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Hop Independent Hop Domination in Graphs

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Abstract. Let G be an undirected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A set $S \subseteq V(G)$ is called a hop independent hop dominating set of G if S is both hop independent and hop dominating set of G. The minimum cardinality of hop independent hop dominating set of G, denoted by $\gamma_{hih}(G)$, is called the hop independent hop domination number of G. In this paper, we show that the hop independent hop domination number of a graph G lies between the hop domination number and the hop independence number of graph G. We characterize these types of sets in the shadow graph, join, corona, and lexicographic product of two graphs. Moreover, either exact values or bounds of the hop independent hop domination numbers of these graphs are given.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Hop domination, hop domination number, hop independent set, hop independence number, hop independent hop domination number

1. Introduction

Since its introduction in 2015 (see [10]), hop domination has been one of the topics of research or investigation in the area. To date, a number of variants of hop domination have been introduced and studied. The newly defined variations and parameters are studied in many classes of graphs including those which result from some binary operations (see [1], [2], [4], [5], [6], [7], [9], and [11]).

Recently, Hassan et al. [3] introduced the concept of hop independent set in a graph and defined the parameter called hop independence number of a graph. The hop independence number may be equal or less or greater than the independence number of graph. A result in [3] shows that the absolute difference between the independence number and the hop independence number of a graph can be made arbitrarily large. Moreover, it was pointed out that the maximum hop independent set in a graph G forms a hop dominating set, that is, the hop independence number of a graph G is at least equal to the hop domination

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number of G. This observation, the way the variation "independent domination" for the standard domination concept was introduced, and the possible future applications it may offer researchers in the field, are some of the motivations the authors have for introducing and studying hop independent hop domination in a graph. The newly defined concept and the corresponding parameter will be studied initially for some graphs including those that result from some binary operations.

2. Terminology and Notation

Two vertices u, v of a graph G are *adjacent*, or *neighbors*, if uv is an edge of G. Moreover, an edge uv of G is *incident* to two vertices u, v of G. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u in G. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{u \in X} N_G(u)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

Let G be a graph. A set $D \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. Any dominating set D of G with cardinality $\gamma(G)$, is called a γ-set of G.

Let G be a graph. A set $D \subseteq V(G)$ is a *total dominating set* of G if for every $v \in V(G)$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G(D) = V(G)$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Any total dominating set with cardinality equal to $\gamma_t(G)$ is called a γ_t-set.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G^2(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G^2[u] = N_G^2(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G^2(X) = \bigcup_{u \in X} N_G^2(u)$. The *closed hop neighborhood* of X in G is the set $N_G^2[X] = N_G^2(X) \cup X$.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if $N_G^2[S] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A nonempty set $S \subseteq V(G)$ is a *hop independent set* of G if $d_G(x, y) \neq 2$ for any two vertices $x, y \in S$. The *hop independence number of* G, denoted by $\alpha_h(G)$, is the largest cardinality of a hop independent set of G. Any hop independent set of G with cardinality $\alpha_h(G)$ is called an α_h-set of G.

A set $S \subseteq V(G)$ is called a *hop independent hop dominating set* of G if S is both hop independent and hop dominating set of G. The minimum cardinality of a hop independent hop dominating set of G, denoted by $\gamma_{hih}(G)$, is called the *hop independent hop domination number* of G. Any hop independent hop dominating set of G with cardinality $\gamma_{hih}(G)$ is called a γ_{hih}-set of G.

A set $S \subseteq V(G)$ is a *clique* if the subgraph $\langle S \rangle$ induced by S is a complete graph. The maximum cardinality of a clique of G, denoted by $\omega(G)$, is called the clique number of G. A set $C \subseteq V(G)$ is a *pointwise non-dominating set* if for every $v \in V(G) \setminus C$, there exists $u \in C$ such that $v \notin N_G(u)$. The minimum cardinality of a pointwise non-dominating set of G, denoted by $pnd(G)$, is called a *pointwise non-domination number* of G.

A set $S \subseteq V(G)$ is a *clique pointwise non-dominating set* if S is both a clique and a pointwise non-dominating set of G. The smallest cardinality of a clique pointwise non-dominating set of G, denoted by $cpnd(G)$, is called the *clique pointwise non-domination number* of G. Any clique pointwise non-dominating set of G with cardinality $cpnd(G)$ is called a $cpnd$-set of G.

A total dominating set $D \subseteq V(G)$ is called a *total dominating hop independent hop dominating set* if it is a hop independent hop dominating set. The minimum cardinality of a total dominating hop independent hop dominating set of G is denoted by $\gamma_t^{hih}(G)$. Any total dominating hop independent hop dominating set D with cardinality $\gamma_t^{hih}(G)$ is referred to as a γ_t^{hih}-set.

The *shadow graph* $S(G)$ of graph G is constructed by taking two copies of G, say G_1 and G_2, and then joining each vertex $u \in G_1$ to the neighbors of its corresponding vertex $u' \in G_2$.

Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} \rangle + H^v$. The *lexicographic product* $G[H]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(v,a)(u,b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$.

Note that any non-empty set $C \subseteq V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Specifically, $T_x = \{a \in V(H) : (x,a) \in C\}$ for each $x \in S$.

3. Results

Proposition 1. *Let G be any graph on n vertices. Then*

$$\gamma_h(G) \leq \gamma_{hih}(G) \leq \alpha_h(G).$$

Proof. Since every hop independent hop dominating set S is a hop dominating set, it follows that $\gamma_h(G) \leq \gamma_{hih}(G)$. Also, since every α_h-set is a hop independent hop dominating set, the right inequality holds. $\square$

Remark 1. *The bounds given in Proposition 1 are tight. Moreover, strict inequalities can also be attained.*

To see this, consider $G = C_4$. Then $\gamma_h(G) = \gamma_{hih}(G) = \alpha_h(G) = 2$.

For the strict inequalities, let $S_1 = \{c, e, g\}$ (see Figure 1). Then S_1 is a minimum hop dominating set of G. Since $d_G(c, g) = 2$, it follows that S_1 is not a hop independent set. Next, let $S_2 = \{a, b, g, h\}$(see Figure 2). Then S_2 is a minimum hop independent hop dominating set of G. However, S_2 is not a maximum hop independent set of G. In fact, $S_3 = \{a, b, h, i, j\}$ (see Figure 3) is a maximum hop independent set of G. Consequently, $\gamma_h(G) = 3 < 4 = \gamma_{hih}(G) < 5 = \alpha_h(G)$.

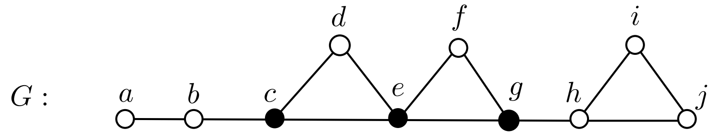

Figure 1: A graph G with $\gamma_h(G) < \gamma_{hih}(G) < \alpha_h(G)$

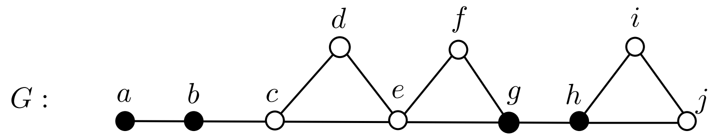

Figure 2: A graph G with $\gamma_h(G) < \gamma_{hih}(G) < \alpha_h(G)$

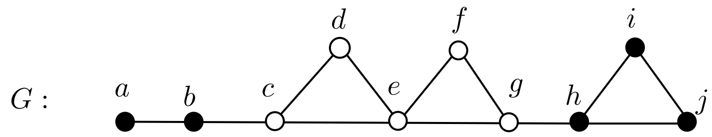

Figure 3: A graph G with $\gamma_h(G) < \gamma_{hih}(G) < \alpha_h(G)$

Theorem 1. *[3] Let G be any graph on n vertices. Then S is a hop independent set of G if and only if every component of $\langle S \rangle$ is complete. Moreover, $\alpha_h(G) = n$ if and only if every component of G is complete.*

Theorem 2. *Let G be a graph. Then $\gamma_h(G) = \gamma_{hih}(G)$ if and only if G has a γ_h-set S such that every component of $\langle S \rangle$ is complete.*

Proof. Suppose $\gamma_h(G) = \gamma_{hih}(G)$ and let S be a γ_{hih}-set of G. Then S is a hop independent set of G. Hence, by Theorem 1, every component of $\langle S \rangle$ is complete. By assumption, S is a γ_h-set of G.

Conversely, suppose G has a γ_h-set S such that every component of $\langle S \rangle$ is complete. Then by Theorem 1, S is a hop independent set. Since S is a hop dominating set, it follows that S is a hop independent hop dominating set. Hence, $\gamma_{hih}(G) \leq |S| = \gamma_h(G)$. By Proposition 1, $\gamma_h(G) = \gamma_{hih}(G)$. $\square$

The next result is immediate from Theorem 2.

Corollary 1. *Let $n \geq 3$ be any positive integer. Then*

(i) $\gamma_{hih}(P_n) = \gamma_h(P_n)$ *and*

(ii) $\gamma_{hih}(C_n) = \gamma_h(C_n)$.

Theorem 3. *Let G be any graph on $n \geq 2$ vertices. Then $2 \leq \gamma_{hih}(G) \leq n$. Moreover, the following statements hold.*

(i) $\gamma_{hih}(G) = 2$ *if and only if* $\gamma_h(G) = 2$.

(ii) $\gamma_{hih}(G) = n$ *if and only if every component C of G is complete.*

Proof. Clearly, $2 \leq \gamma_{hih}(G) \leq n$.

(i) Suppose $\gamma_{hih}(G) = 2$. By Proposition 1, $\gamma_h(G) \leq \gamma_{hih}(G) = 2$. Since $\gamma_h(G) \geq 2$ for any graph of order $n \geq 2$, it follows that $\gamma_h(G) = 2$.

Conversely, suppose $\gamma_h(G) = 2$, say $S = \{a, b\}$ is a γ_h-set. Suppose, on the contrary, that S is not a hop independent set, that is $d_G(a, b) = 2$. Then there exists $x \in V(G)$ such that $x \in N_G(a) \cap N_G(b)$. This implies that $x \notin N_G^2(a) \cup N_G^2(b)$, contrary to the assumption that S is a hop dominating set. Therefore, S is a hop independent set and $\gamma_{hih}(G) \leq 2$. Consequently, $\gamma_{hih}(G) = 2$.

(ii) Suppose $\gamma_{hih}(G) = n$. Then $S = V(G) = \{v_1, \ldots, v_n\}$ is the only hop independent hop dominating set of G. Thus, by Theorem 1, every component of $\langle S \rangle = G$ is complete.

For the converse, suppose that every component C of G is complete. Then by Proposition 1, $n = \gamma_h(G) \leq \gamma_{hih}(G) \leq \alpha_h(G) = n$. Hence, $\gamma_{hih}(G) = n$. □

The next result is a direct consequence of Theorem 3.

Corollary 2. *Let n be a positive integer. Then each of the following statements holds.*

(i) $\gamma_{hih}(\overline{K}_n) = n$.

(ii) $\gamma_{hih}(K_n) = n$.

(iii) $\gamma_{hih}(G) + \gamma_{hih}(\overline{G}) = 2n$ *if* $G = K_n$.

(iv) $\gamma_{hih}(G) \leq n - 1$ *if G is a graph on n vertices and has at least one non-complete component.*

Proposition 2. *Let G be a graph on $n \geq 3$ vertices. If G has at least one non-complete component, then*

(i) $4 \leq \gamma_{hih}(G) + \gamma_{hih}(\overline{G}) \leq 2n - 1$, *and*

(ii) $4 \leq \gamma_{hih}(G) \cdot \gamma_{hih}(\overline{G}) \leq n^2 - n$.

Proof. By Corollary 2(iv), $\gamma_{hih}(G) \leq n-1$ and by Theorem 3, $\gamma_{hih}(\overline{G}) \leq n$. These imply that $\gamma_{hih}(G)+\gamma_{hih}(\overline{G}) \leq n-1+n = 2n-1$ and $\gamma_{hih}(G)\cdot\gamma_{hih}(\overline{G}) \leq (n-1)n = n^2-n$. Since $\gamma_{hih}(G) \geq 2$ for any graph of order at least 2, the left inequalities follow. □

Theorem 4. *Let G be any graph of order $n \geq 3$. Then $\gamma_{hih}(G) = n-1$ if and only if all but one component H of G are cliques, where $H = K_{|V(H)|} \setminus e$ (H is obtained from $K_{|V(H)|}$ by deleting an edge e).*

Proof. Suppose $\gamma_{hih}(G) = n-1$. Let $S = V(G) \setminus \{v\}$ be a γ_{hih}-set. Let $G_1, G_2, \ldots, G_k$ be the components of $\langle S \rangle$. Then each G_i is a clique. Since $v \notin S$ and S is a hop dominating set, there exists $w \in S$ such that $d_G(v,w) = 2$. Assume that $w \in V(G_1)$. Let $z \in N_G(v) \cap N_G(w)$. Then $z \in V(G_1)$. Suppose there exists $j \neq 1$ such that $vp \in E(G)$ for some $p \in V(G_j)$. Then $S^* = V(G) \setminus \{v,p\}$ is a hop independent hop dominating set in G, contrary to the assumption that $\gamma_{hih}(G) = n-1$. This implies that the components of G are $\langle V(G_1) \cup \{v\} \rangle, G_2, \ldots, G_k$. Now, we will show that $xv \in E(G)$ for every $x \in V(G_1) \setminus \{w\}$. Suppose there exists $q \in V(G_1) \setminus \{w\}$ such that $qv \notin E(G)$. Then $d_G(q,v) = 2$. Let $D = \{t \in V(G_1) : vt \notin E(G)\}$. Then $|D| \geq 2$ and $\langle V(G_1) \setminus D \rangle$ is a clique. Notice that $d_G(v,t) = 2$ for every $t \in D$. Now, let $S' = V(G) \setminus D$. Then S' is a hop dominating set. Since the components of $\langle S' \rangle$ are $\langle V(G_1) \setminus D \rangle, G_2, \ldots, G_k$, it follows that S' is a hop independent set. Hence, $\gamma_{hih}(G) \leq |S'| = n - |D| \leq n-2$, a contradiction. Hence, $xv \in E(G)$ for every $x \in V(G_1) \setminus \{w\}$. This implies that $H = \langle V(G_1) \cup \{v\} \rangle = K_{V(H)} \setminus e$, where $e = vw$.

The converse is clear. □

The next result follows from Theorem 4.

Corollary 3. *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{hih}(G) = n-1$ if and only if $G = K_n \setminus e$ for some $e \in E(K_n)$.*

Theorem 5. *Let G be a non-trivial connected graph. Then S is a hop independent hop dominating set of $S(G)$ if and only if one of the following conditions holds:*

(i) *S is a hop independent hop dominating set in G_1.*

(ii) *S is a hop independent hop dominating set in G_2.*

(iii) *$S = S_{G_1} \cup S_{G_2}$ such that $S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are hop independent hop dominating sets in G_1 and G_2, respectively, where $S'_{G_2} = \{a \in V(G_1) : a' \in S_{G_2}\}$ and $S'_{G_1} = \{a \in V(G_2) : a' \in S_{G_1}\}$.*

Proof. Let S be a hop independent hop dominating set of $S(G)$. Set $S_{G_1} = S \cap V(G_1)$ and $S_{G_2} = S \cap V(G_2)$. If $S_{G_2} = \varnothing$, then $S = S_{G_1}$ is a hop independent hop dominating set of G_1. If $S_{G_1} = \varnothing$, then $S = S_{G_2}$ is a hop independent hop dominating set of G_2. Hence, (i) or (ii) holds. Next, suppose S_{G_1} and S_{G_2} are both non-empty. Let $S'_{G_2} = \{a \in V(G_1) : a' \in S_{G_2}\}$. Suppose $S_{G_1} \cup S'_{G_2}$ is not a hop independent set in G_1. Then there exist $p,q \in S_{G_1} \cup S'_{G_2}$ such that $d_{G_1}(p,q) = 2 = d_{S(G)}(p,q)$. Since S_{G_1}

and S_{G_2} are hop independent sets, we may assume that $p \in S_{G_1}$ and $q \in S'_{G_2}$. Then $q' \in S_{G_2}$ and $d_{S(G)}(p, q') = 2$, contrary to the assumption that S is a hop independent set. Thus, $S_{G_1} \cup S'_{G_2}$ is a hop independent set. Next, let $x \in V(G_1) \setminus S_{G_1} \cup S'_{G_2}$. Then $x \in V(S(G)) \setminus S$. Since S is a hop dominating of $S(G)$, there exists $y \in S$ such that $d_{S(G)}(x, y) = 2$. If $y \in S_{G_1}$, then we are done. Suppose $y \in S_{G_2}$, say $y = z'$, where $z \in V(G_1)$. Then $z \in S'_{G_2}$ and $d_{S(G)}(x, z) = d_{G_1}(x, z) = 2$. Therefore, $S_{G_1} \cup S'_{G_2}$ is a hop dominating set. Consequently, $S_{G_1} \cup S'_{G_2}$ is a hop independent hop dominating set of G_1. Similarly, $S'_{G_1} \cup S_{G_2}$ is a hop independent hop dominating set of G_2. Hence, (iii) holds.

For the converse, suppose (i) holds. Then S is a hop independent set of $S(G)$. Let $a \in V(S(G)) \setminus S$. If $a \in V(G_1) \setminus S$, then there exists $b \in S$ such that

$$d_{G_1}(a, b) = d_{S(G)}(a, b) = 2.$$

Suppose $a \in V(G_2)$, say $a = v'$, where $v \in V(G_1)$. If $v \in S$, then

$$d_{G_1}(a, v) = d_{S(G)}(a, v) = 2.$$

If $v \notin S$, then there exists $w \in S$ such that $d_{G_1}(v, w) = 2$. It follows that

$$d_{S(G)}(a, w) = d_{S(G)}(v', w) = 2.$$

Therefore, S is a hop independent hop dominating set of $S(G)$. Similarly, if (ii) holds, then S is a hop independent hop dominating set of $S(G)$. Now, suppose (iii) holds. Suppose further that $S = S_{G_1} \cup S_{G_2}$ is not a hop independent set in $S(G)$. Then there exist $a, b \in S$ such that $d_{S(G)}(a, b) = 2$. Since S_{G_1} and S_{G_2} are hop independent sets, we may assume that $a \in S_{G_1}$ and $b \in S_{G_2}$. Then $d_{G_2}(a', b) = 2$, contrary to the assumption that $S'_{G_1} \cup S_{G_2}$ is a hop independent set in G_2. Therefore, S is a hop independent set $S(G)$. Next, let $y \in V(S(G)) \setminus S$. Then $y \notin S_{G_1} \cup S_{G_2}$. Suppose $y \in V(G_2) \setminus S_{G_2}$, say $y = z'$, where $z \in V(G_1)$. Then $z \notin S'_{G_2}$. If $z \in S_{G_1}$, then $d_{S(G)}(y, z) = d_{S(G)}(z', z) = 2$. Suppose $z \notin S_{G_1}$. Since $S_{G_1} \cup S'_{G_2}$ is a hop dominating set of G_1, there exists $p \in S_{G_1} \cup S'_{G_2}$ such that $d_{G_1}(p, z) = 2 = d_{S(G)}(p, z)$. If $p \in S_{G_1}$, then $p \in S$ and $d_{S(G)}(p, z') = 2$. If $p \in S'_{G_2}$, then $p' \in S_{G_2} \subseteq S$ and $d_{G_2}(p', z') = d_{S(G)}(p', z') = 2$. Therefore, S is a hop dominating set of $S(G)$. Consequently, S is a hop independent hop dominating set of $S(G)$. □

Corollary 4. *Let G be a non-trivial connected graph. Then $\gamma_{hih}(S(G)) = \gamma_{hih}(G)$. In particular, we have*

(i) $\gamma_{hih}(S(K_n)) = \gamma_{hih}(K_n) = n$ *for every* $n \geq 2$,

(ii) $\gamma_{hih}(S(P_n)) = \gamma_{hih}(P_n) = \gamma_h(P_n)$ *for every* $n \geq 2$, *and*

(iii) $\gamma_{hih}(S(C_n)) = \gamma_{hih}(C_n) = \gamma_h(C_n)$ *for evey* $n \geq 3$.

Proof. Let S be a γ_{hih}-set of G. Then by Theorem 5, S is a hop independent hop dominating set of $S(G)$. Thus, $\gamma_{hih}(S(G)) \leq |S| = \gamma_{hih}(G)$.

On the other hand, suppose S^* is a γ_{hih}-set of $S(G)$. If S^* is of type (i) or (ii), then S^* is a hop independent hop dominating set of G by (i) and (ii) of Theorem 5. Hence, $\gamma_{hih}(S(G)) = |S^*| \geq \gamma_{hih}(G)$. Suppose S^* is of type (iii), say $S^* = S_{G_1} \cup S_{G_2}$. Then $S^*_G = S_{G_1} \cup S'_{G_2}$ is a hop independent hop dominating set of G_1 by Theorem 5(iii). This implies that $\gamma_{hih}(S(G)) = |S^*| = |S^*_G| \geq \gamma_{hih}(G)$. Consequently, $\gamma_{hih}(S(G)) = \gamma_{hih}(G)$.

Statements $(i), (ii)$ and (iii) follow from Corollary 1, Theorem 3 and Theorem 5. □

Lemma 1. *[8] Let G be a graph of order n. Then every component of $S(G)$ is a complete graph if and only if $G = \overline{K}_n$.*

Theorem 6. *Let G be a graph on n vertices. Then $2 \leq \gamma_{hih}(S(G)) \leq 2n$. Moreover, $\gamma_{hih}(S(G)) = 2n$ if and only if $G = \overline{K}_n$.*

Proof. Since $2 \leq |V(S(G))| \leq 2n$, it follows that $2 \leq \gamma_{hih}(S(G)) \leq 2n$ by Theorem 3. Suppose $\gamma_{hih}(S(G)) = 2n$. Then every component of $S(G)$ is a complete graph by Theorem 3. Hence, $G = \overline{K}_n$ by Lemma 1.

For the converse, suppose $G = \overline{K}_n$. Then $S(G) = \overline{K}_{2n}$. Hence, $\gamma_{hih}(S(G)) = 2n$ by Theorem 3. □

Theorem 7. *Let G be a graph of order n. Then $1 \leq cpnd(G) \leq n$. Moreover,*

(i) *$cpnd(G) = 1$ if and only if G has an isolated vertex.*

(ii) *$cpnd(G) = 2$ if and only if G has no isolated vertex and there exist adjacent vertices x and y of G such that $N_G(x) \cap N_G(y) = \varnothing$.*

(iii) *$cpnd(G) = n$ if and only if G is a complete graph.*

Proof. Celarly, $1 \leq cpnd(G) \leq n$.

(i) Suppose $cpnd(G) = 1$, say $\{p\}$ is a *cpnd*-set of G. Clearly, p is an isolated vertex of G.

Conversely, suppose G has an isolated vertex, say a. Then $\{a\}$ is a *cpnd*-set of G. Hence, $cpnd(G) = 1$.

(ii) Suppose $cpnd(G) = 2$, say $C = \{x, y\}$ is a clique pointwise non-dominating set of G. Then x and y are adjacent. By (i), G has no isolated vertex. Now, suppose that $N_G(x) \cap N_G(y) \neq \varnothing$. Let $a \in N_G(x) \cap N_G(y)$. Then $a \in V(G) \setminus C$. Since a is adjacent to both x and y, it follows that C is not a pointwise non-dominating set, a contradiction. Hence, $N_G(x) \cap N_G(y) = \varnothing$.

Conversely, suppose G has no isolated vertex and there exist adjacent vertices x and y of G such that $N_G(x) \cap N_G(y) = \varnothing$. Let $C = \{x, y\}$ and $a \in V(G) \setminus C$. By assumption, a is not adjacent to x or y. This implies that C is a clique pointwise non-dominating set, and so $cpnd(G) \leq 2$. Since G has no isolated vertex, $cpnd(G) \geq 2$. Consequently, $cpnd(G) = 2$.

(iii) Suppose $cpnd(G) = n$. Then $C = V(G)$ is the only clique pointwise non-dominating set of G. It follows that G is a complete graph.

The converse is clear. □

The next result follows immediately from Theorem 7(ii).

Corollary 5. *Let n be any positive integer. Then*

(i) $cpnd(P_n) = 2$ *for any* $n \geq 2$.

(ii) $cpnd(C_n) = 2$ *for any* $n \geq 4$.

Theorem 8. *[7] Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets of G and H, respectively.*

Theorem 9. *[3] Let G and H be graphs. Then S is a non-empty hop independent set of $G + H$ if and only if one of the following statements holds:*

(i) *S is a clique in G.*

(ii) *S is a clique in H*

(iii) *$S \cap V(G)$ and $S \cap V(G)$ are cliques in G and H, respectively.*

Theorem 10. *Let G and H be two graphs. A set $S \subseteq V(G + H)$ is a hop independent hop dominating set of $G + H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are clique pointwise non-dominating sets of G and H, respectively.*

Proof. Suppose S is a hop independent hop dominating set of $G + H$. Then S_G and S_H are both non-empty. Since S is a hop dominating set, S_G and S_H are pointwise non-dominating sets of G and H, respectively by Theorem 8. Since S is a hop independent set, S_G and S_H are cliques in G and H, respectively, by Theorem 9. Therefore, S_G and S_H are clique pointwise non-dominating sets of G and H, respectively.

Conversely, suppose that $S = S_G \cup S_H$, where S_G and S_H are clique pointwise non-dominating sets of G and H, respectively. Since S_G and S_H are pointwise non-dominating sets, $S = S_G \cup S_H$ is a hop dominating set of $G + H$ by Theorem 8. Since S_G and S_H are cliques, it follows that $S = S_G \cup S_H$ is a hop independent set of $G + H$ by Theorem 9. Consequently, $S = S_G \cup S_H$ is a hop independent hop dominating set of $G + H$. □

The next result follows from Theorem 7, Corollary 5 and Theorem 10.

Corollary 6. *Let G and H be graphs. Then $\gamma_{hih}(G + H) = cpnd(G) + cpnd(H)$. In particular, we have*

(i) $\gamma_{hih}(K_n + H) = n + cpnd(H)$ *for all* $n \geq 1$,

(ii) $\gamma_{hih}(G + H) = 2$ *if G and H contain isolated vertices,*

(iii) $\gamma_{hih}(W_n) = \gamma_{hih}(K_1 + C_n) = 3$ *for all* $n \geq 4$,

(iv) $\gamma_{hih}(F_n) = \gamma_{hih}(K_1 + P_n) = 3$ *for all* $n \geq 2$,

(v) $\gamma_{hih}(K_n + K_m) = n + m$ *for all* $n, m \geq 1$,

(vi) $\gamma_{hih}(P_n + P_m) = 4$ *for all* $n, m \geq 2$,

(vii) $\gamma_{hih}(C_n + C_m) = 4$ *for all* $n, m \geq 4$, *and*

(viii) $\gamma_{hih}(K_{1,n}) = \gamma_{hih}(K_1 + \overline{K}_n) = 2$ *for all* $n \geq 1$.

Theorem 11. *[3] Let G be a non-trivial connected graph and let H be any graph. Then S is a hop independent set in $G \circ H$ if and only if $S = A \cup (\cup_{v \in V(G)} S_v)$ and satisfies the following conditions:*

(i) *A is a hop independent set of G.*

(ii) *S_v is empty or a clique in H^v for each $v \in V(G) \setminus N_G(A)$.*

(iii) *$S_v = \varnothing$ for each $v \in N_G(A)$.*

Theorem 12. *[7] Let G and H be any two graphs. A set $C \subseteq V(G)$ is a hop dominating set of $G \circ H$ if and only if $C = A \cup (\cup_{v \in V(G) \cap N_G(A)} S_v) \cup (\cup_{w \in V(G) \setminus N_G(A)} E_w)$ and satisfies the following conditions:*

(i) *$A \subseteq V(G)$ such that for each $w \in V(G) \setminus A$, there exists $x \in A$ with $d_G(w, x) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $V(H^y) \cap C \neq \varnothing$.*

(ii) *$S_v \subseteq V(H^v)$ for each $v \in V(G) \cap N_G(A)$.*

(iii) *$E_w \subseteq V(H^w)$ is a pointwise non-dominating set of H^w for each $w \in V(G) \setminus N_G(A)$.*

Theorem 13. *Let G be a non-trivial connected graph and let H be any graph. Then C is a hop independent hop dominating set of $G \circ H$ if and only if $C = A \cup (\cup_{v \in V(G)} C_v)$, where $A \subseteq V(G)$, $C_v \subseteq V(H^v)$ for each $v \in V(G)$, and satisfies the following conditions:*

(i) *$A = \varnothing$ or A is a hop independent set of G.*

(ii) *$C_y = \varnothing$ for each $y \in N_G(A)$.*

(iii) *For each $v \in V(G) \setminus N_G^2[A]$, there exists $w \in N_G(v)$ such that C_w is a hop independent set of H^w.*

(iv) *For each $v \in V(G) \setminus N_G(A)$, C_v is a clique pointwise non-dominating set.*

Proof. Suppose C is a hop independent hop dominating set of $G \circ H$. Let $A = C \cap V(G)$ and $C_v = C \cap V(H^v)$ for each $v \in V(G)$. Suppose further that $A \neq \varnothing$. Since C is a hop independent set, A is a hop independent set of G by Theorem 11(i). This shows that (i) holds. Clearly, (ii) holds by Theorem 11(iii). Next, let $v \in V(G) \setminus N_G^2[A]$. Then $v \notin A$ and $d_G(u, v) \neq 2$ for every $u \in A$. Since C is a hop dominating set of $G \circ H$, there exists $x \in C$ such that $d_{G \circ H}(x, v) = 2$. It follows that $x \in C_w$ for some $w \in N_G(v)$. Since C is a hop independent set, it follows that C_w is a hop independent set of H^w, showing that (iii) holds. Property (iv) follows immediately from Theorem 11(ii) and Theorem 12(iii).

For the converse, suppose that C has the given form and satisfies conditions (i), (ii), (iii) and (iv). Then by Theorem 11, C is a hop independent set of $G \circ H$. Next, let $a \in V(G \circ H) \setminus C$ and $v \in V(G)$ such that $a \in V(v + H^v)$. Consider the following cases:

Case 1: $a = v$.

Then $a \in V(G) \setminus A$. If $a \in N_G^2(A)$, then $d_G(a, b) = d_{G\circ H}(a, b) = 2$ for some $b \in A \subseteq C$. If $a \notin N_G^2(A)$, then there exists $w \in N_G(a)$ such that C_w is a hop independent set by (iii). Choose any $p \in C_w$. Then $p \in C$ and $d_{G\circ H}(a, p) = 2$.

Case 2: $a \neq v$.

Then $a \in V(H^v) \setminus C_v$. If $v \in N_G(A)$, say $cv \in E(G \circ H)$ for some $c \in A \subseteq C$, then $d_G(c, a) = 2$. Suppose $v \notin N_G(A)$. By (iv), there exists $q \in C_v \subseteq C$ such that $d_{G\circ H}(q, a) = 2$. Thus, C is a hop dominating set.

Therefore, C is a hop independent hop dominating set in $G \circ H$. □

Consider the following families of graphs:

$$\mathcal{G}_1 = \{G : G \text{ admits a dominating hop independent hop dominating set}\} \text{ and}$$

$$\mathcal{G}_2 = \{G : G \text{ admits a total dominating hop independent hop dominating set}\}.$$

Let G be a graph. Denote by

$$\mathcal{H} = \{A : A \text{ is a hop independent hop dominating set of } G\},$$

$\mathcal{H}_G^1 = \{B : B$ is a dominating hop independent hop dominating set of $G\}$ if $G \in \mathcal{G}_1$, and

$\mathcal{H}_G^2 = \{T : T$ is a total dominating hop independent hop dominating set of $G\}$ if $G \in \mathcal{G}_2$.

Corollary 7. *Let G be a non-trivial connected graph and let H be any graph. Then*

$$\gamma_{hih}(G \circ H) \leq min\{|A| + |V(G) \setminus N_G(A)|cpnd(H) : A \in \mathcal{H}\}.$$

In particular, we have

(i) $\gamma_{hih}(G \circ H) \leq min\{|B| + |B \setminus N_G(B)|cpnd(H) : B \in \mathcal{H}_G^1\}$ *if* $G \in \mathcal{G}_1$, *and*

(ii) $\gamma_{hih}(G \circ H) \leq \gamma_t^{hih}(G)$ *if* $G \in \mathcal{G}_G^2$.

Proof. Let $A \in \mathcal{H}$. For each $v \in V(G) \setminus N_G(A)$, let C_v be a *cpnd*-set of H^v. Then $C = A \cup (\cup_{v \in V(G)\setminus N_G(A)} C_v)$ is a hop independent hop dominating set of $G \circ H$ by Theorem 13. Thus,

$$\gamma_{hih}(G \circ H) \leq |C| = |A| + \sum_{v \in V(G)\setminus N_G(A)} |C_v| = |A| + |V(G) \setminus N_G(A)|cpnd(H).$$

Since A is arbitrary, it follows that

$$\gamma_{hih}(G \circ H) \leq \min\{|A| + |V(G) \setminus N_G(A)|cpnd(H) : A \in \mathcal{H}\}.$$

Let $G \in \mathcal{G}_1$ and let $B \in \mathcal{H}_G^1$. Since B is a dominating set, it follows that

$$V(G) \setminus N_G(B) = B \setminus N_G(B).$$

Hence, (i) follows from the first inequality. Next, let $G \in \mathcal{G}_2$ and let $T \in \mathcal{H}_G^2$. Since T is a total dominating set of G, it follows that $V(G) = N_G(T)$. Thus,

$$\gamma_{hih}(G \circ H) \leq \min\{|T| : T \in \mathcal{H}_G^2\} = \gamma_t^{hih}(G)$$

by the first inequality. Hence, (ii) holds. □

Theorem 14. *[3] Let G and H be non-trivial connected graphs. Then $C = \bigcup\limits_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a hop independent set of $G[H]$ if and only if the following conditions hold.*

(i) S is a hop independent set of G.

(ii) T_x is a clique in H for each $x \in S$.

Theorem 15. *[7] Let G and H be connected non-trivial graphs. A subset $C \bigcup_{x \in S} [\{x\} \times T_x]$ of $V(G[H]$ is a hop dominating set of $G[H]$ if and only if the following conditions hold.*

(i) S is a hop dominating set of G.

(ii) T_x is a pointwise non-dominating set of H for each $x \in S$ with $|N_G^2(x) \cap S| = 0$.

Theorem 16. *Let G and H be non-trivial connected graphs. Then $C = \bigcup\limits_{x \in A} [\{x\} \times T_x]$, where $A \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in A$, is a hop independent hop dominating set of $G[H]$ if and only if the following conditions hold.*

(i) A is a hop independent hop dominating set of G.

(ii) T_x is a clique pointwise non-dominating set in H for each $x \in A$.

Proof. Suppose $C = \bigcup\limits_{x \in A} [\{x\} \times T_x]$ is a hop independent hop dominating set of $G[H]$. Since C is a hop independent set, it follows that A is a hop independent set of G by Theorem 14. Moreover, since C is a hop dominating set of G, A is a hop dominating set by Theorem 15. Hence, (i) holds. Also, by Theorem 14 and Theorem 15, T_x is a clique pointwise non-dominating set of H for each $x \in A$, respectively. Hence, (ii) holds.

Conversely, suppose that $C = \bigcup\limits_{x \in B} [\{x\} \times T_x]$ and satisfies (i) and (ii). Then by Theorem 14 and Theorem 15, C is a hop independent hop dominating set of $G[H]$. □

Corollary 8. *Let G and H be non-trivial connected graphs. Then*

$$\gamma_{hih}(G[H]) = \gamma_{hih}(G)cpnd(H).$$

In particular, we have

(i) $\gamma_{hih}(K_n[K_m]) = \gamma_{hih}(K_n)cpnd(K_m) = nm$ *for any* $n, m \geq 1$,

(ii) $\gamma_{hih}(K_n[P_m]) = \gamma_{hih}(K_n)cpnd(P_m) = 2n$ *for any* $n \geq 1, m \geq 3$,

(iii) $\gamma_{hih}(K_n[C_m]) = \gamma_{hih}(K_n)cpnd(C_m) = 2n$ *for any* $n \geq 1, m \geq 4$,

(iv) $\gamma_{hih}(P_n[P_m]) = \gamma_{hih}(P_n)cpnd(P_m) = 2\gamma_h(P_n)$ *for any* $n, m \geq 3$, *and*

(v) $\gamma_{hih}(C_n[C_m]) = \gamma_{hih}(C_n)cpnd(C_m) = 2\gamma_h(C_n)$ *for any* $n, m \geq 4$.

Proof. Let A be a γ_{hih}-set of G and let T be a $cpnd$-set of H. For each $x \in A$, set $T_x = T$. Then $C = \bigcup_{x\in A} [\{x\} \times T_x] = A \times T$ is a hop independent hop dominating set of $G[H]$ by Theorem 16. Hence, $\gamma_{hih}(G[H]) \leq |C| = \gamma_{hih}(G)cpnd(H)$.

On the other hand, if $C' = \bigcup_{x\in A'} [\{x\} \times R_x]$ is a γ_{hih}-set of $G[H]$, then A' is a hop independent hop dominating set of G and R_x is a clique pointwise non-dominating set of H for every $x \in A'$ by Theorem 16. Hence,

$$\gamma_{hih}(G[H]) = |C'| = \sum_{x\in A'} |R_x| \geq |A'|cpnd(H) \geq \gamma_{hih}(G)cpnd(H).$$

Consequently, $\gamma_{hih}(G[H]) = \gamma_{hih}(G)cpnd(H)$. The assertions in $(i), (ii), (iii), (iv)$ and (v) follow from Theorem 2, Corollary 1, Theorem 7, Corollary 5, and the first part. □

4. Conclusion

The concept of hop independent hop domination has been introduced and initially investigated in this study. As pointed out, every graph admits a hop independent hop dominating set and the hop independent hop domination number of a graph lies between the hop domination number and the hop independence number of the graph. Graphs which attained some specific values for their hop independent hop domination number have been characterized. Necessary and sufficient conditions for a subset in the shadow graph, join, corona, and lexicographic product of two graphs have been obtained. These characterizations have been used to obtain bounds or exact value of the hop independent hop domination number of each of these graphs. It may be interesting to find bounds for this newly defined parameter in terms of other known parameters, investigate the concept for trees and other interesting graphs, and determine the complexity of the hop independent hop domination problem.

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