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Upper and lower weakly (Λ, sp)-continuous multifunctions

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Abstract. Our main purpose is to introduce the concepts of upper and lower weakly (Λ, sp)-continuous multifunctions. In particular, some characterizations of upper and lower weakly (Λ, sp)-continuous multifunctions are investigated.

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Key Words and Phrases: Upper weakly (Λ, sp)-continuous multifunction, lower weakly (Λ, sp)-continuous multifunction

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1. Introduction

In topology, there has been recently significant interest in characterizing and investigating the characterizations of some weak forms of continuity for functions and multifunctions. As weak forms of continuity in topological spaces, weak continuity [9], quasicontinuity [11], semi-continuity [10] and almost continuity in the sense of Husain [7] are well-known. It is shown in [12] that quasicontinuity is equivalent to semi-continuity. It will be shown that weak continuity, semi-continuity and almost continuity are respectively independent. Popa and Stan [22] introduced weak quasi-continuity which is implied by both weak continuity and quasicontinuity. Janković [8] introduced almost weak continuity as a generalization of both weak continuity and almost continuity. Noiri [13] obtained some characterizations of almost weak continuity and some relations between almost weak continuity and weak continuity. Popa [19] and Smithson [23] independently introduced the notion of weakly continuous multifunctions. The present authors introduced and studied other weak forms of continuous multifunctions: weakly quasicontinuous multifunctions [15], almost weakly continuous multifunctions [16], weakly α-continuous multifunctions [20], weakly β-continuous multifunctions [21]. These multifunctions have similar characterizations. The analogy in their definitions and results suggests the need of formulating a unified theory. Noiri and Popa [17] introduced and studied the notions of upper and lower weakly m-continuous

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multifunctions as a multifunction from a set satisfying certain minimal condition into a topological space. In [5], the present authors introduced and studied the notions of upper and lower (τ_1, τ_2)-precontinuous multifunctions. Viriyapong and Boonpok [25] introduced and investigated the concepts of upper and lower weakly $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Abd El-Monsef et al. [6] introduced a weak form of open sets called β-open sets. The notion of β-open sets is equivalent to that of semi-preopen sets due to Andrijević [1]. Noiri and Hatir [14] introduced the concept of Λ_{sp}-sets in terms of the concept of β-open sets and investigated the notion of Λ_{sp}-closed sets by using Λ_{sp}-sets. In [3], the author introduced the concepts of (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. Moreover, some characterizations of upper and lower (Λ, sp)-continuous multifunctions were provided in [3]. The purpose of the present paper is to introduce the notions of upper and lower weakly (Λ, sp)-continuous multifunctions. Furthermore, several characterizations of upper and lower weakly (Λ, sp)-continuous multifunctions are discussed.

2. Preliminaries

Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *β-open* [6] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [14] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$. A subset A of a topological space (X, τ) is called a *Λ_{sp}-set* [14] if $A = \Lambda_{sp}(A)$. A subset A of a topological space (X, τ) is called *(Λ, sp)-closed* [3] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called *(Λ, sp)-open*. The family of all (Λ, sp)-open sets in a topological space (X, τ) is denoted by $\Lambda_{sp}O(X, \tau)$. Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [3] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [3] of A and is denoted by $A^{(\Lambda,sp)}$. The union of all (Λ, sp)-open sets contained in A is called the *(Λ, sp)-interior* [3] of A and is denoted by $A_{(\Lambda,sp)}$.

Lemma 1. [3] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,sp)}$ and $[A^{(\Lambda,sp)}]^{(\Lambda,sp)} = A^{(\Lambda,sp)}$.

(2) If $A \subseteq B$, then $A^{(\Lambda,sp)} \subseteq B^{(\Lambda,sp)}$.

(3) $A^{(\Lambda,sp)}$ is (Λ, sp)-closed.

(4) A is (Λ, sp)-closed if and only if $A = A^{(\Lambda,sp)}$.

Lemma 2. [3] *For subsets A and B of a topological space (X, τ), the following properties hold:*

(1) $A_{(\Lambda,sp)} \subseteq A$ *and* $[A_{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$.

(2) If $A \subseteq B$*, then* $A_{(\Lambda,sp)} \subseteq B_{(\Lambda,sp)}$.

(3) $A_{(\Lambda,sp)}$ *is* (Λ, sp)*-open.*

(4) A *is* (Λ, sp)*-open if and only if* $A_{(\Lambda,sp)} = A$.

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

A subset A of a topological space (X, τ) is said to be $s(\Lambda, sp)$*-open* (resp. $p(\Lambda, sp)$*-open*, $\beta(\Lambda, sp)$*-open*, $r(\Lambda, sp)$*-open*) if $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ (resp. $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$, $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$) [3].

The complement of a $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $r(\Lambda, sp)$-open) set is said to be $s(\Lambda, sp)$*-closed* (resp. $p(\Lambda, sp)$*-closed*, $\beta(\Lambda, sp)$*-closed*, $r(\Lambda, sp)$*-closed*). The family of all $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $r(\Lambda, sp)$-open) sets in a topological space (X, τ) is denoted by $s\Lambda_{sp}O(X, \tau)$ (resp. $p\Lambda_{sp}O(X, \tau)$, $\beta\Lambda_{sp}O(X, \tau)$, $r\Lambda_{sp}O(X, \tau)$).

Throughout this paper, the spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, following [2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$. A multifunction $F : X \to Y$ is said to be *injective* if $x \neq y$ implies that $F(x) \cap F(y) = \emptyset$. Moreover, $F : X \to Y$ is called *upper semi-continuous* (resp. *lower semi-continuous*) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y [18].

3. Characterizations of upper and lower weakly (Λ, sp)-continuous multifunctions

In this section, we introduce the notions of upper and lower weakly (Λ, sp)-continuous multifunctions. Moreover, several characterizations of upper and lower weakly (Λ, sp)-continuous multifunctions are discussed.

Definition 1. *A multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is said to be:*

(i) upper weakly (Λ, sp)*-continuous if, for each* $x \in X$ *and each* (Λ, sp)*-open set* V *of* Y *containing* $F(x)$*, there exists a* (Λ, sp)*-open set* U *of* X *containing* x *such that* $F(U) \subseteq V^{(\Lambda,sp)}$*;*

(ii) lower weakly (Λ, sp)*-continuous if, for each* $x \in X$ *and each* (Λ, sp)*-open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$*, there exists a* (Λ, sp)*-open set* U *of* X *containing* x *such that* $F(z) \cap V^{(\Lambda,sp)} \neq \emptyset$ *for each* $z \in U$*.*

Theorem 1. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper weakly (Λ, sp)-continuous;

(2) $F^+(V) \subseteq [F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y;

(3) $[F^-(K_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(K)$ for every (Λ, sp)-closed set K of Y;

(4) $[F^-([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$ for every subset B of Y;

(5) $F^+(B_{(\Lambda,sp)}) \subseteq [F^+([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every subset B of Y;

(6) $[F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(V^{(\Lambda,sp)})$ for every (Λ, sp)-open set V of Y;

(7) $[F^-(V)]^{(\Lambda,sp)} \subseteq F^-(V^{(\Lambda,sp)})$ for every (Λ, sp)-open set V of Y;

(8) $[F^-(K_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(K)$ for every $r(\Lambda, sp)$-closed set K of Y.

Proof. (1) ⇒ (2): Let V be any (Λ, sp)-open set of Y such that $x \in F^+(V)$. Then, $F(x) \subseteq V$. There exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq V^{(\Lambda,sp)}$. Thus, $U \subseteq F^+(V^{(\Lambda,sp)})$. Since U is (Λ, sp)-open, we have $x \in [F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ and hence $F^+(V) \subseteq [F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$.

(2) ⇒ (3): Let K be any (Λ, sp)-closed set of Y. Then, $Y - K$ is (Λ, sp)-open in Y and by (2),

$$\begin{aligned} X - F^-(K) &= F^+(Y - K) \\ &\subseteq [F^+([Y - K]^{(\Lambda,sp)})]_{(\Lambda,sp)} \\ &= X - [F^-(K_{(\Lambda,sp)})]^{(\Lambda,sp)}. \end{aligned}$$

Thus, $[F^-(K_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(K)$.

(3) ⇒ (4): Let B be any subset of Y. Then, $B^{(\Lambda,sp)}$ is a (Λ, sp)-closed set of Y and by (3), $[F^-([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$.

(4) ⇒ (5): Let B be any subset of Y. By (4), we have

$$\begin{aligned} X - [F^+([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{(\Lambda,sp)} &= [X - F^+([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \\ &= [F^-([[Y - B]^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \\ &\subseteq F^-([Y - B]^{(\Lambda,sp)}) \\ &= X - F^+(B_{(\Lambda,sp)}) \end{aligned}$$

and hence $F^+(B_{(\Lambda,sp)}) \subseteq [F^+([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{(\Lambda,sp)}$.

(5) ⇒ (1): Let $x \in X$ and V be any (Λ, sp)-open set of Y such that $F(x) \subseteq V$. Then, $x \in F^+(V) \subseteq [F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ and there exists a (Λ, sp)-open set U of X containing x such that $U \subseteq F^+(V^{(\Lambda,sp)})$. Thus, $F(U) \subseteq V^{(\Lambda,sp)}$ and hence F is upper weakly (Λ, sp)-continuous.

(4) ⇒ (6) and (6) ⇒ (7): The proofs are obvious.

(7) ⇒ (8): Let K be any $r(\Lambda, sp)$-closed set of Y. Thus, by (7),

$$\begin{aligned}[F^-(K_{(\Lambda,sp)})]^{(\Lambda,sp)} &\subseteq F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)}) \\ &= F^-(K).\end{aligned}$$

(8) ⇒ (3): Let K be any (Λ, sp)-closed set of Y. Then, $[K_{(\Lambda,sp)}]^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed in Y and $[[K_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [K^{(\Lambda,sp)}]_{(\Lambda,sp)} = K_{(\Lambda,sp)}$, by (8),

$$\begin{aligned}[F^-(K_{(\Lambda,sp)})]^{(\Lambda,sp)} &= [F^-([[K_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \\ &\subseteq F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)}) \\ &\subseteq F^-(K).\end{aligned}$$

Theorem 2. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower weakly (Λ, sp)-continuous;

(2) $F^-(V) \subseteq [F^-(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y;

(3) $[F^+(K_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(K)$ for every (Λ, sp)-closed set K of Y;

(4) $[F^+([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(B^{(\Lambda,sp)})$ for every subset B of Y;

(5) $F^-(B_{(\Lambda,sp)}) \subseteq [F^-([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every subset B of Y;

(6) $[F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(V^{(\Lambda,sp)})$ for every (Λ, sp)-open set V of Y;

(7) $[F^+(V)]^{(\Lambda,sp)} \subseteq F^+(V^{(\Lambda,sp)})$ for every (Λ, sp)-open set V of Y;

(8) $[F^+(K_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(K)$ for every $r(\Lambda, sp)$-closed set K of Y.

Proof. The proof is similar to that of Theorem 1.

Definition 2. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly (Λ, sp)-continuous if, for each $x \in X$ and each (Λ, sp)-open set V of Y containing $f(x)$, there exists a (Λ, sp)-open set U of X containing x such that $f(U) \subseteq V^{(\Lambda,sp)}$.*

Corollary 1. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is weakly (Λ, sp)-continuous;

(2) $f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y;

(3) $[f^{-1}(K_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(K)$ for every (Λ, sp)-closed set K of Y;

(4) $[f^{-1}([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(B^{(\Lambda,sp)})$ *for every subset* B *of* Y*;*

(5) $f^{-1}(B_{(\Lambda,sp)}) \subseteq [f^{-1}([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{(\Lambda,sp)}$ *for every subset* B *of* Y*;*

(6) $[f^{-1}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ *for every* (Λ, sp)*-open set* V *of* Y*;*

(7) $[f^{-1}(V)]^{(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ *for every* (Λ, sp)*-open set* V *of* Y*;*

(8) $[f^{-1}(K_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(K)$ *for every* $r(\Lambda, sp)$*-closed set* K *of* Y*.*

Theorem 3. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is upper weakly* (Λ, sp)*-continuous;*

(2) $[F^{-}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{-}(V^{(\Lambda,sp)})$ *for every* $\beta(\Lambda, sp)$*-open set* V *of* Y*;*

(3) $[F^{-}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{-}(V^{(\Lambda,sp)})$ *for every* $s(\Lambda, sp)$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): This follows from (4) of Theorem 1.
(2) ⇒ (3): The proof is obvious since $s\Lambda_{sp}O(Y, \sigma) \subseteq \beta\Lambda_{sp}O(Y, \sigma)$.
(3) ⇒ (1): Since $\Lambda_{sp}O(Y, \sigma) \subseteq s\Lambda_{sp}O(Y, \sigma)$, the proof is obvious by (7) of Theorem 1.

Theorem 4. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is lower weakly* (Λ, sp)*-continuous;*

(2) $[F^{+}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{+}(V^{(\Lambda,sp)})$ *for every* $\beta(\Lambda, sp)$*-open set* V *of* Y*;*

(3) $[F^{+}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{+}(V^{(\Lambda,sp)})$ *for every* $s(\Lambda, sp)$*-open set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 3.

Corollary 2. *For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) f *is weakly* (Λ, sp)*-continuous;*

(2) $[f^{-1}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ *for every* $\beta(\Lambda, sp)$*-open set* V *of* Y*;*

(3) $[f^{-1}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ *for every* $s(\Lambda, sp)$*-open set* V *of* Y*.*

Theorem 5. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is upper weakly* (Λ, sp)*-continuous;*

(2) $[F^{-}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{-}(V^{(\Lambda,sp)})$ *for every* $p(\Lambda, sp)$*-open set* V *of* Y*;*

(3) $[F^{-}(V)]^{(\Lambda,sp)} \subseteq F^{-}(V^{(\Lambda,sp)})$ *for every* $p(\Lambda,sp)$*-open set* V *of* Y*;*

(4) $F^{+}(V) \subseteq [F^{+}(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ *for every* $p(\Lambda,sp)$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $p(\Lambda,sp)$-open set of Y. Since $[V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is (Λ,sp)-open, by Theorem 1(7),

$$\begin{aligned}[F^{-}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} &\subseteq F^{-}([[V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \\ &\subseteq F^{-}(V^{(\Lambda,sp)}).\end{aligned}$$

(2) ⇒ (3): Let V be any $p(\Lambda,sp)$-open set of Y. By (2), we have

$$\begin{aligned}[F^{-}(V)]^{(\Lambda,sp)} &\subseteq [F^{-}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \\ &\subseteq F^{-}(V^{(\Lambda,sp)}).\end{aligned}$$

(3) ⇒ (4): Let V be any $p(\Lambda,sp)$-open set of Y. Thus, by (3),

$$\begin{aligned}X - [F^{+}(V^{(\Lambda,sp)})]_{(\Lambda,sp)} &= [X - F^{+}(V^{(\Lambda,sp)})]^{(\Lambda,sp)} \\ &= [F^{-}(Y - V^{(\Lambda,sp)})]^{(\Lambda,sp)} \\ &\subseteq F^{-}([Y - V^{(\Lambda,sp)}]^{(\Lambda,sp)}) \\ &= X - F^{+}([V^{(\Lambda,sp)}]_{(\Lambda,sp)}) \\ &\subseteq X - F^{+}(V)\end{aligned}$$

and hence $F^{+}(V) \subseteq [F^{+}(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$.

(4) ⇒ (1): Let V be any (Λ,sp)-open set of Y. Then, V is $p(\Lambda,sp)$-open, by (4), $F^{+}(V) \subseteq [F^{+}(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$. By Theorem 1, F is upper weakly (Λ,sp)-continuous.

Theorem 6. *For a multifunction* $F : (X,\tau) \to (Y,\sigma)$*, the following properties are equivalent:*

(1) F *is lower weakly* (Λ,sp)*-continuous;*

(2) $[F^{+}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{+}(V^{(\Lambda,sp)})$ *for every* $p(\Lambda,sp)$*-open set* V *of* Y*;*

(3) $[F^{+}(V)]^{(\Lambda,sp)} \subseteq F^{+}(V^{(\Lambda,sp)})$ *for every* $p(\Lambda,sp)$*-open set* V *of* Y*;*

(4) $F^{-}(V) \subseteq [F^{-}(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ *for every* $p(\Lambda,sp)$*-open set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 5.

Corollary 3. *For a function* $f : (X,\tau) \to (Y,\sigma)$*, the following properties are equivalent:*

(1) f *is weakly* (Λ,sp)*-continuous;*

(2) $[f^{-1}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ *for every* $p(\Lambda,sp)$*-open set* V *of* Y*;*

(3) $[f^{-1}(V)]^{(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ *for every* $p(\Lambda, sp)$*-open set* V *of* Y*;*

(4) $f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$ *for every* $p(\Lambda, sp)$*-open set* V *of* Y*.*

Definition 3. [3] *Let* A *be a subset of a topological space* (X, τ)*. The* $\theta(\Lambda, sp)$*-closure of* A*,* $A^{\theta(\Lambda,sp)}$*, is defined as follows:*

$$A^{\theta(\Lambda,sp)} = \{x \in X \mid A \cap\ U^{(\Lambda,sp)} \neq \emptyset \textit{ for each } U \in \Lambda_{sp}O(X, \tau) \textit{ containing } x\}.$$

A subset A of a topological space (X, τ) is said to be $\theta(\Lambda, sp)$*-closed* [3] if $A = A^{\theta(\Lambda,sp)}$. The complement of a $\theta(\Lambda, sp)$-closed set is said to be $\theta(\Lambda, sp)$*-open*. The union of all $\theta(\Lambda, sp)$-open sets contained in A is called the $\theta(\Lambda, sp)$*-interior* of A and is denoted by $A_{\theta(\Lambda,sp)}$.

Lemma 3. [3] *For a subset* A *of a topological space* (X, τ)*, the following properties hold:*

(1) If A *is* (Λ, sp)*-open in* X*, then* $A^{(\Lambda,sp)} = A^{\theta(\Lambda,sp)}$*.*

(2) $A^{\theta(\Lambda,sp)}$ *is* (Λ, sp)*-closed.*

Definition 4. [3] *A multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is said to be:*

(i) upper (Λ, sp)*-continuous if, for each* $x \in X$ *and each* (Λ, sp)*-open set* V *of* Y *such that* $F(x) \subseteq V$*, there exists a* (Λ, sp)*-open set* U *of* X *containing* x *such that* $F(U) \subseteq V$*;*

(ii) lower (Λ, sp)*-continuous if, for each* $x \in X$ *and each* (Λ, sp)*-open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$*, there exists a* (Λ, sp)*-open set* U *of* X *containing* x *such that* $F(z) \cap V \neq \emptyset$ *for each* $z \in U$*.*

Definition 5. [3] *A topological space* (X, τ) *is said to be* Λ_{sp}*-regular if, for each* (Λ, sp)*-closed set* F *and each* $x \notin F$*, there exist disjoint* (Λ, sp)*-open sets* U *and* V *such that* $x \in U$ *and* $F \subseteq V$*.*

Lemma 4. [3] *Let* (Y, σ) *be a* Λ_{sp}*-regular space. For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is upper* (Λ, sp)*-continuous;*

(2) $F^-(B^{\theta(\Lambda,sp)})$ *is* (Λ, sp)*-closed in* X *for every subset* B *of* Y*;*

(3) $F^-(K)$ *is* (Λ, sp)*-closed in* X *for every* $\theta(\Lambda, sp)$*-closed set* K *of* Y*;*

(4) $F^+(V)$ *is* (Λ, sp)*-open in* X *for every* $\theta(\Lambda, sp)$*-open set* V *of* Y*.*

Lemma 5. [3] *Let* (X, τ) *be a* Λ_{sp}*-regular space. Then, the following properties hold:*

(1) $A^{(\Lambda,sp)} = A^{\theta(\Lambda,sp)}$ *for every subset* A *of* X*.*

(2) Every (Λ, sp)*-open set is* $\theta(\Lambda, sp)$*-open.*

Theorem 7. *Let (Y, σ) be a Λ_{sp}-regular space. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower (Λ, sp)-continuous;

(2) $F^+[B^{\theta(\Lambda,sp)}]$ is (Λ, sp)-closed in X for every subset B of Y;

(3) $F^+(K)$ is (Λ, sp)-closed in X for every $\theta(\Lambda, sp)$-closed set K of Y;

(4) $F^-(V)$ is (Λ, sp)-open in X for every $\theta(\Lambda, sp)$-open set V of Y;

(5) F is lower weakly (Λ, sp)-continuous.

Proof. The proofs of the implications: (1) ⇒ (2) ⇒ (3) ⇒ (4) are similar as in Lemma 4.

(4) ⇒ (5): Let V be any (Λ, sp)-open set of Y. Since (Y, σ) is Λ_{sp}-regular, by Lemma 5, V is $\theta(\Lambda, sp)$-open in Y and by (4), $F^-(V) = [F^-(V)]_{(\Lambda,sp)} \subseteq [F^-(V^{(\Lambda,sp)})]_{(\Lambda,sp)}$. Thus, by Theorem 2, F is lower weakly (Λ, sp)-continuous.

(5) ⇒ (1): Let $x \in X$ and V be any (Λ, sp)-open set of Y such that $F(x) \cap V \neq \emptyset$. Since (Y, σ) is Λ_{sp}-regular, there exists a (Λ, sp)-open set W of Y such that $F(x) \cap W \neq \emptyset$ and $W^{(\Lambda,sp)} \subseteq V$. Since F is is lower weakly (Λ, sp)-continuous, there exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that $F(z) \cap W^{(\Lambda,sp)} \neq \emptyset$; hence $F(z) \cap V \neq \emptyset$ for each $z \in U$. This shows that F is lower (Λ, sp)-continuous.

Definition 6. [4] *A topological space (X, τ) is said to be Λ_{sp}-normal if, for any pair of disjoint (Λ, sp)-closed sets F and H, there exist disjoint (Λ, sp)-open sets U and V such that $F \subseteq U$ and $H \subseteq V$.*

Lemma 6. [4] *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_{sp}-normal.

(2) For every pair of (Λ, sp)-open sets U and V whose union is X, there exist (Λ, sp)-closed sets F and H such that $F \subseteq U$, $H \subseteq V$ and $F \cup H = X$.

(3) For every (Λ, sp)-closed set F and every (Λ, sp)-open set G containing F, there exists a (Λ, sp)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G$.

(4) For every pair of disjoint (Λ, sp)-closed sets F and H, there exist disjoint (Λ, sp)-open sets U and V such that $F \subseteq U$ and $H \subseteq V$ and $U^{(\Lambda,sp)} \cap V^{(\Lambda,sp)} = \emptyset$.

Theorem 8. *Let (Y, σ) be a Λ_{sp}-normal space. For a multifunction $F : (X, \tau) \to (Y, \sigma)$ such that $F(x)$ is (Λ, sp)-closed in Y for each $x \in X$, the following properties are equivalent:*

(1) F is upper (Λ, sp)-continuous;

(2) F is upper weakly (Λ, sp)-continuous.

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (1): Suppose that F is upper weakly (Λ, sp)-continuous. Let $x \in X$ and V be any (Λ, sp)-open set of Y containing $F(x)$. Since $F(x)$ is (Λ, sp)-closed in Y, by the Λ_{sp}-normality of (Y, σ), there exists a (Λ, sp)-open set U of Y such that

$$F(x) \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq V.$$

Since F is upper weakly (Λ, sp)-continuous, there exists $W \in \Lambda_{sp}O(X, \tau)$ containing x such that $F(W) \subseteq U^{(\Lambda,sp)} \subseteq V$. This shows that F is upper (Λ, sp)-continuous.

Definition 7. [24] *A topological space (X, τ) is said to be Λ_{sp}-compact if every cover of X by (Λ, sp)-open sets of X has a finite subcover.*

A subset K of a topological space (X, τ) is said to be Λ_{sp}-*compact* if every cover of X by (Λ, sp)-open sets of X has a finite subcover.

Definition 8. *A topological space (X, τ) is called Λ_{sp}-Urusohn if, for each distinct points x and y in X, there exist $U, V \in \Lambda_{sp}O(X, \tau)$ containing x and y, respectively, such that $U^{(\Lambda,sp)} \cap V^{(\Lambda,sp)} = \emptyset$.*

Lemma 7. *If A and B are disjoint Λ_{sp}-compact subsets of a Λ_{sp}-Urusohn space (X, τ), then there exist $U, V \in \Lambda_{sp}O(X, \tau)$ such that $A \subseteq U$, $B \subseteq V$ and $U^{(\Lambda,sp)} \cap V^{(\Lambda,sp)} = \emptyset$.*

Definition 9. *A topological space (X, τ) is called Λ_{sp}-T_2 if, for each distinct points x and y in X, there exist $U, V \in \Lambda_{sp}O(X, \tau)$ containing x and y, respectively, such that $U \cap V = \emptyset$.*

Theorem 9. *If $F : (X, \tau) \to (Y, \sigma)$ is an upper weakly (Λ, sp)-continuous injective multifunction into a Λ_{sp}-Urusohn space (Y, σ) and $F(x)$ is Λ_{sp}-compact for each $x \in X$, then (X, τ) is Λ_{sp}-T_2.*

Proof. For any distinct points x_1, x_2 of X, we have $F(x_1) \cap F(x_2) = \emptyset$ since F is injective. Since $F(x)$ is Λ_{sp}-compact for each $x \in X$ and (Y, σ) is Λ_{sp}-Urusohn, by Lemma 7, there exist $V_1, V_2 \in \Lambda_{sp}O(Y, \sigma)$ such that $V_1^{(\Lambda,sp)} \cap V_2^{(\Lambda,sp)} = \emptyset$. Since F is upper weakly (Λ, sp)-continuous, there exist $U_1, U_2 \in \Lambda_{sp}O(X, \tau)$ containing x_1 and x_2, respectively, such that $F(U_1) \subseteq V_1^{(\Lambda,sp)}$ and $F(U_2) \subseteq V_2^{(\Lambda,sp)}$. Thus, $U_1 \cap U_2 = \emptyset$ and hence (X, τ) is Λ_{sp}-T_2.

4. Conclusion

The branch of mathematics called topology is concerned with all questions directly or indirectly related to continuity. This paper deals with the concept of upper (resp. lower) weak (Λ, sp)-continuity. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is called upper (resp. lower) weakly (Λ, sp)-continuous multifunctions if, for each $x \in X$ and each (Λ, sp)-open set V of Y such that $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there exists a (Λ, sp)-open set U of

X containing x such that $U \subseteq F^+(V^{(\Lambda,sp)})$ (resp. $U \subseteq F^-(V^{(\Lambda,sp)})$). Moreover, some characterizations and several properties concerning upper (resp. lower) weakly (Λ, sp)-continuous multifunctions are explored. The ideas and results of this paper may motivate further research.

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