EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 1, 2023, 84-96
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Almost (Λ, sp)-continuity for multifunctions

Chawalit Boonpok[1], Nongluk Viriyapong[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*

Abstract. This paper deals with the concepts of upper and lower almost (Λ, sp)-continuous multifunctions. Moreover, several characterizations of upper and lower almost (Λ, sp)-continuous multifunctions are investigated.

2020 Mathematics Subject Classifications: 54C08, 54C60

Key Words and Phrases: Upper almost (Λ, sp)-continuous multifunction, lower almost (Λ, sp)-continuous multifunction

---

1. Introduction

It is well-known that the branch of mathematics called topology is related to all questions directly or indirectly concerned with continuity. Semi-open sets, preopen sets, α-open sets, β-open sets and δ-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of weak forms of continuity for functions and multifunctions. In 1968, Singal and Singal [18] introduced and studied the notion of almost continuous functions. In 1982, Popa [12] introduced the concepts of upper and lower almost continuous multifunctions. The notion of almost quasi-continuous multifunctions was introduced by Popa and Noiri [13]. Noiri and Popa [8] investigated several characterizations of upper and lower almost quasi-continuous multifunctions. In 1993, Popa et al. [16] introduced the concepts of upper and lower almost precontinuous multifunctions. Moreover, Popa et al. [17] obtained some characterizations of upper and lower almost precontinuous multifunctions. In 1996, Popa and Noiri [14] introduced and investigated the notions of upper and lower almost α-continuous multifunctions. In 1999, Noiri and Popa [9] introduced the concepts of upper and lower almost β-continuous multifunctions. Popa and Noiri [15] further studied some characterizations of upper and lower almost β-continuous multifunctions. In 2006, Ekici and Park [5] introduced and studied almost γ-continuous multifunctions. In 2010, Noiri and Popa [10] introduced and studied the notions of upper and lower almost

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i1.4576

Email addresses: chawalit.b@msu.ac.th (C. Boonpok), nongluk.h@msu.ac.th (N. Viriyapong)

m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space. In 2018, Boonpok et al. [4] introduced and investigated the concepts of upper and lower almost (τ_1, τ_2)-precontinuous multifunctions.

The concept of β-open sets due to Abd El-Monsef et al. [6] or semi-preopen sets in the sense of Andrijević [1] plays a significant role in general topology. Noiri and Hatir [7] introduced the concept of Λ_{sp}-sets in terms of the concept of β-open sets and investigated the notion of Λ_{sp}-closed sets by using Λ_{sp}-sets. In [3], the author introduced the concepts of (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. The notion of (Λ, sp)-continuous multifunctions was studied in [3]. The purpose of the present paper is to introduce the concepts of upper and lower almost (Λ, sp)-continuous multifunctions. Moreover, some characterizations of upper and lower almost (Λ, sp)-continuous multifunctions are discussed.

2. Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *β-open* [6] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [7] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$. A subset A of a topological space (X, τ) is called a *Λ_{sp}-set* [7] if $A = \Lambda_{sp}(A)$. A subset A of a topological space (X, τ) is called *(Λ, sp)-closed* [3] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called *(Λ, sp)-open*. The family of all (Λ, sp)-open sets in a topological space (X, τ) is denoted by $\Lambda_{sp}O(X, \tau)$. Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [3] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [3] of A and is denoted by $A^{(\Lambda,sp)}$. The union of all (Λ, sp)-open sets contained in A is called the *(Λ, sp)-interior* [3] of A and is denoted by $A_{(\Lambda,sp)}$.

Lemma 1. [3] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,sp)}$ *and* $[A^{(\Lambda,sp)}]^{(\Lambda,sp)} = A^{(\Lambda,sp)}$.

(2) If $A \subseteq B$, *then* $A^{(\Lambda,sp)} \subseteq B^{(\Lambda,sp)}$.

(3) $A^{(\Lambda,sp)}$ *is* (Λ, sp)*-closed.*

(4) A *is* (Λ, sp)*-closed if and only if* $A = A^{(\Lambda,sp)}$.

Lemma 2. [3] *For subsets A and B of a topological space (X, τ), the following properties hold:*

(1) $A_{(\Lambda,sp)} \subseteq A$ *and* $[A_{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$.

(2) If $A \subseteq B$*, then* $A_{(\Lambda,sp)} \subseteq B_{(\Lambda,sp)}$.

(3) $A_{(\Lambda,sp)}$ *is* (Λ, sp)*-open.*

(4) A *is* (Λ, sp)*-open if and only if* $A_{(\Lambda,sp)} = A$.

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

A subset A of a topological space (X, τ) is said to be $s(\Lambda, sp)$*-open* (resp. $p(\Lambda, sp)$*-open*, $\beta(\Lambda, sp)$*-open*, $r(\Lambda, sp)$*-open*) if $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ (resp. $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$, $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$) [3]. The complement of a $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $r(\Lambda, sp)$-open) set is said to be $s(\Lambda, sp)$*-closed* (resp. $p(\Lambda, sp)$*-closed*, $\beta(\Lambda, sp)$*-closed*, $r(\Lambda, sp)$*-closed*).

The family of all $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $r(\Lambda, sp)$-open) sets in a topological space (X, τ) is denoted by $s\Lambda_{sp}O(X, \tau)$ (resp. $p\Lambda_{sp}O(X, \tau)$, $\beta\Lambda_{sp}O(X, \tau)$, $r\Lambda_{sp}O(X, \tau)$). Let A be a subset of a topological space (X, τ). The intersection of all $s(\Lambda, sp)$-closed sets containing A is called the $s(\Lambda, sp)$*-closure* of A and is denoted by $A^{s(\Lambda,sp)}$. The union of all $s(\Lambda, sp)$-open sets contained in A is called the $s(\Lambda, sp)$*-interior* of A and is denoted by $A_{s(\Lambda,sp)}$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then, F is said to be a *surjection* if $F(X) = Y$, or equivalently, if for each $y \in Y$, there exists $x \in X$ such that $y \in F(x)$. Moreover, $F : X \to Y$ is called *upper semi-continuous* (resp. *lower semi-continuous*) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y [11].

3. Upper and lower almost (Λ, sp)-continuous multifunctions

In this section, we introduce the notions of upper and lower almost (Λ, sp)-continuous multifunctions. In particular, several characterizations of upper and lower almost (Λ, sp)-continuous multifunctions are discussed.

Definition 1. *A multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is said to be:*

(i) upper almost (Λ, sp)*-continuous at a point* $x \in X$ *if, for each* (Λ, sp)*-open set* V *of* Y *containing* $F(x)$*, there exists a* (Λ, sp)*-open set* U *of* X *containing* x *such that* $F(U) \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$*;*

(ii) *lower almost* (Λ, sp)*-continuous at a point* $x \in X$ *if, for each* (Λ, sp)*-open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$*, there exists a* (Λ, sp)*-open set* U *of* X *containing* x *such that* $F(z) \cap [V^{(\Lambda,sp)}]_{(\Lambda,sp)} \neq \emptyset$ *for each* $z \in U$*;*

(iii) *upper (lower) almost* (Λ, sp)*-continuous if* F *has this property at each point of* X*.*

Lemma 3. [21] *For a subset* A *of a topological space* (X, τ)*, the following properties hold:*

(1) $A^{s(\Lambda,sp)} = A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$*;*

(2) $A_{s(\Lambda,sp)} = A \cap [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$*.*

Lemma 4. *Let* A *be a subset of a topological space* (X, τ)*. If* A *is* (Λ, sp)*-open in* X*, then* $A^{s(\Lambda,sp)} = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$*.*

Theorem 1. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is upper almost* (Λ, sp)*-continuous at* $x \in X$*;*

(2) $x \in [F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$ *for every* (Λ, sp)*-open set* V *of* Y *containing* $F(x)$*;*

(3) $x \in [F^+(V^{s(\Lambda,sp)})]_{(\Lambda,sp)}$ *for every* (Λ, sp)*-open set* V *of* Y *containing* $F(x)$*;*

(4) $x \in [F^+(V)]_{(\Lambda,sp)}$ *for every* $r(\Lambda, sp)$*-open set* V *of* Y *containing* $F(x)$*;*

(5) *for each* $r(\Lambda, sp)$*-open set* V *of* Y *containing* $F(x)$*, there exists a* (Λ, sp)*-open set* U *of* X *containing* x *such that* $F(U) \subseteq V$*.*

Proof. (1) ⇒ (2): Let V be any (Λ, sp)-open set of Y containing $F(x)$. There exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Thus,

$$x \in U \subseteq F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})$$

and hence $x \in [F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$.

(2) ⇒ (3): This follows from Lemma 4.

(3) ⇒ (4): Let V be any $r(\Lambda, sp)$-open set of Y containing $F(x)$. Then, it follows from Lemma 4 that $V = [V^{(\Lambda,sp)}]_{(\Lambda,sp)} = V^{s(\Lambda,sp)}$.

(4) ⇒ (5): Let V be any $r(\Lambda, sp)$-open set of Y containing $F(x)$. Thus, by (4), $x \in [F^+(V)]_{(\Lambda,sp)}$ and there exists a (Λ, sp)-open set U of X containing x such that $x \in U \subseteq F^+(V)$; hence $F(U) \subseteq V$.

(5) ⇒ (1): Let V be any (Λ, sp)-open set of Y containing $F(x)$. Since $[V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $r(\Lambda, sp)$-open, there exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$. This shows that F is upper almost (Λ, sp)-continuous at $x \in X$.

Theorem 2. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F is lower almost (Λ, sp)-continuous at $x \in X$;

(2) $x \in [F^{-}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y such that

$$F(x) \cap V \neq \emptyset;$$

(3) $x \in [F^{-}(V^{s(\Lambda,sp)})]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$;

(4) $x \in [F^{-}(V)]_{(\Lambda,sp)}$ for every $r(\Lambda, sp)$-open set V of Y containing $F(x)$;

(5) for each $r(\Lambda, sp)$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a (Λ, sp)-open set U of X containing x such that $U \subseteq F^{-}(V)$.

Proof. The proof is similar to that of Theorem 1.

Definition 2. *A function $f : (X, \tau) \to (Y, \sigma)$ is called almost (Λ, sp)-continuous at a point $x \in X$ if, for each (Λ, sp)-open set V of Y containing $f(x)$, there exists a (Λ, sp)-open set U of X containing x such that $f(U) \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$. A function $f : (X, \tau) \to (Y, \sigma)$ is called almost (Λ, sp)-continuous if f has this property at each point of X.*

Corollary 1. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost (Λ, sp)-continuous at $x \in X$;

(2) $x \in [f^{-1}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y containing $f(x)$;

(3) $x \in [f^{-1}(V^{s(\Lambda,sp)})]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y containing $f(x)$;

(4) $x \in [f^{-1}(V)]_{(\Lambda,sp)}$ for every $r(\Lambda, sp)$-open set V of Y containing $f(x)$;

(5) for each $r(\Lambda, sp)$-open set V of Y containing $f(x)$, there exists a (Λ, sp)-open set U of X containing x such that $U \subseteq f^{-1}(V)$.

Theorem 3. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost (Λ, sp)-continuous;

(2) $F^{+}(V) \subseteq [F^{+}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y;

(3) $[F^{-}([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{-}(K)$ for every (Λ, sp)-closed set K of Y;

(4) $[F^{-}([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{-}(B^{(\Lambda,sp)})$ for every subset B of Y;

(5) $F^{+}(B_{(\Lambda,sp)}) \subseteq [F^{+}([[B_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every subset B of Y;

(6) $F^{+}(V)$ is (Λ, sp)-open in X for every $r(\Lambda, sp)$-open set V of Y;

(7) $F^{-}(K)$ is (Λ, sp)-closed in X for every $r(\Lambda, sp)$-closed set K of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any (Λ, sp)-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$. Thus, by Theorem 1, $x \in [F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$ and hence $F^+(V) \subseteq [F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$.

(2) $\Rightarrow$ (3): Let K be any (Λ, sp)-closed set of Y. Then, $Y - K$ is (Λ, sp)-open in Y and by (2),

$$\begin{aligned} X - F^-(K) &= F^+(Y - K) \\ &\subseteq [F^+([[Y - K]^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)} \\ &= [X - F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{(\Lambda,sp)} \\ &= X - [F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)}. \end{aligned}$$

Thus, $[F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then, $B^{(\Lambda,sp)}$ is a (Λ, sp)-closed set of Y and by (3), $[F^-([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then, we have

$$\begin{aligned} F^+(B_{(\Lambda,sp)}) &= X - F^-([Y - B]^{(\Lambda,sp)}) \\ &\subseteq X - [F^-([[[Y - B]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \\ &= X - [F^-(Y - [[B_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \\ &= [F^+([[B_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}. \end{aligned}$$

(5) $\Rightarrow$ (6): Let V be any $r(\Lambda, sp)$-open set of Y. By (5), we have $F^+(V) \subseteq [F^+(V)]_{(\Lambda,sp)}$ and hence $F^+(V)$ is (Λ, sp)-open in X.

(6) $\Rightarrow$ (7): The proof is obvious.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be any $r(\Lambda, sp)$-open set of Y containing $F(x)$. Since $Y - V$ is $r(\Lambda, sp)$-closed and by (7), $X - F^+(V) = F^-(Y - V)$ is (Λ, sp)-closed in X. Thus, $F^+(V)$ is (Λ, sp)-open and hence $x \in [F^+(V)]_{(\Lambda,sp)}$. Then, there exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq V$. It follows from Theorem 1 that F is upper almost (Λ, sp)-continuous.

Theorem 4. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost (Λ, sp)-continuous;

(2) $F^-(V) \subseteq [F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y;

(3) $[F^+([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(K)$ for every (Λ, sp)-closed set K of Y;

(4) $[F^+([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(B^{(\Lambda,sp)})$ for every subset B of Y;

(5) $F^-(B_{(\Lambda,sp)}) \subseteq [F^-([[B_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$ for every subset B of Y;

(6) $F^{-}(V)$ *is* (Λ, sp)*-open in* X *for every* $r(\Lambda, sp)$*-open set* V *of* Y*;*

(7) $F^{+}(K)$ *is* (Λ, sp)*-closed in* X *for every* $r(\Lambda, sp)$*-closed set* K *of* Y*.*

Proof. The proof is similar to that of Theorem 3.

Corollary 2. *For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) f *is almost* (Λ, sp)*-continuous;*

(2) $f^{-1}(V) \subseteq [f^{-1}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$ *for every* (Λ, sp)*-open set* V *of* Y*;*

(3) $[f^{-1}([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(K)$ *for every* (Λ, sp)*-closed set* K *of* Y*;*

(4) $[f^{-1}([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(B^{(\Lambda,sp)})$ *for every subset* B *of* Y*;*

(5) $f^{-1}(B_{(\Lambda,sp)}) \subseteq [f^{-1}([[B_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{(\Lambda,sp)}$ *for every subset* B *of* Y*;*

(6) $f^{-1}(V)$ *is* (Λ, sp)*-open in* X *for every* $r(\Lambda, sp)$*-open set* V *of* Y*;*

(7) $f^{-1}(K)$ *is* (Λ, sp)*-closed in* X *for every* $r(\Lambda, sp)$*-closed set* K *of* Y*.*

Theorem 5. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is upper almost* (Λ, sp)*-continuous;*

(2) $[F^{-}(V)]^{(\Lambda,sp)} \subseteq F^{-}(V^{(\Lambda,sp)})$ *for every* $\beta(\Lambda, sp)$*-open set* V *of* Y*;*

(3) $[F^{-}(V)]^{(\Lambda,sp)} \subseteq F^{-}(V^{(\Lambda,sp)})$ *for every* $s(\Lambda, sp)$*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\beta(\Lambda, sp)$-open set of Y. Then, $V^{(\Lambda,sp)}$ is a $r(\Lambda, sp)$-closed set of Y. Since F is upper almost (Λ, sp)-continuous and by Theorem 3, $F^{-}(V^{(\Lambda,sp)})$ is (Λ, sp)-closed in X. Thus, $[F^{-}(V)]^{(\Lambda,sp)} \subseteq F^{-}(V^{(\Lambda,sp)})$.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (1): Let K be any $r(\Lambda, sp)$-closed set of Y. Then, K is $s(\Lambda, sp)$-open in Y. Thus, by (3), $[F^{-}(K)]^{(\Lambda,sp)} \subseteq F^{-}(K^{(\Lambda,sp)}) = F^{-}(K)$ and hence $F^{-}(K)$ is (Λ, sp)-closed in X. By Theorem 3, F is upper almost (Λ, sp)-continuous.

Theorem 6. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is lower almost* (Λ, sp)*-continuous;*

(2) $[F^{+}(V)]^{(\Lambda,sp)} \subseteq F^{+}(V^{(\Lambda,sp)})$ *for every* $\beta(\Lambda, sp)$*-open set* V *of* Y*;*

(3) $[F^{+}(V)]^{(\Lambda,sp)} \subseteq F^{+}(V^{(\Lambda,sp)})$ *for every* $s(\Lambda, sp)$*-open set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 5.

Corollary 3. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost (Λ, sp)-continuous;

(2) $[f^{-1}(V)]^{(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ for every $\beta(\Lambda, sp)$-open set V of Y;

(3) $[f^{-1}(V)]^{(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ for every $s(\Lambda, sp)$-open set V of Y.

Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a $\delta(\Lambda, sp)$-*cluster point* [19] of A if $A \cap [U^{(\Lambda,sp)}]_{(\Lambda,sp)} \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all $\delta(\Lambda, sp)$-cluster points of A is called the $\delta(\Lambda, sp)$-*closure* [19] of A and is denoted by $A^{\delta(\Lambda,sp)}$. If $A = A^{\delta(\Lambda,sp)}$, then A is said to be $\delta(\Lambda, sp)$-*closed* [19]. The complement of a $\delta(\Lambda, sp)$-closed set is said to be $\delta(\Lambda, sp)$-*open*. The union of all $\delta(\Lambda, sp)$-open sets contained in A is called the $\delta(\Lambda, sp)$-*interior* [19] of A and is denoted by $A_{\delta(\Lambda,sp)}$.

Lemma 5. [19] *Let A be a subset of a topological space (X, τ). Then, the following properties hold:*

(1) If A is (Λ, sp)-open in X, then $A^{(\Lambda,sp)} = A^{\delta(\Lambda,sp)}$.

(2) $A^{\delta(\Lambda,sp)}$ is (Λ, sp)-closed.

Theorem 7. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost (Λ, sp)-continuous;

(2) $[F^{-}([[B^{\delta(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{-}(B^{\delta(\Lambda,sp)})$ for every subset B of Y;

(3) $[F^{-}([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{-}(B^{\delta(\Lambda,sp)})$ for every subset B of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 5, $B^{\delta(\Lambda,sp)}$ is (Λ, sp)-closed in Y and by Theorem 3, $[F^{-}([[B^{\delta(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{-}(B^{\delta(\Lambda,sp)})$.

(2) ⇒ (3): This is obvious since $B^{(\Lambda,sp)} \subseteq B^{\delta(\Lambda,sp)}$.

(3) ⇒ (1): Let K be any $r(\Lambda, sp)$-closed set of Y. Thus, by (3), we have $[F^{-}(K)]^{(\Lambda,sp)} = [F^{-}([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} = [F^{-}([[K^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{-}(K^{\delta(\Lambda,sp)}) = F^{-}(K)$ and hence $F^{-}(K)$ is (Λ, sp)-closed in X. By Theorem 3, F is upper almost (Λ, sp)-continuous.

Theorem 8. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost (Λ, sp)-continuous;

(2) $[F^{+}([[B^{\delta(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{+}(B^{\delta(\Lambda,sp)})$ for every subset B of Y;

(3) $[F^{+}([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^{+}(B^{\delta(\Lambda,sp)})$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 7.

Corollary 4. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost (Λ, sp)-continuous;

(2) $[f^{-1}([[B^{\delta(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(B^{\delta(\Lambda,sp)})$ for every subset B of Y;

(3) $[f^{-1}([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(B^{\delta(\Lambda,sp)})$ for every subset B of Y.

Definition 3. [3] *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is called lower (Λ, sp)-continuous at a point $x \in X$ if, for each (Λ, sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a (Λ, sp)-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is called lower (Λ, sp)-continuous if F has this property at each point of X.*

Lemma 6. *If $F : (X, \tau) \to (Y, \sigma)$ is lower almost (Λ, sp)-continuous, then for each $x \in X$ and each subset B of Y with $F(x) \cap B_{\delta(\Lambda,sp)} \neq \emptyset$, there exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that $U \subseteq F^{-}(B)$.*

Proof. Let $x \in X$ and let B be a subset of Y with $F(x) \cap B_{\delta(\Lambda,sp)} \neq \emptyset$. Since $F(x) \cap B_{\delta(\Lambda,sp)} \neq \emptyset$, there exists a nonempty $r(\Lambda, sp)$-open set V of Y such that $V \subseteq B$ and $F(x) \cap V \neq \emptyset$. Since F is lower almost (Λ, sp)-continuous, there exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$; hence $U \subseteq F^{-}(B)$.

Theorem 9. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost (Λ, sp)-continuous;

(2) $[F^{+}(B)]^{(\Lambda,sp)} \subseteq F^{+}(B^{\delta(\Lambda,sp)})$ for every subset B of Y;

(3) $F(A^{(\Lambda,sp)}) \subseteq [F(A)]^{\delta(\Lambda,sp)}$ for every subset A of X;

(4) $F^{+}(K)$ is (Λ, sp)-closed in X for every $\delta(\Lambda, sp)$-closed set K of Y;

(5) $F^{-}(V)$ is (Λ, sp)-open in X for every $\delta(\Lambda, sp)$-open set V of Y;

(6) $F^{-}(B_{\delta(\Lambda,sp)}) \subseteq [F^{-}(B)]_{(\Lambda,sp)}$ for each subset B of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. Suppose that $x \notin F^{+}(B^{\delta(\Lambda,sp)})$. Then, we have $x \in F^{-}(Y - B^{\delta(\Lambda,sp)}) = F^{-}([Y - B]_{\delta(\Lambda,sp)})$. By Lemma 6, there exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that $U \subseteq F^{-}(Y - B) = X - F^{+}(B)$. Thus, $U \cap F^{+}(B) = \emptyset$ and hence $x \in X - [F^{+}(B)]^{(\Lambda,sp)}$. This shows that $[F^{+}(B)]^{(\Lambda,sp)} \subseteq F^{+}(B^{\delta(\Lambda,sp)})$.

(2) ⇒ (3): Let A be any subset of X. By (2), we have $A^{(\Lambda,sp)} \subseteq [F^{+}(F(A))]^{(\Lambda,sp)} \subseteq F^{+}([F(A)]^{\delta(\Lambda,sp)})$ and hence $F(A^{(\Lambda,sp)}) \subseteq [F(A)]^{\delta(\Lambda,sp)}$.

(3) ⇒ (1): Let B be any subset of Y. Then, by the hypothesis and Lemma 5,

$$F([F^{+}([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)}) \subseteq [F(F^{+}([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}))]^{\delta(\Lambda,sp)}$$

$$\subseteq [[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$$
$$\subseteq B^{(\Lambda,sp)}$$

and hence $[F^+([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(B^{(\Lambda,sp)})$. By Theorem 4, F is lower almost (Λ, sp)-continuous.

(2) ⇒ (4): Let K be any $\delta(\Lambda, sp)$-closed set of Y. Then, $K^{\delta(\Lambda,sp)} = K$. By (2), we have $[F^+(K)]^{(\Lambda,sp)} \subseteq F^+(K^{\delta(\Lambda,sp)}) = F^+(K)$ and hence $F^+(K)$ is (Λ, sp)-closed in X.

(4) ⇒ (5): The proof is obvious.

(5) ⇒ (6): Let B be any subset of Y. By (5), we have

$$F^-(B_{\delta(\Lambda,sp)}) = [F^-(B_{\delta(\Lambda,sp)})]_{(\Lambda,sp)}$$
$$\subseteq [F^-(B)]_{(\Lambda,sp)}.$$

(6) ⇒ (1): Let V be any $r(\Lambda, sp)$-open set of Y. Then, we have V is $\delta(\Lambda, sp)$-open and $V_{\delta(\Lambda,sp)} = V$. Thus, by (6), $F^-(V) \subseteq [F^-(V)]_{(\Lambda,sp)}$ and hence $F^-(V)$ is (Λ, sp)-open in X. By Theorem 4, F is lower almost (Λ, sp)-continuous.

Definition 4. [19] *A topological space (X, τ) is said to be $s(\Lambda, sp)$-regular if, for each $s(\Lambda, sp)$-closed set F and each $x \notin F$, there exist disjoint $s(\Lambda, sp)$-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 7. [19] *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, where (Y, σ) is a $s(\Lambda, sp)$-regular space, the following properties are equivalent:*

(1) F is lower (Λ, sp)-continuous;

(2) $F^+(B^{\delta(\Lambda,sp)})$ is (Λ, sp)-closed in X for every subset B of Y;

(3) $F^+(K)$ is (Λ, sp)-closed in X for every $\delta(\Lambda, sp)$-closed set K of Y;

(4) $F^-(V)$ is (Λ, sp)-open in X for every $\delta(\Lambda, sp)$-open set V of Y.

Lemma 8. [19] *Let (X, τ) be a $s(\Lambda, sp)$-regular space. Then, the following properties hold:*

(1) $A^{(\Lambda,sp)} = A^{\delta(\Lambda,sp)}$ for every subset A of X.

(2) Every (Λ, sp)-open set is $\delta(\Lambda, sp)$-open.

Theorem 10. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, where (Y, σ) is a $s(\Lambda, sp)$-regular space, the following properties are equivalent:*

(1) F is lower (Λ, sp)-continuous;

(2) $F^+(B^{\delta(\Lambda,sp)})$ is (Λ, sp)-closed in X for every subset B of Y;

(3) $F^+(K)$ is (Λ, sp)-closed in X for every $\delta(\Lambda, sp)$-closed set K of Y;

(4) $F^-(V)$ is (Λ, sp)-open in X for every $\delta(\Lambda, sp)$-open set V of Y;

(5) F is lower almost (Λ, sp)-continuous.

Proof. The proofs of the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are similar to those in Lemma 7.

(4) ⇒ (5): Let V be any $r(\Lambda, sp)$-open set of Y. Then, V is (Λ, sp)-open in Y and by Lemma 8, V is $\delta(\Lambda, sp)$-open. Thus, by (4), $F^{-}(V)$ is (Λ, sp)-open in X and by Theorem 4, F is lower almost (Λ, sp)-continuous.

(5) ⇒ (1): Let $x \in X$ and V be any (Λ, sp)-open set of Y such that $F(x) \cap V \neq \emptyset$. Since (Y, σ) is $s(\Lambda, sp)$-regular, there exists a $r(\Lambda, sp)$-open set W such that $F(x) \cap W \neq \emptyset$ and $W \subseteq V$. Since F is lower almost (Λ, sp)-continuous, there exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that $F(z) \cap W \neq \emptyset$ for every $z \in U$. Thus, $F(z) \cap V \neq \emptyset$ for every $z \in U$. This shows that F is lower (Λ, sp)-continuous.

Definition 5. [20] *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (Λ, sp)-continuous at a point $x \in X$ if, for each (Λ, sp)-open set V of Y containing $f(x)$, there exists a (Λ, sp)-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (Λ, sp)-continuous if f has this property at each point $x \in X$.*

Corollary 5. *For a function $f : (X, \tau) \to (Y, \sigma)$, where (Y, σ) is a $s(\Lambda, sp)$-regular space, the following properties are equivalent:*

(1) f is (Λ, sp)-continuous;

(2) $f^{-1}(B^{\delta(\Lambda, sp)})$ is (Λ, sp)-closed in X for every subset B of Y;

(3) $f^{-1}(K)$ is (Λ, sp)-closed in X for every $\delta(\Lambda, sp)$-closed set K of Y;

(4) $f^{-1}(V)$ is (Λ, sp)-open in X for every $\delta(\Lambda, sp)$-open set V of Y;

(5) f is almost (Λ, sp)-continuous.

4. Conclusion

In topology, there has been recently significant interest in characterizing and investigating the properties of several weak forms of continuity for functions and multifunctions. The development of such a theory is in fact very well motivated. This work is concerned with the concept of upper (resp. lower) almost (Λ, sp)-continuous multifunctions. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is called upper (resp. lower) almost (Λ, sp)-continuous multifunctions if, for each $x \in X$ and each (Λ, sp)-open set V of Y such that $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there exists a (Λ, sp)-open set U of X containing x such that $U \subseteq F^{+}([V^{(\Lambda, sp)}]_{(\Lambda, sp)})$ (resp. $U \subseteq F^{-}([V^{(\Lambda, sp)}]_{(\Lambda, sp)})$). Several characterizations and some properties concerning upper (resp. lower) almost (Λ, sp)-continuous multifunctions are established. The ideas and results of this work may motivate further research.

Acknowledgements

This research project was financially supported by Mahasarakham University.

References

[1] D. Andrijević. On b-open sets. *Matematički Vesnik*, 48:59–64, 1996.

[2] C. Berge. *Espaces topologiques fonctions multivoques.* Dunod, Paris, 1959.

[3] C. Boonpok. (Λ, sp)-closed sets and related topics in topological spaces. *WSEAS Transactions on Mathematics*, 19:312–322, 2020.

[4] C. Boonpok, C. Viriyapong, and M. Thongmoon. On upper and lower (τ_1, τ_2)- precontinuous multifunctions. *Journal of Mathematics and Computer Science*, 18:282–293, 2018.

[5] E. Ekici and J. H. Park. A weak form of some types of continuous multifunctions. *Filomat*, 20(2):13–32, 2006.

[6] M. E. Abd El-Monsef, S. N. El-Deeb, and R. A. Mahmoud. β-open sets and β-continuous mappings. *Bulletin of the Faculty of Science. Assiut University.*, 12:77–90, 1983.

[7] T. Noiri and E. Hatir. Λ_{sp}-sets and some weak separation axioms. *Acta Mathematica Hungarica*, 103(3):225–232, 2004.

[8] T. Noiri and V. Popa. Characterizations of almost quasi-continuous multifunctions. *Research Reports of Yatsushiro National College of Technology*, 15:97–101, 1993.

[9] T. Noiri and V. Popa. On upper and lower almost β-continuous multifunctions. *Acta Mathematica Hungarica*, 82:57–73, 1999.

[10] T. Noiri and V. Popa. A unified theory of almost continuity for multifunctions. *Scientific Studies and Research, Series Mathematics and Informatics*, 20(1):185–214, 2010.

[11] V. I. Ponomarev. Properties of topological spaces preserved under multivalued continuous mappings on compacta. *American Mathematical Society Translations*, 38(2):119–140, 1964.

[12] V. Popa. Almost continuous multifunctions. *Matematički Vesnik*, 6(9)(34):75–84, 1982.

[13] V. Popa and T. Noiri. On upper and lower almost quasi-continuous multifunctions. *Bulletin of the Institute of Mathematics, Academia Sinica*, 21:337–349, 1993.

[14] V. Popa and T. Noiri. On upper and lower almost α-continuous multifunctions. *Demonstratio Mathematica*, 29:381–396, 1996.

[15] V. Popa and T. Noiri. On upper and lower weakly β-continuous multifunctions. *Annales Universitatis Scientiarum Budapestinensis*, 43:25–48, 2000.

[16] V. Popa, T. Noiri, and M. Ganster. On upper and lower almost precontinuous multifunctions. *Far East Journal of Mathematical Sciences*, Special Volume(Part I):49–68, 1997.

[17] V. Popa, T. Noiri, M. Ganster, and K. Dlaska. On upper and lower θ-irresolute multifunctions. *Journal of Institute of Mathematics & Computer Sciences. Mathematics Series*, 6:137–149, 1993.

[18] M. K. Singal and A. R. Singal. Almost continuous mappings. *Yokohama Mathematical Journal*, 16:63–73, 1968.

[19] C. Viriyapong and C. Boonpok. (Λ, sp)-continuity and $\delta(\Lambda, sp)$-closed sets. *International Journal of Mathematics and Computer Science*, 17(4):1685–1689, 2022.

[20] C. Viriyapong and C. Boonpok. (Λ, sp)-continuous functions. *WSEAS Transactions on Mathematics*, 21:380–385, 2022.

[21] C. Viriyapong and C. Boonpok. Weak quasi (Λ, sp)-continuity for multifunctions. *International Journal of Mathematics and Computer Science*, 17(3):1201–1209, 2022.