $[B \setminus \{v\}] \cup \{u_v\}$ is a $\gamma_{ch,\text{coi}}$-set of $G$.

Proof: Suppose that $f_{\gamma_{ch,\text{coi}}}(G) = \gamma_{ch,\text{coi}}(G)$. Let $B$ be a $\gamma_{ch,\text{coi}}$-set of $G$ such that $f_{\gamma_{ch,\text{coi}}}(G) = |B| = \gamma_{ch,\text{coi}}(G)$, that is, $B$ is the only forcing subset for itself. Let $v \in B$. Since $B \setminus \{v\}$ is not a forcing subset for $B$, there exists a $u_v \in V(G) \setminus B$ such that $[B \setminus \{v\}] \cup \{u_v\}$ is a $\gamma_{ch,\text{coi}}$-set of $G$.

Conversely, suppose that every $\gamma_{ch,\text{coi}}$-set $B'$ of $G$ satisfies the given condition. Let $B$ be a $\gamma_{ch,\text{coi}}$-set of $G$ such that $f_{\gamma_{ch,\text{coi}}}(G) = \gamma_{ch,\text{coi}}(G)$. Suppose further that $B$ has a forcing subset $Q$ with $|Q| < |B|$, that is, $B = Q \cup P$ where $P = \{z \in B : z \notin Q\}$. Pick $z \in P$. By assumption, there exists $u_z \in V(G) \setminus B$ such that $[B \setminus \{z\}] \cup \{u_z\} = T$ is a $\gamma_{ch,\text{coi}}$-set of $G$. Thus, $T = Q \cup R$, where $R = [P \setminus \{z\}] \cup \{u_z\}$, is a $\gamma_{ch,\text{coi}}$-set containing $Q$, a contradiction. Hence, $B$ is the only forcing subset for $B$. Therefore, $f_{\gamma_{ch,\text{coi}}}(G) = \gamma_{ch,\text{coi}}(G)$. □

Proposition 1. For any complete graph $K_n$ with $n \geq 1$ vertices, $f_{\gamma_{ch,\text{coi}}}(K_n) = 0$.

Proof: By definition of $K_n$, $V(K_n)$ is the only $\gamma_{ch,\text{coi}}$-set of $K_n$. By Remark 3(i), $f_{\gamma_{ch,\text{coi}}}(K_n) = 0$. □

Proposition 2. For any path $P_n$ with $n \geq 1$ vertices,

$$
f_{\gamma_{ch,\text{coi}}}(P_n) = \begin{cases} 0, & \text{if } n \neq 3, \\ 1, & \text{if } n = 3. \end{cases}
$$

Proof: Suppose that $P_n = [v_1, v_2, \ldots, v_n]$. It can be seen that $f_{\gamma_{ch,\text{coi}}}(P_1) = f_{\gamma_{ch,\text{coi}}}(P_2) = 0$. Moreover, if $n = 4$, then $P_4$ has $\gamma_{ch,\text{coi}}$-set $B_1 = \{v_2, v_3\}$ which is the only $\gamma_{ch,\text{coi}}$-set of $P_4$. By Remark 3(i), $f_{\gamma_{ch,\text{coi}}}(P_4) = 0$. Suppose that $n > 4$, then clearly $B_2 = \{v_2, v_3, v_4, \ldots, v_{n-1}\}$ is the only $\gamma_{ch,\text{coi}}$-set of $P_n$. Thus, by Remark 3(i),
Theorem 3. Let $f_{\text{ch,coi}}(B_2) = 0 = f_{\text{ch,coi}}(P_n)$.

Suppose that $n = 3$. Then $P_3$ has $\gamma_{\text{ch,coi}}$-sets $B_3 = \{v_1, v_2\}$ and $B_4 = \{v_2, v_3\}$ which are the only $\gamma_{\text{ch,coi}}$-sets of $P_3$ with $v_1 \in B_3$ and $v_1 \notin B_4$. Hence, by Remark 3(ii), $f_{\text{ch,coi}}(B_3) = 1 = f_{\text{ch,coi}}(P_3)$. 

\[ \text{Proposition 3.} \] For any cycle $C_n$ with $n \geq 3$ vertices,

\[ f_{\text{ch,coi}}(C_n) = \begin{cases} 0, & \text{if } n = 3, \\ n - 1, & \text{if } n \geq 4. \end{cases} \]

\textbf{Proof:} Suppose that $C_n = [v_1, v_2, \ldots, v_n, v_1]$. Since $C_3 = K_3$, by Proposition 1, $f_{\text{ch,coi}}(C_3) = 0$. Suppose that $n \geq 4$. Then the $\gamma_{\text{ch,coi}}$-sets of $C_n$ are $B_1 = \{v_1, v_2, \ldots, v_{n-1}\}$, $B_2 = \{v_2, v_3, \ldots, v_n\}$, $B_3 = \{v_3, v_4, \ldots, v_n, v_1\}$, ..., $B_n = \{v_n, v_1, v_2, \ldots, v_{n-2}\}$. Clearly, for each $v_i \in B_j$ where $i, j \in \{1, 2, 3, \ldots, n\}$, there exists $v_k \in V(C_n) \setminus B_j$ such that $[B_j \setminus \{v_i\}] \cup \{v_k\}$ is a $\gamma_{\text{ch,coi}}$-set of $G$. Hence, by Theorem 2, $f_{\text{ch,coi}}(C_n) = n - 1$. 

4. Forcing Connected Co-Independent Hop Domination in the Edge Corona of Graphs

The following two results are taken from the Masteral’s Thesis of Sandra A. Nanding. To verify the result, the proof is provided.

\textbf{Theorem 3.} Let $G$ be a connected graph of order $n \geq 3$ and $H$ be any graph. Then $C \subseteq V(G \circ H)$ is a connected co-independent hop dominating set of $G \circ H$ if and only if $C = A \cup (\bigcup_{uv \in E(G)} S_{uv})$ where

(i) $A \subseteq V(G)$ is a connected co-independent set of $G$ containing all vertices incident to all the edges of $G$.

(ii) $S_{uv} = V(H^{uw})$ if $uv \in E(G)$ such that $u \in V(G) \setminus A$ or $v \in V(G) \setminus A$.

(iii) For every $a, b \in A$ such that $ab \in E(G)$ and $S_{ab} \neq V(H^{ab})$, $V(H^{ab}) \setminus S_{ab}$ is an independent set in $H^{ab}$.

\textbf{Proof:} Suppose that $C$ is a connected co-independent hop dominating set of $G \circ H$. Let $A = C \cap V(G)$ and let $S_{uv} = C \cap V(H^{uw})$ for each $uv \in E(G)$. Then $C = A \cup (\bigcup_{uv \in V(G)} S_{uv})$ where $A \subseteq V(G)$. First, we show that $\langle A \rangle$ is connected. Let $x, y \in A$ with $x \neq y$. If $xy \in E(G)$, then we are done. Suppose that $xy \notin E(G)$. Since $\langle C \rangle$ is connected and $x, y \in C$, there exists an $x$-$y$ path $[x_1, x_2, \ldots, x_n]$ in $\langle C \rangle$ where $x = x_1$, $y = x_n$ and $n > 2$. If $x_i \in A$ for all $i \in \{1, 2, \ldots, n\}$, then the path $[x_1, x_2, \ldots, x_n]$ in $A$. Suppose there exists $x_i \notin A$. Then $x_i \in S_{uv}$ for some edge $uv \in E(G)$. By definition of $G \circ H$, $u, v \in A$. Hence, $[x_1, \ldots, u, v, \ldots, x_n]$ is a path in $A$, showing that $\langle A \rangle$ is connected. Next, let $u, v \in V(G) \setminus A$ with $u \neq v$. Then $u, v \in V(G \circ H) \setminus C$. Since $V(G \circ H) \setminus C$ is independent, $uv \notin E(G \circ H)$.

Since $u, v \in V(G)$, $uv \notin E(G)$ implying that $V(G) \setminus A$ is independent. Now, suppose $v$ is a vertex incident to all the edges of $G$ and $v \notin A$. Then $v \in N_G(w) \cap N_{G \circ H}(p)$ for all $w \in V(G)$ and for all $p \in V(H^{uw})$. Thus, $N_{G \circ H}(v, 2) \cap C = \emptyset$, a contradiction since $C$
is a hop dominating set. Hence, $A$ is a connected co-independent set of $G$ containing all vertices incident to all edges of $G$, showing that (i) holds.

Let $uv \in E(G)$ with $u \notin A$. Suppose $S_{uv} \neq V(H_{uv})$. Then there exists $x \in V(H_{uv}) \setminus S_{uv}$. Hence, $x, u \in V(G \circ H) \setminus C$ and $xu \in E(G \circ H)$, a contradiction to the independence of $V(G \circ H) \setminus C$. Thus, $S_{uv} = V(H_{uv})$ and (ii) holds.

Lastly, let $a, b \in A$ such that $ab \in E(G)$ and $S_{ab} \neq V(H_{ab})$. Since $V(G \circ H) \setminus C$ is independent and $(V(H_{ab}) \setminus S_{ab}) \subseteq V(G \circ H) \setminus C$, $V(H_{ab}) \setminus S_{ab}$ is an independent set in $H_{ab}$. Hence, (iii) holds.

For the converse, suppose $C = A \cup \left( \bigcup_{uv \in E(G)} S_{uv} \right)$ where (i), (ii) and (iii) hold. First, we show that $C$ is connected. Let $u, v \in C$ with $u \neq v$. If $uv \in E(G \circ H)$, then we are done. So, suppose that $uv \notin E(G \circ H)$. Consider the following cases.

**Case 1.** $u, v \in A$

By (i), $(A)$ is connected. Hence, there exists a $u$-$v$ path $P[u, v]$ in $A$. Since $A \subseteq C$, the path $P[u, v]$ is in $C$.

**Case 2.** $u \in A$ and $v \in S_{xy}$ for some $xy \in E(G)$

Since $uv \notin E(G \circ H)$, $u \neq x$ and $u \neq y$. Since $V(G) \setminus A$ is independent by (i), $x \in A$ or $y \in A$, say $x \in A$. If $ux \in E(G)$, then the path $[u, x, v]$ is a $u$-$v$ path in $C$. Suppose $ux \notin E(G)$. Since $(A)$ is connected by (i) and $u, x \in A$, there exists $u$-$x$ path $[y_1, y_2, ..., y_k]$ in $A$ where $u = y_1$, $x = y_k$ and $k > 2$. Hence, the path $[y_1, y_2, ..., y_k, v]$ is a $u$-$v$ path in $C$.

**Case 3.** $u, v \in S_{pq}$ for some edge $pq \in E(G)$

Since $V(G) \setminus A$ is independent by (i), $p \in A$ or $q \in A$. Hence, the path $[u, p, v]$ or $[u, q, v]$ is in $C$.

In any case, $(C)$ is connected.

Next, we show that $V(G \circ H) \setminus C$ is independent. Let $p, q \in V(G \circ H) \setminus C$ with $p \neq q$. Consider the following cases.

**Case 1.** $p \in V(G) \setminus A$ and $q \in V(G) \setminus A$

Since $V(G) \setminus A$ is independent by (i), $pq \notin E(G)$. Thus, $pq \notin E(G \circ H)$.

**Case 2.** $p \in V(G) \setminus A$, $q \in V(H_{xy}) \setminus S_{xy}$ for some $xy \in E(G)$

Since $S_{xy} \neq V(H_{xy})$, $x, y \in A$ by (ii). Hence, $p \neq x$ and $p \neq y$. By definition of $G \circ H$, $pq \notin E(G \circ H)$.

**Case 3.** $p \in V(H_{xy}) \setminus S_{xy}$ and $q \in V(H_{rs}) \setminus S_{rs}$ for some distinct edges $xy, rs \in E(G)$

Then, by definition of $G \circ H$, $pq \notin E(G \circ H)$.

**Case 4.** $p, q \in V(H_{zt}) \setminus S_{zt}$ for some edge $zt \in E(G)$

Since $V(H_{zt}) \setminus S_{zt}$ is independent by (iii), $pq \notin E(G \circ H)$.

Therefore, in any case, $V(G \circ H) \setminus C$ is an independent set in $G \circ H$.

Lastly, we show that $C$ is a hop dominating set of $G \circ H$. Let $u \in V(G \circ H) \setminus C$.

Consider the following cases.

**Case 1.** $u \in V(G) \setminus A$

Let $deg_G(u) = 1$. Since $|V(G)| \geq 3$, there exists $vw \in E(G)$ with $u \in N_G(v) \setminus N_G(w)$ or $u \in N_G(w) \setminus N_G(v)$. If $w \in A$, then $w \in N_G(u, 2) \cap A$. If $w \notin A$, then $S_{uv} = V(H_{uv})$ by (ii). Thus, a vertex $p \in N_{G \circ H}(u, 2) \cap S_{uv}$ exists. Hence, $p \in N_{G \circ H}(u, 2) \cap C$. 
Therefore, $\gamma_{f\text{coi}}$ is a by Theorem 3. Hence, $w \in \text{coi} = z$. If $S_{yz} \neq \emptyset$, then $S_{yz} = V(H^y)$. Hence, a vertex $w \in N_{G \cup H}(u, 2) \cap S_{yz}$ or $w \in N_{G \cup H}(u, 2) \cap S_{zx}$.

Therefore, in any case, $C$ is a hop dominating set of $G \circ H$.

Accordingly, $C$ is a connected co-independent hop dominating set of $G \circ H$. □

Corollary 2. Let $G$ be a connected graph of order $n \geq 3$ of size $p$ and $H$ be any graph of order $m$. Then $\gamma_{ch,\text{coi}}(G \circ H) = n + p(m - \beta(H))$.

Proof: Let $C_o = A \cup (\bigcup_{uv \in V(G)} S_{uv})$ be a $\gamma_{ch,\text{coi}}$-set of $G \circ H$. Then conditions $(i)$, $(ii)$ and $(iii)$ of Theorem 3 hold where $A = V(G)$ and $S_{uv} = V(H^u) \setminus S^*$ where $S^*$ is any independent set of $H^u$. Thus,

$$\gamma_{ch,\text{coi}}(G \circ H) = |A| + p|S_{uv}|$$

$$= n + p(|V(H^u)| - |S^*|)$$

$$\geq n + p(m - \beta(H)).$$

Let $T$ be a $\beta$-set of $H$ and $S_{uv} = V(H^u) \setminus T$ for each $uv \in E(G)$. Then $C = V(G) \cup (\bigcup_{uv \in E(G)} S_{uv})$ is a connected co-independent hop dominating set of $G \circ H$ by Theorem 3. Hence,

$$\gamma_{ch,\text{coi}}(G \circ H) \leq |C|$$

$$= |V(G)| + p|S_{uv}|$$

$$= n + p|V(H^u) \setminus T|$$

$$= n + p(m - \beta(H)).$$

Therefore, $\gamma_{ch,\text{coi}}(G \circ H) = n + p(m - \beta(H))$. □

Remark 3 and Theorem 2 hold if the $\gamma_{ch,\text{coi}}$-set is replace by $\text{coi}$-set. Thus, we have the following Remark and Theorem.

Remark 4. Let $G$ be a connected graph. Then

(i) $f_{\text{coi}}(G) = 0$ if and only if $G$ has a unique $\text{coi}$-set, and

(ii) $f_{\text{coi}}(G) = 1$ if and only if $G$ has at least two $\text{coi}$-sets, one of which, say $D$, contains an element which is not found in any $\text{coi}$-set of $G$.

Theorem 4. Let $G$ be a connected graph. Then $f_{\text{coi}}(G) = \text{coi}(G)$ if and only if for all $\text{coi}$-set $D$ of $G$ and for each $z \in D$, there exists $u_z \in V(G) \setminus D$ such that $[D \setminus \{z\}] \cup \{u_z\}$ is a $\text{coi}$-set of $G$.

Proof: Suppose that $f_{\text{coi}}(G) = \text{coi}(G)$. Let $D$ be a $\text{coi}$-set of $G$ such that $f_{\text{coi}}(G) = |D| = \text{coi}(G)$, that is, $D$ is the only forcing subset for itself. Let $z \in D$. 

Since $D \setminus \{z\}$ is not a forcing subset for $D$, there exists a $u_z \in V(G) \setminus D$ such that $[D \setminus \{z\}] \cup \{u_z\}$ is a coi-set of $G$.

Conversely, suppose that every coi-set $D'$ of $G$ satisfies the given condition. Let $D$ be a coi-set of $G$ such that $f_{coi}(G) = f_{coi}(D)$. Suppose further that $D$ has a forcing subset $A$ with $|A| < |D|$, that is, $D = A \cup C$ where $C = \{v \in D : v \notin A\}$. Pick $v \in C$. By assumption, there exists $u_v \in V(G) \setminus D$ such that $[D \setminus \{v\}] \cup \{u_v\} = R$ is a coi-set of $G$. Thus, $R = A \cup T$, where $T = [C \setminus \{v\}] \cup \{u_v\}$, is a coi-set containing $A$, a contradiction. Hence, $D$ is the only forcing subset for $D$. Therefore, $f_{coi}(G) = coi(G)$. \hfill \qed

Example 2. The formulas below give the co-independent number of the complete graph $K_n$, path $P_n$ and cycle $C_n$.

\[
coi(K_n) = \begin{cases} 
1, & \text{if } n = 1, \\
n - 1, & \text{if } n \geq 2.
\end{cases}
\]

\[
coi(P_n) = \begin{cases} 
1, & \text{if } n = 1, \\
n/2, & \text{if } n \text{ is even}, \\
(n - 1)/2, & \text{if } n \text{ is odd}.
\end{cases}
\]

\[
coi(C_n) = \begin{cases} 
n/2, & \text{if } n \geq 4 \text{ and } n \text{ is even}, \\
n + 1/2, & \text{if } n \geq 3 \text{ and } n \text{ is odd}.
\end{cases}
\]

Proposition 4. For any complete graph $K_n$ with $n \geq 1$ vertices,

\[
f_{coi}(K_n) = \begin{cases} 
0, & \text{if } n = 1, \\
n - 1, & \text{if } n \geq 2.
\end{cases}
\]

Proof: Suppose that $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. It can be seen that $f_{coi}(K_1) = 0$. If $n = 2$, then $K_2$ has only two coi-sets $R_1 = \{v_1\}$ and $R_2 = \{v_2\}$ with $v_1 \in R_1$ and $v_1 \notin R_2$. By Remark 4(ii), $f_{coi}(K_2) = n - 1 = 1$.

Suppose that $n > 2$. Then the coi-sets of $K_n$ are $B_1 = \{v_1, v_2, \ldots, v_{n-1}\}$, $B_2 = \{v_2, v_3, \ldots, v_n\}$, $B_3 = \{v_3, v_4, \ldots, v_n, v_1\}$, $\ldots$, $B_n = \{v_n, v_1, v_2, \ldots, v_{n-2}\}$. Clearly, for each $v_i \in B_j$ where $i, j \in \{1, 2, 3, \ldots, n\}$, there exists $v_k \in V(K_n) \setminus B_j$ such that $[B_j \setminus \{v_i\}] \cup \{v_k\}$ is a coi-set of $G$. Hence, by Theorem 4, $f_{coi}(K_n) = n - 1$. \hfill \qed

Proposition 5. For any path $P_n$ with $n \geq 1$ vertices,

\[
f_{coi}(P_n) = \begin{cases} 
0, & \text{if } n = 1, 3 \text{ and } n \geq 5 \text{ is odd}, \\
1, & \text{if } n = 2, 4 \text{ and } n \geq 6 \text{ is even}.
\end{cases}
\]
Proof: Suppose that $P_n = [v_1, v_2, \ldots, v_n]$. It can be verified that $f_{coi}(P_3) = f_{coi}(P_4) = 0$ and $f_{coi}(P_2) = 1$. If $n = 4$, then $P_4$ has coi-sets $B_1 = \{v_1, v_3\}$, $B_2 = \{v_2, v_4\}$ and $B_3 = \{v_2, v_3\}$ which are the only coi-sets of $P_4$ with $v_4 \notin B_1, B_3$. Thus, by Remark 4(ii), $f_{coi}(P_4) = 1$.

Now, suppose that $n \geq 5$ and $n$ is odd, then clearly $B = \{v_2, v_4, v_6, \ldots, v_{n-3}, v_{n-1}\}$ is the only coi-set of $P_n$. Thus, by Remark 4(i), $f_{coi}(B) = 0 = f_{coi}(P_n)$. Next, suppose that $n \geq 6$ and $n$ is even. Then $P_n$ has coi-sets $S_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ and $S_2 = \{v_2, v_4, v_6, \ldots, v_n\}$ which are the only coi-set of $P_n$ with $v_3 \in S_1$ and $v_3 \notin S_2$. Hence, by Remark 4(ii), $f_{coi}(P_n) = 1$.

**Proposition 6.** For any cycle $C_n$ with $n \geq 3$ vertices,

$$f_{coi}(C_n) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Suppose that $C_n = [v_1, v_2, \ldots, v_n, v_1]$. If $n = 3$, then the coi-sets of $C_3$ are $Q_1 = \{v_1, v_2\}$, $Q_2 = \{v_2, v_3\}$ and $Q_3 = \{v_1, v_3\}$. Clearly, for each $v_i \in Q_j$ where $i, j \in \{1, 2, 3\}$, there exists $v_k \in V(C_3) \setminus Q_j$ where $k \in \{1, 2, 3\}$ and $i \neq k$ such that $[Q_j \setminus \{v_i\}] \cup \{v_k\}$ is a coi-set of $G$. Thus, by Theorem 4, $f_{coi}(C_3) = 2$. Now, suppose that $n$ is even. Then $B_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ and $B_2 = \{v_2, v_4, v_6, \ldots, v_n\}$ are the only coi-sets of $C_n$ with $v_3 \in B_1$ and $v_3 \notin B_2$. Thus, by Remark 4(ii), $f_{coi}(B_1) = 1 = f_{coi}(C_n)$. Next, suppose that $n > 3$ and $n$ is odd. Then

$$R_1 = \{v_1, v_3, v_5, \ldots, v_{n-2}, v_n\},$$
$$R_2 = \{v_1, v_3, v_5, \ldots, v_{n-2}, v_{n-1}\},$$
$$R_3 = \{v_2, v_4, v_6, \ldots, v_{n-1}, v_1\} \text{ and},$$
$$R_4 = \{v_2, v_4, v_6, \ldots, v_{n-1}, v_1\}$$

are coi-sets of $C_n$. Hence, no vertex of $C_n$ is contained in a unique coi-set. Thus, $f_{coi}(C_n) \geq 2$. Clearly, $\{v_1, v_n\}$ is uniquely contained in $R_1$. Therefore, $f_{coi}(R_1) = 2 = f_{coi}(C_n)$.  

The next result is a restatement of Corollary 2.

**Corollary 3.** Let $G$ be a connected graph of order $n \geq 3$ of size $p$ and $H$ be any graph. Then $C \subseteq V(G \circ H)$ is a $\gamma_{ch,coi}$-set of $G \circ H$ if and only if $C = V(G) \cup \left( \bigcup_{uv \in E(G)} S_{uv} \right)$ where $S_{uv}$ is a co-independent set of $H^{uv}$ for each $u, v \in V(G)$ such that $uv \in E(G)$, and $S_{uv} \neq V(H^{uv})$. In particular, $\gamma_{ch,coi}(G \circ H) = |V(G)| + p(coi(H))$.

**Theorem 5.** Let $G$ be a nontrivial connected graph of order $n \geq 3$ of size $p$ and $H$ be any graph. Then

$$f_{\gamma_{ch,coi}}(G \circ H) = \begin{cases} 0, & \text{if } H \text{ has a unique coi-set,} \\ p[f_{coi}(H)], & \text{if } H \text{ has no unique coi-set.} \end{cases}$$
Proof: Suppose \( H \) has a unique \( \text{coi} \)-set. For each \( uv \in E(G) \), where \( u, v \in V(G) \), let \( R_{uv} \subseteq V(H_{uv}) \) be the unique \( \text{coi} \)-set of \( H_{uv} \). By Corollary 3, \( C = V(G) \cup \left( \bigcup_{uv \in E(G)} R_{uv} \right) \) is the unique \( \gamma_{\text{ch,coi}} \)-set of \( G \odot H \). Thus, by Remark 3(i), \( f_{\gamma_{\text{ch,coi}}}(G \odot H) = 0 \). On the other hand, suppose that \( H \) does not have a unique \( \text{coi} \)-set. For every \( uv \in E(G) \), where \( u, v \in V(G) \), let \( W_{uv} \subseteq V(H_{uv}) \) be a \( \text{coi} \)-set of \( H_{uv} \) with \( f_{\text{coi}}(H_{uv}) = f_{\text{coi}}(W_{uv}) \), and let \( R_{W_{uv}} \subseteq W_{uv} \) be a forcing subset for \( W_{uv} \) with \( f_{\text{coi}}(W_{uv}) = |R_{W_{uv}}| \). Then by Corollary 3, \( S_W = V(G) \cup \left( \bigcup_{uv \in E(G)} W_{uv} \right) \) is a \( \gamma_{\text{ch,coi}} \)-set of \( G \odot H \). Let \( D = \bigcup_{uv \in E(G)} R_{W_{uv}} \). Then \( D \) is a forcing subset for \( S_W \). Thus,

\[
f_{\gamma_{\text{ch,coi}}}(G \odot H) \leq f_{\gamma_{\text{ch,coi}}}(S_W) \leq |D| = p\left[f_{\text{coi}}(H)\right].
\]

Next, let \( C' \) be a \( \gamma_{\text{ch,coi}} \)-set of \( G \odot H \) such that \( f_{\gamma_{\text{ch,coi}}}(G \odot H) = f_{\gamma_{\text{ch,coi}}}(C') \). Then by Corollary 3, let \( C' = V(G) \cup \left( \bigcup_{uv \in E(G)} Q_{uv} \right) \) where \( Q_{uv} \) is a \( \text{coi} \)-set of \( H_{uv} \) for each \( uv \in E(G) \) for which \( u, v \in V(G) \). Now, we let \( D' \) be a forcing subset for \( C' \) such that \( f_{\gamma_{\text{ch,coi}}}(C') = |D'| \). Suppose that there exists \( ab \in E(G) \), for each \( a, b \in V(G) \) such that \( D' \cap Q_{ab} = D_{ab} \) is not a forcing subset for \( Q_{ab} \). Let \( Q_{ab}' \) be a \( \text{coi} \)-set of \( H_{ab} \) with \( Q_{ab}' \neq Q_{ab} \). Then

\[
C'' = V(G) \cup \left( \bigcup_{uv \in E(G) \setminus \{ab\}} Q_{uv} \right) \cup Q_{ab}'
\]

is a \( \gamma_{\text{ch,coi}} \)-set of \( G \odot H \) with \( C' \neq C'' \) and \( D' \subseteq C'' \), a contradiction. Thus, \( C_{uv} = D' \cap Q_{uv} \) is a forcing subset for \( Q_{uv} \) for each \( uv \in E(G) \) where \( u, v \in V(G) \). Let \( C_0 = \bigcup_{uv \in E(G)} C_{uv} \). Then

\[
f_{\gamma_{\text{ch,coi}}}(G \odot H) = |D'| \geq |C_0| = \sum_{uv \in E(G)} |C_{uv}| \geq \sum_{uv \in E(G)} f_{\text{coi}}(H_{uv}) = |E(G)|f_{\text{coi}}(H).
\]

Therefore, \( f_{\gamma_{\text{ch,coi}}}(G \odot H) = p\left[f_{\text{coi}}(H)\right] \).

\( \square \)

**Example 3.** For complete graph \( K_3 \) and path \( P_3 \), \( f_{\gamma_{\text{ch,coi}}}(K_3 \odot P_3) = 0 \) since \( P_3 \) has a unique \( \text{coi} \)-set.

**Example 4.** For cycle \( C_3 \) and path \( P_2 \), \( f_{\gamma_{\text{ch,coi}}}(C_3 \odot P_2) = 3 f_{\text{coi}}(P_2) = 3 \cdot 1 = 3 \) since \( P_2 \) has no unique \( \text{coi} \)-sets.

5. **Forcing Connected Co-Independent Hop Domination in the Lexicographic Product of Graphs**

The following is a restatement of Corollary 1.

**Theorem 6.** Let \( G \) and \( H \) be any nontrivial connected graphs of orders \( m > 2 \) and \( n > 2 \), respectively with \( \gamma(H) \neq 1 \). Then \( \gamma_{\text{ch,coi}}(G[H]) = mn - \beta(G)\beta(H) \).
Proof: Suppose that $M$ is a $\beta$-set of $G$ and $N$ is a $\beta$-set of $H$. Set $T_x = V(H) \setminus N$ if $x \in M$ and else set $T_x = V(H)$. Then in view of Theorem 1,

$$S = \bigcup_{x \in V(G)} \{x\} \times T_x$$

is a connected co-independent hop dominating set of $G[H]$. Hence,

$$\gamma_{ch,coi}(G[H]) \leq |S|$$

$$= \left| \bigcup_{x \in M} \{x\} \times (V(H) \setminus N) \right| + \left| \bigcup_{x \in V(G) \setminus M} \{x\} \times V(H) \right|$$

$$= |M|(|V(H)| - |N|) + (|V(G)| - |M|)|V(H)|$$

$$= \beta(G)(n - \beta(H)) + (m - \beta(G))n$$

$$= (\beta(G))n - \beta(G)\beta(H) + mn - (\beta(G))n$$

$$= mn - \beta(G)\beta(H).$$

Conversely, suppose that $S_0 = \bigcup_{x \in V(G)} \{x\} \times R_x$ is a $\gamma_{ch,coi}$-set of $G[H]$. Let $D = \{x \in V(G) : R_x = V(H)\}$. We claim that $V(G) \setminus D$ is an independent set of $G$. Suppose on the contrary. Then there exists vertices $x, y \in V(G) \setminus D$ such that $xy \in E(G)$ and pick $a, b \in V(H) \setminus R_x$ for which $ab \in E(H)$. This implies that $(x, a)(x, b), (y, a)(y, b) \in E(G[H])$, a contradiction to the fact that $S_0$ is a $\gamma_{ch,coi}$-set of $G[H]$. Thus, $V(G) \setminus D$ is an independent set of $G$. Hence, $|V(G) \setminus D| \leq \beta(G)$ and so

$$|D| \geq m - \beta(G).$$

On the other hand, by condition (ii) in Theorem 1, we have $V(H) \setminus R_x$ is an independent set of $H$. Thus, $|V(H) \setminus R_x| \leq \beta(H)$ and so

$$|R_x| \geq n - \beta(H).$$

Therefore, inequalities (1) and (2) imply

$$\gamma_{ch,coi}(G[H]) = |S_0|$$

$$= \left| \bigcup_{x \in D} \{x\} \times V(H) \right| + \left| \bigcup_{x \in V(G) \setminus D} \{x\} \times R_x \right|$$

$$= |D||V(H)| + (|V(G)| - |D|)|R_x|$$

$$\geq |D|n + (m - |D|)(n - \beta(H))$$

$$= |D|n + mn - m\beta(H) - |D|n + |D|\beta(H)$$

$$\geq mn - m\beta(H) + (m - \beta(G))\beta(H)$$

$$= mn - m\beta(H) + m\beta(H) - \beta(G)\beta(H).$$
Therefore, $\gamma_{ch,coi}(G[H]) = mn - \beta(G)\beta(H)$.

As a consequence of Theorem 6, the next result follows.

**Corollary 4.** Let $G$ and $H$ be nontrivial connected graphs with orders $m > 2$ and $n > 2$, respectively, and $\gamma(H) \neq 1$. Then $S \subseteq V(G[H])$ is a $\gamma_{ch,coi}$-set of $G[H]$ if and only if

$$S = \bigcup_{x \in A} (\{x\} \times T_x) \cup [(V(G) \setminus A) \times V(H)]$$

for some $\beta$-set $A$ of $G$ such that $V(H) \setminus T_x$ is a $\beta$-set of $H$ for all $x \in A$.

**Theorem 7.** Let $G$ and $H$ be connected graphs of orders $m > 2$ and $n > 2$, respectively, where $H$ has a unique $\beta$-set and $\gamma(H) \neq 1$. Then

$$f_{\gamma_{ch,coi}}(G[H]) = \begin{cases} 0, & \text{if } G \text{ has a unique } \beta\text{-set,} \\ f\beta^c(G), & \text{if } G \text{ has no unique } \beta\text{-set.} \end{cases}$$

**Proof:** By Corollary 4, $S = [A \times B] \cup [A^c \times V(H)]$ is a $\gamma_{ch,coi}$-set of $G[H]$ for some $\beta$-set $A$ of $G$ such that $V(H) \setminus B$ is a $\beta$-set of $H$. Suppose that $H$ has unique $\beta$-set, say $R$. If $G$ has a unique $\beta$-set $M$, then $G[H]$ has a unique $\gamma_{ch,coi}$-set

$$[M \times R] \cup [M^c \times V(H)].$$

By Remark 3(i), $f_{\gamma_{ch,coi}}(G[H]) = 0$. Now, suppose that $G$ has no unique $\beta$-set. Let $A$ be a $\beta$-set of $G$ and let $D_{A^c}$ be a forcing subset for the complement $A^c$ of $A$ such that $f\beta^c(G) = f\beta^c(A) = |D_{A^c}|$. By Corollary 4,

$$S = [A \times R] \cup [A^c \times V(H)]$$

is a $\gamma_{ch,coi}$-set of $G[H]$. We claim that $D_{A^c} \times \{a\}$ is a forcing subset for $S$ for every $a \in V(H) \setminus R$. Suppose on the contrary. Then there exists a $\gamma_{ch,coi}$-set $S'$ of $G[H]$ with $S' \neq S$ such that $D_{A^c} \times \{a\} \subseteq S'$. Then $S' = [A' \times R] \cup [(A')^c \times V(H)]$ for some $\beta$-set $A'$ of $G$. Since $S' \neq S$, $A'$ is not $A$ and $D_{A^c} \times \{a\} \subseteq (A')^c \times V(H)$. Thus, $D_{A^c} \subseteq (A')^c$ a contradiction since $D_{A^c}$ is a forcing subset for $A^c$. Hence, $D_{A^c} \times \{a\}$ is a forcing subset for $S$. Thus,

$$f_{\gamma_{ch,coi}}(G[H]) \leq f_{\gamma_{ch,coi}}(S) \leq |D_{A^c} \times \{a\}| = |D_{A^c}| = f\beta^c(G).$$

Let $S_0 = [A_0 \times R] \cup [A_0^c \times V(H)]$ be a $\gamma_{ch,coi}$-set of $G[H]$ such that $f_{\gamma_{ch,coi}}(G[H]) = f_{\gamma_{ch,coi}}(S_0)$. By Corollary 4, $A_0$ is $\beta$-set of $G$. Let $F_0$ be a forcing subset for $S_0$ with $f_{\gamma_{ch,coi}}(S_0) = |F_0|$. Let $F_0 = \bigcup_{x \in K} (\{x\} \times T_x)$. We claim that $K \subseteq A_0^c$.
and \( T_x \subseteq V(H) \). Let \( K = K_1 \cup K_2 \) where \( K_1 \cap K_2 = \emptyset \). Suppose that \( K \subseteq A_0 \) or \( K_1 \subseteq A_0 \) and \( K_2 \subseteq A_0^c \). Let \( B_0 \) be a \( \beta \)-set of \( G \) with \( B_0 \neq A_0 \). Consider the \( \gamma_{ch,coi} \)-set

\[
S'_0 = [B_0 \times R] \cup [B_0^c \times V(H)].
\]

Then \( S'_0 \neq S_0 \). If \( A_0 \cap B_0 = \emptyset \), then \( K \subseteq B_0^c \) and

\[
F_0 \subseteq K \times R \subseteq B_0^c \times V(H) \subseteq S'_0
\]
or

\[
F_0 \subseteq [K \times R] \cup [K_2 \times V(H)]
\subseteq (K_1 \cup K_2) \times V(H)
\subseteq B_0^c \times V(H)
\subseteq S'_0.
\]

On the other hand, if \( A_0 \cap B_0 \neq \emptyset \) and \( K_1 \subseteq B_0 \) and \( K_2 \subseteq B_0^c \), then

\[
F_0 \subseteq [K \times R] \cup [K_1 \times R] \cup [K_2 \times V(H)]
\subseteq [B_0 \times R] \cup [B_0^c \times V(H)]
\subseteq S'_0.
\]

In either case, we have a contradiction since \( F_0 \) is a forcing subset for \( S_0 \). Hence, \( K \subseteq A_0^c \) and \( T_x \subseteq V(H) \). This implies that \( K \) is a forcing subset for \( A_0^c \). Choose any \( x \in K \) and \( a \in T_x \). Then \( F_a = K \times \{a\} \subseteq F_0 \). Thus,

\[
f_{\gamma_{ch,coi}}(G[H]) = f_{\gamma_{ch,coi}}(S_0) = |F_0| \geq |F_a| = |K| \geq f_{\beta^c}(A_0) = f_{\beta^c}(G).
\]

Therefore, \( f_{\gamma_{ch,coi}}(G[H]) = f_{\beta^c}(G) \). \( \square \)

**Example 5.** For paths \( P_3 \) and \( P_5 \), \( f_{\gamma_{ch,coi}}(P_3[P_5]) = 0 \) since \( P_5 \) has a unique \( \beta \)-set.

**Example 6.** For paths \( P_4 \) and \( P_3 \), \( f_{\gamma_{ch,coi}}(P_4[P_3]) = f_{\beta^c}(P_4) = 2 \) since \( P_4 \) has no unique \( \beta \)-sets.

**Theorem 8.** Let \( G \) and \( H \) be connected graphs of orders \( m > 2 \) and \( n > 2 \), respectively, where \( G \) and \( H \) have no unique \( \beta \)-sets and \( \gamma(H) \neq 1 \). Then

\[
f_{\gamma_{ch,coi}}(G[H]) \leq f_{\beta^c}(H) + f_{\beta^c}(G)[f_{\beta^c}(H) + 1].
\]

**Proof:** Suppose that \( G \) and \( H \) do not have unique \( \beta \)-sets. Let \( A \) be a \( \beta \)-set of \( G \) and \( F_{Ac} \) be a forcing subset for the complement \( A^c \) of \( A \) such that \( f_{\beta^c}(G) = f_{\beta^c}(A) = |F_{Ac}| \). Then for all \( x \in A \), let \( B = V(H) \setminus T_x \) be a \( \beta \)-set of \( H \) and \( F_{Bc} \) be a forcing subset for the complement \( B^c \) of \( B \) such that \( f_{\beta^c}(H) = f_{\beta^c}(B) = |F_{Bc}| \). By Corollary 4,

\[
S = \bigcup_{x \in A} (\{x\} \times T_x) \cup [A^c \times V(H)]
\]
is a $\gamma_{ch,coi}$-set of $G[H]$. We claim that $S_0 = (\{x\} \times F_{B^c}) \cup (F_A \times [F_{B^c} \cup \{y\}])$ for each $x \in A$ and $y \in B$ is a forcing subset for $S$. Suppose on the contrary. Then there exists a $\gamma_{ch,coi}$-set $S' \neq S$ such that $S_0 \subseteq S'$. By Corollary 4,

$$S' = \bigcup_{x \in A'} (\{x\} \times T'_x) \cup [(A')^c \times V(H)]$$

for some $\beta$-set $A'$ of $G$ such that $B' = V(H) \setminus T'_x$ is a $\beta$-set of $H$ for all $x \in A'$. Since $S' \neq S$, either one of the following holds:

(i) $A' = A$ and $B' \neq B$
(ii) $A' \neq A$ and $B' = B$
(iii) $A' \neq A$ and $B' \neq B$.

Suppose (i) holds. Then $S' = \bigcup_{x \in A} (\{x\} \times T'_x) \cup [A^c \times V(H)]$. Since $S_0 \subseteq S'$, $F_{B^c} \subseteq T'_x = V(H) \setminus B' = (B')^c$, a contradiction since $F_{B^c}$ is a forcing subset for $B^c$. If (ii) holds, then $F_A \subseteq (A')^c$. This is again a contradiction since $F_A$ is a forcing subset for $A^c$. If (iii) holds, then $F_{B^c} \subseteq (B')^c$ and $F_A \subseteq (A')^c$, a contradiction. Therefore, $S_0$ is a forcing subset for $S$. Thus,

$$f_{\gamma_{ch,coi}}(G[H]) \leq f_{\gamma_{ch,coi}}(S) = |S_0| = f_{\beta^c}(H) + f_{\beta^c}(G) [f_{\beta^c}(H) + 1].$$

**Example 7.** For cycle $C_4$ and path $P_4$,

$$f_{\gamma_{ch,coi}}(C_4[P_4]) \leq f_{\beta^c}(P_4) + f_{\beta^c}(C_4) [f_{\beta^c}(P_4) + 1] = 2 + 2[2 + 1] = 8$$

since both $C_4$ and $P_4$ have no unique $\beta$-sets.

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**References**


REFERENCES


