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Acyclic and Star Coloring of Powers of Paths and Cycles

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Abstract. Let $G = (V, E)$ be a graph. The $k^{th}-$ power of G denoted by G^k is the graph whose vertex set is V and in which two vertices are adjacent if and only if their distance in G is at most k. A vertex coloring of G is acyclic if each bichromatic subgraph is a forest. A star coloring of G is an acyclic coloring in which each bichromatic subgraph is a star forest. The minimum number of colors such that G admits an acyclic (star) coloring is called the acyclic (star) chromatic number of G and is denoted by $\chi_a(G)(\chi_s(G))$. In this paper we prove that for $n \geq k+1$, $\chi_s(P_n^k) = \min\{\lfloor \frac{k+n+1}{2} \rfloor, 2k+1\}$ and $\chi_a(P_n^k) = k+1$. Further, we show that for $n \geq (k+1)^2$, $2k+1 \leq \chi_s(C_n^k) \leq 2k+2$ and $k+2 \leq \chi_a(C_n^k) \leq k+3$. Finally, we derive the formula $\chi_a(C_n^k) = k+2$ for $n \geq (k+1)^3$.

2020 Mathematics Subject Classifications: 05C15, 05C38

Key Words and Phrases: Acyclic Coloring, Powers of Cycles, Powers of Paths, Star Coloring

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1. Introduction

Graph Theory is widely used in many areas such as the study of molecules and construction of bonds in chemistry, operations research, modeling transport networks, activity networks, computational biochemistry, map coloring, and GSM mobile phone networks, and others [8]. Graph coloring is a branch of graph theory that deals with such applications. Coloring of a graph is an assignment of colors to the elements like vertices, edges, or faces (regions) of a graph. A coloring is called proper coloring if no two adjacent elements are assigned the same color. The most common types of graph colorings are vertex coloring, edge coloring, and face coloring. A $k-coloring$ of a graph $G = (V(G), E(G))$ is a function $c : V(G) \to \{1, 2, ..., k\}$. An acyclic coloring of a graph G is a proper coloring such that all induced bicolored subgraphs of G contain no cycles, in other words, every two color classes induce a forest. Star coloring is acyclic coloring where every bicolored subgraph induces a star forest. The chromatic number of G, denoted $\chi(G)$, is the minimum α such that G admits a $k-proper$ coloring; the acyclic chromatic number of a graph G, denoted $\chi_a(G)$, is the minimum number k such that G admits a $k-acyclic$ coloring;

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and the star chromatic number of a graph G, denoted $\chi_s(G)$, is the minimum number k such that G admits a $k-star$ coloring.

All graphs considered in this article are finite undirected and simple (no loops or multiple edges). All coloring considered in this article is vertex coloring. The following is an obvious observation that we use in our work.

Observation: For a proper coloring c of a graph G the following hold:

(1) c is an acyclic coloring of G if and only if every cycle in G admits at least three colors.

(2) c is a star coloring of G if and only if every path on four vertices in G admits at least three colors.

Acyclic and star coloring were introduced in the early seventies by Grünbaum [3]. Grünbaum showed that a graph with a maximum degree 3 has $4-acyclic$ colorings. Burnstein [2] proved that a graph with a maximum degree 4 has $5-acyclic$ colorings. Wood [10] studied the star and acyclic chromatic numbers of subdivision graph G' of a graph G. A great deal of research has been conducted since then. Recently, Wang et al. [9] studied the acyclic choosability of graphs with bounded degrees. Acyclic and star coloring problems are specialized vertex coloring problems that arise in the efficient computation of Hessians using automatic differentiation or finite differencing when both sparsity and symmetry are exploited.

The k^{th} power of a graph G is defined on the same set of vertices as G and has an edge between any pair of vertices of distance at most k in G. The problem of the coloring of squares of graphs has applications to frequency allocation. Transceivers in a radio network communicate using channels at given radio frequencies. Graph coloring formalizes this problem. When the constraint is that nearby pairs of transceivers cannot use the same channel due to interference. However, if two transceivers are using the same channel and both are adjacent to a third station, a clashing of signals is experienced at that third station. This can be avoided by additionally requiring all neighbors of a node to be assigned different colors, i.e., that vertices of distance at most 2 receive different colors. This is equivalent to coloring the square of the underlying network.

We attempt here to contribute to both of these perspectives, graph powers and acyclic (star) colorings. We focus on the powers of paths and cycles. As usual, P_n denotes the path on n vertices; and C_n denotes the cycle on n vertices. Acyclic colorings are hereditary in the sense that the restriction of an acyclic coloring to a subgraph is an acyclic coloring. Thus, the acyclic chromatic number is nondecreasing from subgraph to supergraph.

The main purpose of this article is to bound and determine the star and acyclic chromatic numbers of powers of paths and cycles. We prove that for large graph sizes, the star and acyclic chromatic numbers of powers of paths and cycles tend to have exact formulas in terms of the power k. As a consequence, we find the value of $\chi_a(P_n^k)$ and

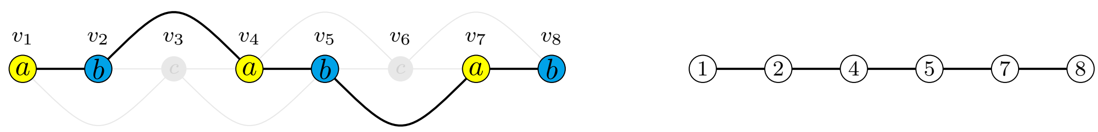
$\chi_s(P_n^k)$ in terms of k, we give an upper bound and a sharp lower bound of $\chi_a(C_n^k)$ in terms of k when $(k+1)^2 \leq n < (k+1)^3$. We derived the exact value of $\chi_a(C_n^k)$ in terms of k for $n \geq (k+1)^3$. Additionally, we give an upper bound and sharp lower bound $\chi_s(C_n^k)$ in terms of k for $(k+1)^2 \leq n$. The underlying common technique is the exploitation of the structure of bicolored induced subgraphs, the bounds that we reach in this article are tight with intervals of two values only. Our results are summarized below, [**Bold**] bounds are sharp.

Graph G	Range of n	$\chi_s(G)$	$\chi_a(G)$
P_n^k	$1 \leq n \leq k+1$	$\mathbf{n}$	$\mathbf{n}$
	$k+2 \leq n \leq 3k+1$	$\lfloor \frac{\mathbf{k+n+1}}{\mathbf{2}} \rfloor$	$\mathbf{k+1}$
	$n \geq 3k+1$	$\mathbf{2k+1}$	
C_n^k	$1 \leq n \leq 2k+1$	$\mathbf{n}$	$\mathbf{n}$
	$2k+2 \leq n < (k+1)^2$	$\mathbf{k+2} \leq \chi_s(G) \leq \mathbf{n}$	$k+2 \leq \chi_a(G) \leq \mathbf{n}$
	$(k+1)^2 \leq n < (k+1)^3$	$\mathbf{2k+1} \leq \chi_s(G) \leq 2k+2$	$\mathbf{k+2} \leq \chi_a(G) \leq k+3$
	$n \geq (k+1)^3$		$\mathbf{k+2}$

2. Acyclic Coloring of P_n^k

Let P_n^k denote the path of order n with vertex set $V(P_n^k) = \{v_0, v_1, ..., v_{n-1}\}$ and edge set $E(P_n^k)$=$\{v_iv_j : 1 \leq |i-j| \leq k\}$. Clearly, P_n^k is a complete graph when $n \leq k+1$ and hence $\chi_a(P_n^k) = n$.

Example 1. *In Figure 1 (a), P_8^2 admits an acyclic coloring as shown, the induced subgraph over the vertices colored by the color classes $\{a, b\}$ is P_6, which is not a star.*



(a) Acyclic coloring of P_8^2 (b) The induced subgraph over the color classes {a,b}

Figure 1: Acyclic Coloring but not Star Coloring

Obviously, every star coloring is acyclic coloring while the converse need not be true in general.

In this section we calculate the acyclic chromatic number as well as the star chromatic number of P_n^k.

Definition 1. ***Chordless cycles:*** *A chordless cycle in a graph, also called a hole or an induced cycle, is a cycle such that no two vertices of the cycle are connected by an edge that does not itself belong to the cycle.*

Definition 2. ***Chordal graph:*** *A chordal graph is a graph in which all cycles of four or more vertices have a chord.*

Proposition 1. *([1]). For every chordal graph G,* $\chi_a(G) = \chi(G)$. □

In the following theorem, we will determine $\chi_a(P_n^k)$ for $n \geq k+2$.

Theorem 1. *For* $n \geq k+2$, $\chi_a(P_n^k) = k+1$.

Proof. Observing that for $k \geq 3$, P_n^k contains no cordless cycle, and so by Proposition 1 we have $\chi_a(P_n^k) = \chi(P_n^k) = k+1$. □

3. Star Coloring of P_n^k

In this section, we divide the vertices of paths P_n into three partitions Prefix, Main and Suffix. By that, we were able to determine the value of star chromatic number of P_n^k.

Lemma 1. *For* $n \geq k+1$, $\chi_s(P_n^k) \leq \min\{2k+1, \lfloor \frac{n+k+1}{2} \rfloor\}$.

Proof. Define $c : V(P_n^k) \to \{c_0, c_1, ..., c_{2k}\}$ by $c(v_j) = c_{j \ mod \ (2k+1)}$. If $v_a - v_b - v_c - v_d$ is a bicolored path in P_n^k, then $2k+1 = d_{P_n}(v_a, v_c) \leq d_{P_n}(v_a, v_b) + d_{P_n}(v_b, v_c) \leq 2k$, a contradiction. So $\chi_s(P_n^k) \leq 2k+1$Clearly, $\chi_s(P_n^k) = n \leq \lfloor \frac{n+k+1}{2} \rfloor$ for $n = k+1$. And $\chi_s(P_n^k) \leq 2k+1 \leq \lfloor \frac{n+k+1}{2} \rfloor$ for $n \geq 3k+2$. Now for $k+1 < n \leq 3k+1$ we have two cases:

Case 1. $n = k + (2i+1)$ for $1 \leq i \leq k$. Then $\chi_s(P_n^k) \leq k+i+1 = \lfloor \frac{k+n+1}{2} \rfloor$.
To see that, define the $(k+i+1)-$ proper coloring $c : V(P_n^k) \to \{c_0, c_1, ..., c_{k+i}\}$ by $c(v_j) = c_{j \ mod(k+i+1)}$, and suppose that $v_a - v_b - v_c - v_d$ is a bicolored path in P_n^k where $a < b < c < d$, then $d_{P_n}(v_a, v_d) = d_{P_n}(v_a, v_c) + d_{P_n}(v_b, v_d) - d_{P_n}(v_b, v_c) \leq k+2i = n-1$. So, $d_{p_n}(v_b, v_c) \geq k+2$, a contradiction.

Case 2. $n = k + (2i)$ for $1 \leq i \leq k$. Then $\chi_s(P_n^k) \leq k+i = \lfloor \frac{k+n+1}{2} \rfloor$.
To see that, define the $(k+i)-$ proper coloring $c : V(P_n^k) \to \{c_0, c_1, ..., c_{k+i-1}\}$ by $c(v_j) = c_{j \ mod(k+i)}$, and suppose that $v_a - v_b - v_c - v_d$ is a bicolored path in P_n^k. Since $d_{P_n}(v_a, v_d) = d_{P_n}(v_a, v_c) + d_{P_n}(v_b, v_d) - d_{P_n}(v_b, v_c) \leq k+2i-1$, we have $d_{P_n}(v_b, v_c) \geq k+1$, a contradiction. □

Lemma 2. *For $n = k + 2i + 1$ where $0 \leq i \leq k$, $\chi_s(P_n^k) \geq k + i + 1$.*

Proof. *Let P_n denote the path of order n with $V(P_n) = \{p_i, p_{i-1}, ..., p_2, p_1, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{k+1}, s_1, s_2, ..., s_i\}$, and edge set $E(P_n) = \{p_x p_{x+1} : x = 1, 2, ..., i-1\} \cup \{v_x v_{x+1} : x = 1, 2, ..., k\} \cup \{s_x s_{x+1} : x = 1, 2, ..., i-1\} \cup \{p_1 v_1, v_{k+1} s_1\}$. Define three induced cliques of P_n^k,* ***Prefix*** *$(P_r(P_n^k))$,* ***Main*** *$(M_a(P_n^k))$ and* ***Suffix*** *$(S_u(P_n^k))$ with vertex sets $V(P_r(P_n^k)) = \{p_1, p_2, ..., p_i\}$, $V(M_a(P_n^k)) = \{v_1, v_2, ..., v_{k+1}\}$ and $V(S_u(P_n^k)) = \{s_1, s_2, ..., s_i\}$. Let $M = \{m_1, m_2, ..., m_{k+1}\}$ and $N = \{n_1, n_2, ..., n_{i-1}\}$ be two disjoint sets of colors, and let $c : V(P_n^k) \to M \cup N$ be a $(k+i)-$proper coloring with $c(v_\alpha) = m_\alpha$. Then $c(P_r(P_n^k)) \cap M \neq \phi$ and $c(S_u(P_n^k)) \cap M \neq \phi$. Let j and h be the least indices where $c(p_j) = m_x$ and $c(s_h) = m_y$ for some $x, y \in \{1, 2, ..., k+1\}$. Then $d_{P_n}(p_j, v_x), d_{P_n}(v_y, s_h) \geq k + 1$, and hence $x \geq k - j + 2$, $y \leq h$.*

Claim (1) *$h > i - j + 1$.*

Let $x, y \in \{1, 2, ..., k+1\}$ where $c(p_j) = m_x$. Suppose that there exists $l \leq i - j + 1$ such that $c(s_l) = m_y$, then $k + 1 \leq d_{p_n}(v_y, s_l) = k + 1 - y + l$ and hence $y \leq l$. But $d_{p_n}(p_j, v_y) = j + y - 1 \leq j + l - 1 \leq i$ and $d_{p_n}(v_x, s_l) = k + 1 - x + l \leq i$, so $p_j - v_y - v_x - s_l$ is a bicolored path in P_n^k. Hence $c(s_l) \in N$ for all $l = 1, 2, ..., i - j + 1$.

Claim (2) *If $A = \{c(p_1), c(p_2), ..., c(p_{j-1})\}$ and $B = \{c(s_1), c(s_2), ..., c(s_{h-1})\}$, then $A \cap B$ has at most $h - i + j - 2$ distinct colors.*

Let $t = h - (i - j + 1)$. If $i + t < k$, then $h + j - 1 \leq k$, so $y + j - 1 \leq k$. Since $d_{p_n}(p_j, v_y) = j + y - 1 \leq j + h - 1 \leq j + (k - j + 1) - 1 = k$ and $d_{p_n}(v_x, s_h) = (k + 1 - x) + h \leq (k + 1 - (k - j + 2)) + (t + i - j + 1) = t + i < k$, we have a bicolored path $p_j - v_y - v_x - s_h$ in P_n^k. So $i + t \geq k$.

Now suppose that $x \geq i + t + 2 - j$ and $y \leq k + 1 - j$, then $p_j - v_y - v_x - s_h$ is a bicolored path in P_n^k. Since $x \geq k - j + 2$ and $y \leq h = t + i - j + 1$, either x or $y \in \{k + 2 - j, k + 3 - j, ..., i + t + 1 - j\}$.

Case 1. *$k + 2 - j \leq y \leq i + t + 1 - j$.*

Let $y = k + 2 - j + w$ for $0 \leq w \leq i + t - k - 1$ and let $w_1 \in \{1, 2, ..., j-1\}$, $w_2 \in \{1, 2, ..., h-1\}$. Clearly, $c(p_{w_1}), c(s_{w_2}) \in N$. If there exist p_{w_1}, s_{w_2} adjacent to v_y such that $c(p_{w_1}) = c(s_{w_2})$, then $p_{w_1} - v_y - s_{w_2} - s_h$ is a bicolored path in P_n^k. Therefore, $c(p_{w_1}) \neq c(s_{w_2})$ when p_{w_1} and s_{w_2} are adjacent to v_y.

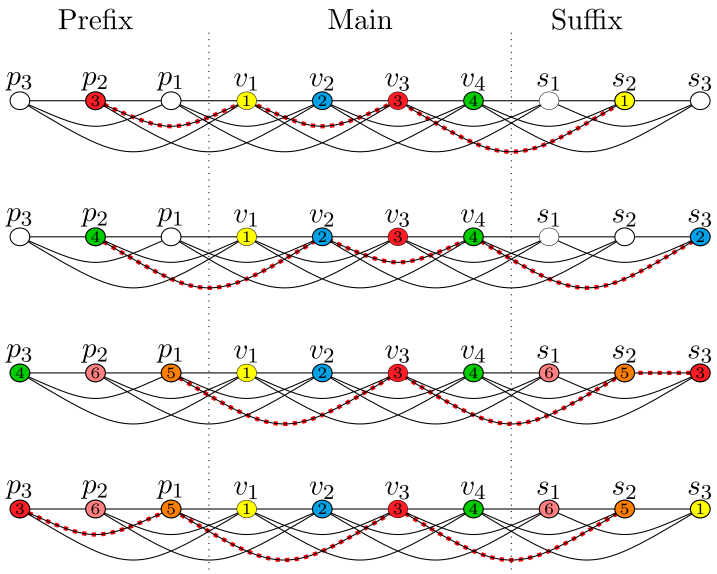
Let $|A \cap B| = t + \alpha$ for some integer α. If a is the number of vertices in $P_r(P_n^k)$ that are adjacent to v_y, then $d_{P_n}(v_y, p_a) = a + y - 1 = a + (k + 2 - j + w) - 1 = k$, and thus $a = j - w - 1$. So there exist $w = j - 1 - a$ vertices that are colored from the set A and are not adjacent to v_y. Therefore, there exist at least $(t + \alpha) - w$ vertices from $P_r(P_n^k)$ that are adjacent to v_y and colored using colors from $A \cap B$.

Also, if b is the number of vertices in $S_u(P_n^k)$ that are adjacent to v_y, then $k = d_{P_n}(v_y, s_b)$. So, $b = k - j + 1 + w$, and thus the number of vertices in $S_u(P_n^k)$ colored from the set B and are not adjacent to v_y is $(h - 1) - b = t + i - k - 1 - w$. Then there exist at least $(t + \alpha) - (t + i - k - w - 1) = \alpha + k + w + 1 - i$ vertices from $S_u(P_n^k)$ that are adjacent to v_y and colored from $A \cap B$. Hence the total number of distinct colors is greater than or equal $(\alpha + k + 1 + w - i) + (t + \alpha - w) = t + 2\alpha + (k - i) + 1 \geq 1 + t + 2\alpha$, but, $1 + t + 2\alpha \leq |(A \cap B)| = t + \alpha$ yields to $\alpha \leq -1$.

Case 2. *$x \leq i + t + 1 - j$*

If we mimic the proof of case (1), we get $|A \cap B| \leq t-1$*.So the number of distinct colors to color* $S_u(P_n^k)$ *and* $P_r(P_n^k)$ *from* N *is* $|A|+|B|-|A \cap B| \geq (j-1)+(h-1)-(t-1) = i$ *colors. Therefore, for* $n = k+2i+1$, $\chi_s(P_n^k) \geq k+i+1$. □

Example 2. *Figure 2 shows an example of the cases in Lemma 2. Let* $M = \{k+2-j, i+t+1-j\}$.



Case(a)	$h \leq i-j+1 = 2$		
j	2	h	2
A	p_1	B	s_1
Path	$p_2 - v_1 - v_3 - s_2$		

Case(b)	$x > i+t+1-j = 3$ $y < k+2-j = 3$		
j	2	h	3
A	p_1	B	s_1, s_2
Path	$p_2 - v_2 - v_4 - s_3$		

Case(c)	$\lvert A \cap B\rvert > h-i+j-2$ $\lvert A \cap B\rvert > 1$ and $y = 3$		
j	3	h	3
A	p_1, p_2	B	s_1, s_2
Path	$p_1 - v_3 - s_2 - s_3$		

Case(d)	$\lvert A \cap B\rvert > h-i+j-2$ $\lvert A \cap B\rvert > 1$ and $x = 3$		
j	3	h	3
A	p_1, p_2	B	s_1, s_2
Path	$p_3 - p_1 - v_3 - s_3$		

Graph Parameters: $n = 10$ $k = 3$ $i = 3$	$t = h-i+j-1$
j is the smallest index in $Prefix$ such that $v(p_j) \in m$, $c(p_j) = c(v_x)$	$A = \{c(p_1), c(p_2), ..., c(p_{j-1})\}$
h is the smallest index in $Suffix$ such that $v(s_h) \in m$, $c(s_h) = c(v_y)$	$B = \{c(s_1), c(s_2), ..., c(s_{h-1})\}$

Figure 2: Cases of star coloring P_{10}^3

Lemma 3. *For* $n = k+2i$ *where* $1 \leq i \leq k$, $\chi_s(P_n^k) \geq k+i$.

Proof. *Let* $n_1 = n-1$. *Then* $\chi_s(P_{n_1}^k) \leq \chi_s(P_n^k)$ *since* P_{n1}^k *is a subgraph of* P_n^k. *But* $n_1 = k+2(i-1)+1$, *so by using Lemma 2 we get* $\chi_s(P_{n_1}^k) \geq k+(i-1)+1 = k+i$. □

As a consequence of Lemmas 1- 2 we have $\chi_s(P_n^k) \leq \min\{\lfloor \frac{n+k+1}{2} \rfloor, 2k+1\}$ for $n \geq k+1$, and $\chi_s(P_n^k) \geq \lfloor \frac{n+k+1}{2} \rfloor$ for $n \in \{k+2i, k+2i+1\}$, where $0 \leq i \leq k$. Moreover, $2k+1 \leq \chi_s(P_n^k) \leq \chi_s(P_{n'}^k)$ for all $n' \geq n$. So, we can conclude the following theorem.

Theorem 2. *For* $n \geq k+1$, $\chi_s(P_n^k) = \min\{\lfloor \frac{n+k+1}{2} \rfloor, 2k+1\}$.

Example 3. $\chi_s(P_{10}^2) = \min\{\lfloor \frac{10+2+1}{2} \rfloor, 2(2)+1\} = \min\{6, 5\} = 5$

Example 4. $\chi_s(P_8^3) = \min\{\lfloor \frac{8+3+1}{2} \rfloor, 2(3)+1\} = \min\{6, 7\} = 6$

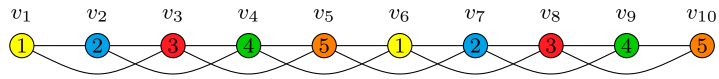


Figure 3: Star Coloring of P_{10}^2

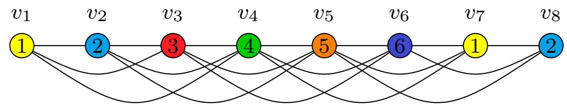


Figure 4: Star Coloring of P_8^3

4. Acyclic Coloring of C_n^k

The technique that we followed in this section was to squeeze $\chi_a(C_n^k)$ between upper and lower bounds until we reached the exact value of $\chi_a(C_n^k)$ for a wide range of cases, and to find an upper bound and a sharp lower bound for $\chi_a(C_n^k)$ for other cases. Then we built on previous studies on the proper coloring of C_n^k, and added some colors to break one of these conditions, by which we were able to determine upper bounds for $\chi_a(C_n^k)$.

Let C_n^k denote the cycle of order n with vertex set $V(C_n^k) = \{v_0, v_1, ..., v_{n-1}\}$ and edge set $E(C_n^k)$=$\{v_iv_j : 1 \leq |i-j|, n-|i-j| \leq k\}$. Clearly, C_n^k is a complete graph when $n \leq 2k+1$ and hence $\chi_a(C_n^k) = n$.

We will start by determining a lower bound for $\chi_a(C_n^k)$ when $n > 2k+1$.

Theorem 3. *For $n > 2k+1$, $\chi_a(C_n^k) \geq k+2$.*

Proof. Let $c : V(C_n^k) \to \{c_0, c_1, ..., c_k\}$ be a $(k+1)-coloring$ of C_n^k.
Since $\{v_0, v_1, ..., v_k\}$ induces a $(k+1)-clique$ in C_n^k, without loss of generality, define c by $c(v_j) = c_{j \mod (k+1)}$ for $j = 0, 1, ..., n-1$. Let $n = q(k+1)+r$ where $q \geq 2$ and $0 \leq r \leq k$. Then we have two cases:

Case 1. $r > 0$. Then $c(v_{r-1}) = c(v_{n-1})$ and $d_{C_n}(v_{n-1}, v_{r-1}) = r \leq k$, a contradiction.

Case 2. $r = 0$. For $1 \leq i \leq k$, the induced subgraph C_i of C_n^k where $V(C_i) = \{v_0, v_i, v_{(k+1)}, v_{(k+1)+i}, ..., v_{(q-1)(k+1)}, v_{(q-1)(k+1)+i}\}$, is a bicolored cycle of C_n^k using two colors c_0 and c_i, a contradiction.

Hence $\chi_a(C_n^k) \geq k+2$. □

Lemma 4. *Let c =$\{c_1, c_2, ..., c_r\}$ be a proper coloring of C_n^k and $P = \{T_i : i = 1, 2, ..., r\}$ be the color classes of c. If C_L is a bicolored cycle in C_n^k, then the following holds. If $T_i \cap V(C_L) \neq \phi$ then $T_i \subseteq V(C_L)$.*

Proof. Clearly, L is even, $L \geq 4$. Obviously, when $|T_i| = 2$ $T_i \subseteq V(C_L)$. Now let $|T_i| \geq 3$ and C_L be a bicolored cycle with $V(C_L) = \{v_{x_1}, v_{y_1}, v_{x_2}, v_{y_2}, ..., v_{x_{\frac{L}{2}}}, v_{y_{\frac{L}{2}}}\}$ and

$E(C_L) = \{v_{x_i}v_{y_i} : i = 1, 2, ..., \frac{L}{2}\} \cup \{v_{y_i}v_{x_{i+1}} : i = 1, 2, ..., \frac{L}{2} - 1\} \cup \{v_{y_{\frac{L}{2}}}v_{x_1}\}$. Assume that there exists $v_h \in T_i - V(C_L)$. If v_h lies between v_{x_i} and $v_{x_{i+1}}$ for some $i = 1, 2, ..., \frac{L}{2} - 1$, then $d_{C_n}(v_{x_i}, v_h) < d_{C_n}(v_{x_i}, v_{y_i}) \leq k$ or $d_{C_n}(v_h, v_{x_{i+1}}) < d_{C_n}(v_{y_i}, v_{x_{i+1}}) \leq k$, a contradiction. And if v_h lies between $v_{x_{\frac{L}{2}}}$ and v_{x_1}, then $d_{C_n}(v_{x_{\frac{L}{2}}}, v_h) < d_{C_n}(v_{x_{\frac{L}{2}}}, v_{y_i}) \leq k$ or $d_{C_n}(v_h, v_{x_1}) < d_{C_n}(v_{y_i}, v_{x_1}) \leq k$, a contradiction. Thus $T_i \subseteq V(C_L)$. □

Since C_L is an induced bicolored cycle of C_n^k, we have $k < d_{C_n}(v_{x_i}, v_{x_{i+1}}), d_{C_n}(v_{y_i}, v_{y_{i+1}}) \leq 2k$ for $i = 1, 2, ..., \frac{L}{2} - 1$, and $k < d_{C_n}(v_{x_{\frac{L}{2}}}, v_{x_1}), d_{C_n}(v_{y_{\frac{L}{2}}}, v_{y_1}) \leq 2k$. Using 4 to get $T_i, T_j \subseteq V(C_L)$ are induced cycles of C_n^{2k}, and $|T_i| = |T_j| \geq \frac{n}{2k}$ for some i and j.

Definition 3. *A clique of C_n^k is called consecutive if it is composed of vertices of consecutive integer indices (module n).*

Lemma 5. *[7] Let $n \geq \max\{3, k+1\}$, write $n = q(k+1) + r$ where $q \geq 1$ and $0 \leq r \leq k$. Then $\chi(C_n^k) = k + 1 + \lceil \frac{r}{q} \rceil$.*

Lemma 6. *[6] For any two integers n,k if $\alpha | n$ and $k < h$ then C_n^k is $\alpha-$colorable if and only if C_n^k contains no $\alpha + 1$ consecutive clique..*

Lemma 7. *[5] Let $n = q(k+1) + r$ where $q \geq 1$ and $0 \leq r \leq k$. Then*

(1) $r = 0$ implies that c'_1 defined by $c'_1(v_i) = i \mod (k+1)$ is a $(k+1)-$proper coloring of C_n^k.

(2) $r \neq 0$, $k_1 = \lceil \frac{r}{q} \rceil$, and $t = \lfloor \frac{r}{k_1} \rfloor$, $w = k + 1 + r - k_1 t$, and $\alpha = k + 1 + k_1$ imply that

$$c'_2 \text{ defined by } c'_2(v_i) = \begin{cases} c_{i \mod \alpha} & if \quad i \in \{0, 1, 2, ..., t\alpha + w - 1\} \\ c_{(i-(t\alpha+w)) \mod (k+1)} & if \quad i \in \{t\alpha + w, ..., n - 1\} \end{cases}$$

is a proper-coloring of C_n^k using H colors only.

Proof. (1) $r = 0$ implies that $\alpha = k + 1$ divides n and by Lemma 6, C_n^k can be colored using $k + 1$ colors.

(2) Let $r \neq 0$, $k_1 = \lceil \frac{r}{q} \rceil$, $t = \lfloor \frac{r}{k_1} \rfloor$, and $\alpha = k + 1 + k_1$. Then we have two cases,

Case 1. *$q = t$. In this case we have $q = t \leq \frac{r}{k_1} \leq \frac{r}{\frac{r}{q}} = q$, and thus $k_1 = \frac{r}{q}$, which implies that $\alpha = k + 1 + \frac{r}{q}$ and so, $q\alpha = q(k+1) + r = n$. Therefore, α divides n and by Lemma 6, c'_2 is a $\alpha - $proper coloring of C_n^k.*

Case 2. *$q \neq t$. Let $w = k + 1 + r - k_1 t$, and color C_n^k using c'_2. Notice that $r \geq k_1 \lfloor \frac{r}{k_1} \rfloor = k_1 t$ and $r - k_1 = k_1(\frac{r}{k_1} - 1) \leq k_1 \lfloor \frac{r}{k_1} \rfloor = k_1 t$ which leads to $0 \leq r - k_1 t \leq k_1$. Thus $k + 1 \leq w \leq \alpha$. Moreover, the cardinality of the subset of vertices $\{v_{t\alpha+w}, ..., v_{n-1}\}$ is a multiple of $(k+1)$ since $n - t\alpha - w = (q - t - 1)(k+1)$ and $t \leq q - 1$. Finally, note that $c'_2(v_{t\alpha+w-1}) = w - 1$, and so $k \leq c'_2(v_{t\alpha+w-1}) \leq \alpha - 1$. Also $c'_2(v_{n-1}) = k$. Hence c'_2 is a proper coloring of C_n^k that uses at most α colors.* □

In general when $n < (k+1)^2$, $\chi_a(C_n^k)$ does not have a lower bound in terms of k since the value of $\chi(C_n^k)$ changes as the ratio $\frac{r}{q}$ change, to illustrate this consider the following example.

Example 5. *Let $k = 5$. Then:*
If $n = 2(k+1)+1$, then $\chi(C_n^k) = 5+1+\lceil \frac{1}{2} \rceil = k+2$, and $\chi_a(C_n^k) \geq k+2$.
If $n = 2(k+1)+5$, then $\chi(C_n^k) = 5+1+\lceil \frac{5}{2} \rceil = k+4$, and $\chi_a(C_n^k) \geq k+4$.

Remark 1. *: Let $n = q(k+1)+r$ where $q \geq k+1$ and $0 \leq r \leq k$. Apply Lemma 7 to get $k_1 = 1$, $t = r$, $w = k+1$, $t \neq q$, $\alpha = k+2$, and*

$$c_2'(v_i) = \begin{cases} c_{i \mod (k+2)} & if \quad i \in \{0, 1, ..., r(k+2)+k\} \\ c_{\{i-r(k+2)+k+1\} \mod (k+1)} & if \quad i \in \{r(k+2)+k+1, ..., n-1\} \end{cases}$$

is a $(k+2)-$proper coloring of C_n^k.

In the following lemmas, we will consider $n \geq (k+1)^2$.

Lemma 8. *Let $n = q(k+1)$ where $q \geq k+1$, and*

$$c_3'(v_i) = \begin{cases} c_{k+1} & if \quad i = h(k+2) \text{ and } h = 0, 1, ..., k-1 \\ c_1'(v_i) & otherwise. \end{cases}$$

Then:
(1) c_3' is a $(k+2)-$ proper coloring of C_n^k.
(2) T_i does not induce a cycle in C_n^{2k} for $0 \leq i \leq k-1$.
(3) c_3' is a $(k+2)-$ acyclic coloring of C_n^k.

Proof. (1) Since c_1' is a proper coloring it is enough to check the vertices with color c_{k+1}. Since $d_{C_n}(v_{h(k+2)}, v_{(h+1)(k+2)}) = k+2$ for $0 \leq h \leq k-1$, and $d_{C_n}(v_{(k-1)(k+2)}, v_0) \geq k+2$, we have c_3' is a proper coloring of C_n^k.
(2) If $q \leq 3$, then $|T_0| = q-1 < \frac{q(k+1)}{2k}$. Now, consider $q > 3$, then T_0 does not induce a cycle in C_n^{2k} since $d_{C_n}(v_{(q-1)(k+1)}, v_{(k+1)}) \geq 2(k+1)$. Also T_i where $1 \leq i \leq k-1$ doesn't induce a cycle in C_n^{2k} since $v_{i(k+1)+i} \notin T_i$ and $d_{C_n}(v_{(i-1)(k+1)+i}, v_{(i+1)(k+1)+i}) = 2(k+1)$.
Note that the coloring c_1' colors the vertices of C_n^k by repeating $c_1, c_2, ..., c_k$, which makes the distance between any two vertices having the same color be $(k+1)$. Adding c_{k+1} in c_3' denies one occurrence of each color of $c_1, c_2, ..., c_{k-1}$ which makes a distance between two vertices having the same color become $2(k+1)$. Accordingly, the color classes $T_1, T_2, ..., T_{k-1}$ will not induce a cycle in C_n^{2k} as shown in the below table:

Rule	$i \; mod(k+1)$																				
i	0	1	...	$k+1$	$k+2$	$k+3$	...	$2k+3$	$2(k+2)$	$2k+5$	...	$3k+5$	$3(k+2)$	$3k+7$	...	$(k-1)(k+2)-1$	$(k-1)(k+2)$	$(k-1)(k+2)+1$	...	$q(k+1)-2$	$n-1$
$c_1(v_i)$	c_0	c_1	...	c_0	c_1	c_2	...	c_1	c_2	c_3	...	c_2	c_3	c_4	...	c_{k-2}	c_{k-1}	c_k	...	c_{k-1}	c_k
$c_3(v_i)$	c_{k+1}	c_1	...	c_0	c_{k+1}	c_2	...	c_1	c_{k+1}	c_3	...	c_2	c_{k+1}	c_4	...	c_{k-2}	c_{k+1}	c_k	...	c_{k-1}	c_k

$\llcorner \quad 2(k+1) \quad \lrcorner$

$\llcorner \quad 2(k+1) \quad \lrcorner$

(3) Suppose that C_L is a bicolored cycle in C_n^k, then $V(C_L) = T_i \cup T_j$ for some i and j. Note that $k = |T_{k+1}| \neq |T_k| \geq k+1$. So $V(C_L) \neq T_{k+1} \cup T_k$ and hence i *or* $j \leq k-1$. From part (2) and Lemma 4, we get a contradiction. Therefore, c_3' is a $(k+2)-$ acyclic coloring of C_n^k. $\square$

Lemma 9. *Let* $n = (k+1)^2 + k$, *and*

$$c_4'(v_i) = \begin{cases} c_{k+2} & if \quad i = j(k+3) \text{ for } j \in \{0,1,...,k-1\} \\ c_{k+1} & if \quad i = k(k+3) \\ c_2'(v_i) & otherwise. \end{cases}$$

Then

(1) c_4' *is a* $(k+3)-$*proper coloring of* C_n^k.

(2) For $0 \leq i \leq k$, T_i *is not an induced cycle of* C_n^{2k}.

(3) c_4' *is a* $(k+3)-$ *acyclic coloring of* C_n^k.

Proof. (1) Since c_2' is a proper coloring, $d_{C_n}(v_{h(k+3)}, v_{(h+1)(k+3)}) = k+3$ for $0 \leq h \leq k-1$, and $d_{C_n}(v_{(k-1)(k+3)}, v_0) = k+4$, we have c_4' is a $(k+3)-$proper coloring of C_n^k. Moreover, to keep c_4' proper coloring, c_{k+1} was assigned to v_{n-1} instead c_{k+2} since $d_{C_n}(v_0, v_{n-1}) = 1$.

(2) If $k \leq 2$, then $|T_0| = k < \frac{(k+1)^2+k}{2k}$. If $k > 2$, then $d_{C_n}(v_{k(k+2)}, v_{(k+2)}) \geq 2(k+1)$. If $0 < i \leq k$, then $v_{i(k+2)+i} \notin T_i$ and thus $d_{C_n}(v_{(i-1)(k+2)+i}, v_{(i+1)(k+2)+i}) = 2(k+2)$. Adding c_{k+2} in c_4' denies one occurrence of each color of $c_1, c_2, ..., c_k$ which makes a distance between two vertices having the same color become $2(k+1)$. Accordingly, the color classes $T_1, T_2, ..., T_k$ will not induce a cycle in C_n^{2k} as shown in the below table:

Rule	$i\ mod(k+2)$																						
i	0	1	...	$k+2$	$k+3$	$k+4$	...	$2k+5$	$2(k+2)+2$	$2k+7$	...	$3k+8$	$3(k+2)+3$	...	$(k-1)(k+2)+k-2$	$(k-1)(k+2)+k-1$	$(k-1)(k+2)+k$	$k(k+2)-1$	...	$k(k+2)+k-1$	$n-1$
$c_2(v_i)$	c_0	c_1	...	c_0	c_1	c_2	...	c_1	c_2	c_3	...	c_2	c_3	...	c_{k-2}	c_{k-1}	c_k	c_{k+1}	...	c_{k-1}	c_k
$c_4(v_i)$	c_{k+2}	c_1	...	c_0	c_{k+2}	c_2	...	c_1	c_{k+2}	c_3	...	c_2	c_{k+2}	...	c_{k-2}	c_{k+2}	c_k	c_{k+1}	...	c_{k-1}	c_{k+1}

└ $2(k+2)$ ┘

└ $2(k+2)$ ┘ └ $k+1$ ┘

(3) Assume that C_L is a bicolored cycle in C_n^k, then $V(C_L) = T_i \cup T_j$ for some i, j. Clearly $k+1 = |T_{k+1}| \neq |T_{k+2}| = k$, so i or $j \leq k$ say i, then T_i is not an induced cycle of C_n^{2k}. $\square$

Lemma 10. *Let* $n = q(k+1) + r$, $0 < r < k+1$, $q \geq k+1$ *and* $n \neq (k+1)^2 + k$. *Let*

$$c_5'(v_i) = \begin{cases} c_{k+2} & if \quad i = j(k+3) \text{ for } j \in \{0,1,...,r\} \\ c_{k+1} & if \quad i = r(k+3) + j(k+2) \text{ for } j \in \{1,2,...,k-r\} \\ c_2'(v_i) & otherwise. \end{cases}$$

Then:

(1) c'_5 *is a* $(k+3)-$*proper coloring of* C_n^k.

(2) For $i \in \{0, ..., k\}$, T_i *is not an induced cycle of* C_n^{2k}.

(3) c'_5 *is a* $(k+3)-$ *acyclic coloring of* C_n^k.

Define $P_1 = \{0, k+3, 2(k+3), ..., r(k+3)\}$ *and* $P_2 = \{r(k+3)+(k+2), r(k+3)+2(k+2), ..., r(k+3)+(k-r)(k+2)\}$.

Proof. (1) Since $d_{C_n}(v_{h(k+3)}, v_{(h+1)(k+3)}) = k+3$ *for* $0 \leq h \leq r-1$, $d_{C_n}(v_{r(k+3)}, v_0) \geq k+3$, $d_{C_n}(v_{r(k+3)+h(k+2)}, v_{r(k+3)+(h+1)(k+2)}) = k+2$ *for* $1 \leq h \leq k-r-1$, $d_{C_n}(v_{n-((k-r)(k+2)-q)}, v_{k+1}) \geq k+2$ *and* c'_2 *is a proper coloring, we have* c'_5 *is a proper coloring of* C_n^k.

(2) To show that T_i *is not an induced cycle of* C_n^{2k} *for* $i \in \{0, ..., k\}$, *consider the following cases*

Case 1. $i = 0$. *Then* $v_0 \notin T_0$ *and* $d_{C_n}(v_{n-k}, v_{k+2}) = 2k+2$.

Case 2. $0 < i < r$. *Then* $v_{i(k+2)+i} \notin T_i$ *and* $d_{C_n}(v_{(i-1)(k+2)+i}, v_{(i+1)(k+2)+i}) = 2(k+2)$.

Case 3. $i = r$. *Then* $v_{r(k+2)+i} \notin T_i$ *and* $d_{C_n}(v_{(r-1)(k+2)+i}, v_{r(k+2)+(k+1)+i}) = 2k+3$.

Case 4. $r < i < k$. *Then* $v_{r(k+3)+(i-r)(k+2)} \notin T_i$ *and* $d_{C_n}(v_{r(k+3)+(i-r-1)(k+2)}, v_{r(k+3)+(i-r+1)(k+2)}) = 2(k+2)$.

Case 5. $i = k$. *Then* $v_{r(k+3)+(k-r)(k+2)} \notin T_k$ *and* $d_{C_n}(v_{r(k+3)+(k-r-1)(k+2)}, v_k) \geq 2k+3$ *when* $q \leq k+2$. *while* $d_{C_n}(v_{r(k+3)+(k-r-1)(k+2)}, v_{r(k+3)+(k-r+1)(k+2)}) = 2(k+2)$ *when* $q \geq k+3$.

Therefore, T_k *is not an induced cycle of* C_n^{2k}.

Adding c_{k+2} *to* P_1 *denies a turn of each color of* $c_1, c_2, ..., c_r$, *and adding* c_{k+1} *to* P_2 *does the same for colors* $c_{r+1}, c_{r+2}, ..., c_k$. *Also the distance between the last vertex in* P_1 *that is colored with* c_{k+1} *and the first vertex in* P_2 *that is colored with* c_{k+1} *is* $(k+1)$ *which keeps the coloring proper as shown in the below table:*

Rule	$i\ mod(k+2)$																					$(i-x)mod(k+1)$								
i	0	1	...	$k+2$	$k+3$	$k+4$	...	$2k+5$	$2(k+3)$	$2k+7$	...	$3k+8$	$3(k+3)$	...	$(r-1)(k+2)+r$	...	$r(k+3)$	$r(k+3)+1$	...	$x-1$	x	...	$x+r$	$x+r+1$	...	$r(k+3)+k+1$	$r(k+3)+k+2$	...	n-1
$c_2(v_i)$	c_0	c_1	...	c_0	c_1	c_2	...	c_1	c_2	c_3	...	c_2	c_3	...	c_r	...	c_r	c_{r+1}	...	c_k	c_0	...	c_r	c_{r+1}	...	c_r	c_{r+1}	...	c_{k+1}
$c_5(v_i)$	c_{k+2}	c_1	...	c_0	c_{k+2}	c_2	...	c_1	c_{k+2}	c_3	...	c_2	c_{k+2}	...	c_r	...	c_{k+2}	c_{r+1}	...	c_k	c_0	...	c_r	c_{r+1}	...	c_r	c_{k+1}	...	c_{k+1}

$2(k+2)$ $\quad$ $2(k+1)+1$

$2(k+2)$ $\quad$ $k+1$

$x = r(k+2)+k+1$

(3) Since $d_{C_n}(v_0, v_{k+1}) = k+1$ *and* T_i *not an induced cycle of* C_n^{2k} *for* $i \leq k$, c'_5 *is a* $(k+3)-$ *acyclic coloring for* C_n^k. □

Lemma 11. *Let* $n = q(k+1)+r$, $0 < r \leq k$, $q \geq k+1$, $q-r \geq k+1$ *and*

$$c'_6(v_i) = \begin{cases} c_{k+1} & \text{if } i = (r+j)(k+2)+(k+1) \text{ for } j \in \{0, 1, ..., k-1\} \\ c'_2(v_i) & \text{otherwise.} \end{cases}$$

Then:
(1) c'_6 is a $(k+2)-$proper coloring of C_n^k.
(2) For $i \in \{0, ..., k-1\}$, T_i is not an induced cycle of C_n^{2k}.
(3) c'_6 is a $(k+2)-$ acyclic coloring of C_n^k.

Proof. *(1) Note that $d_{C_n}(v_{(r+h)(k+2)+k+1}, v_{(r+h+1)(k+2)+k+1}) = k+2$ for $0 \le h \le k-2$, $d_{C_n}(v_{(r+k-1)(k+2)+k+1}, v_{k+1}) \ge k+3$, and $d_{C_n}(v_{r(k+2)-1}, v_{r(k+2)+k+1}) = k+2$. Therefore, c'_6 is a proper coloring of C_n^k.*
(2) For $0 \le i < k-1$, $v_{(r+i)(k+2)+(k+1)} \notin T_i$ and $d_{C_n}(v_{(r+i)(k+2)}, v_{(r+i)(k+2)+2(k+1)}) = 2(k+1)$.
For $i = k-1$ we have the following two cases:
Case 1. *$q - r > k+1$. Then $v_{(r+k-1)(k+2)+(k+1)} \notin T_{k-1}$ and $d_{C_n}(v_{(r+k-1)(k+2)}, v_{(r+k-1)(k+2)+2(k+1)}) = 2(k+1)$.*
Case 2. *: $q - r = k+1$. Then $v_{(r+k-1)(k+2)+(k+1)} \notin T_{k-1}$ and $d_{C_n}(v_{(r+k-1)(k+2)}, v_{k-1}) = 2(k+1)$.*
Hence T_i is not an induced cycle of C_n^{2k} for $i \in \{0, ..., k-1\}$.
(3) Note that $r+k = |T_{k+1}| \ne |T_k| \ge r+k+1$. Moreover, T_i is not an induced cycle in C_n^{2k} for $i \le k-1$, so c'_6 is a $(k+2)-$ acyclic coloring for C_n^k. □

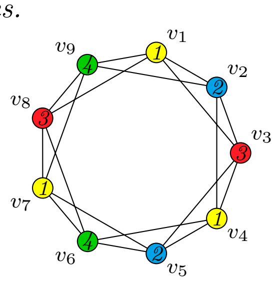
As a consequence of Lemmas 4-11 we get the following theorem.

Theorem 4. *Let C_n^k be the k^{th}-power of a cycle of order n. Then*
(1) $k+2 \le \chi_a(C_n^k) \le k+3$ if $n \ge (k+1)^2$.
(2) $\chi_a(C_n^k) = k+2$ if $n = q(k+1)+r$ and $q-r \ge k+1$.
(3) $\chi_a(C_n^k) = k+2$ if $n \ge (k+1)^3$.

According to Theorem 4 when n is between $(k+1)^2$ and $(k+1)^3$, $\chi_a(C_n^k)$ varies between $k+2$ and $k+3$, while for $n \ge (k+1)^3$, $\chi_a(C_n^k) = k+2$.

The following example shows that $k+2$ is a sharp lower bound for $\chi_a(C_n^k)$ when $n = (k+1)^2$.

Example 6. *Let $k = 2$ and $n = (k+1)^2$, then $c'_2(C_n^k)$ uses only 4 colors to acyclic color C_n^k, $\chi_a(C_n^k) = k+2$. The union of any two color classes induces a disjoint collection of trees.*



T_1,T_2	v_1	v_2	v_4	v_5	v_7
T_1,T_3	v_1	v_3	v_4	v_7	v_8
T_1,T_4	v_4	v_6	v_7	v_9	v_1
T_2,T_3	v_2	v_3	v_5	v_8	
T_2,T_4	v_9	v_2	v_5	v_6	
T_3,T_4	v_6	v_8	v_9	v_3	

5. Star Coloring of C_n^k

In this section we bound $\chi_s(C_n^k)$ between two values by combining some results from previous sections with the relation between $\chi_s(G)$ and $\chi(G^2)$.

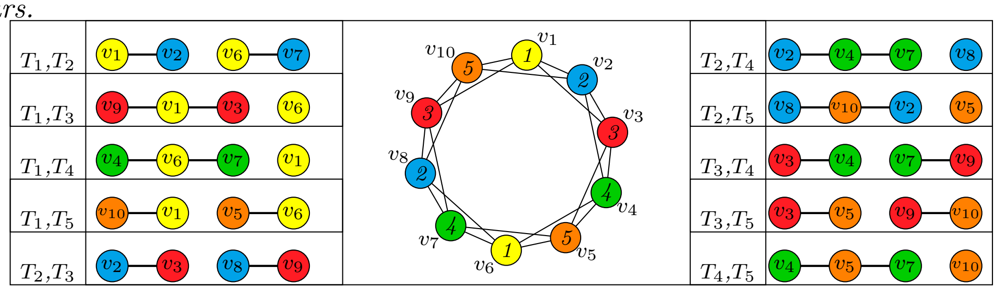
Lemma 12. *[4] Let G be a graph of order n and G^2 be the square graph of G. Then, $\chi_s(G) \leq \chi(G^2)$, where $\chi(G)$ denotes the (proper) chromatic number of G.*

Theorem 5. *For $n \geq (k+1)^2$, $2k+1 \leq \chi_s(C_n^k) \leq 2k+2$.*

Proof. Let $n = q(k+1)+r$. Using Lemmas 5 and 12 to get $\chi(C_n^{2k}) = 2k+1+\left\lceil \frac{r}{q} \right\rceil = 2k+2$ and $\chi_s(C_n^k) \leq 2k+2$. Moreover, P_n^k is a subgraph of C_n^k, so $\chi_s(P_n^k) \leq \chi_s(C_n^k)$. According to Lemma 1 $\chi_s(P_n^k) = 2k+1$, so $2k+1 \leq \chi_s(C_n^k)$. □

The following example shows that $k+2$ is a sharp lower bound for $\chi_a(C_n^k)$.

Example 7. *Let $k = 2$ and $n = (k+1)^2 + 1$, then $c(C_n^k)$ uses only 5 colors to star color C_{10}^2, $\chi_s(C_n^k) = 2k+1$. The union of any two color classes induces a disjoint collection of stars.*



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