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Strong Resolving Hop Domination in Graphs

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Abstract. A vertex w in a connected graph G strongly resolves two distinct vertices u and v in $V(G)$ if v is in any shortest u-w path or if u is in any shortest v-w path. A set W of vertices in G is a strong resolving set G if every two vertices of G are strongly resolved by some vertex of W. A set S subset of $V(G)$ is a *strong resolving hop dominating set* of G if S is a strong resolving set in G and for every vertex $v \in V(G) \setminus S$ there exists $u \in S$ such that $d_G(u,v) = 2$. The smallest cardinality of such a set S is called the strong resolving hop domination number of G. This paper presents the characterization of the strong resolving hop dominating sets in the join, corona and lexicographic product of graphs. Furthermore, this paper determines the exact value or bounds of their corresponding strong resolving hop domination number.

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1. Introduction

Domination in graphs have been widely studied in Graph Theory. Over the years, many variations of domination have been studied. Hop domination in graphs was defined and characterized by Natarajan and Ayyaswamy [8], they also determined the hop domination number of some graphs. Variations of the domination and hop domination in graphs were studied in [4, 10, 11].

Resolving sets was studied by Slater in [12]. Variations of resolving sets and resolving dominating sets were studied in [2, 6, 7]. The strong resolving sets and strong metric dimension were introduced and characterized by Oellermann and Peters-Fransen [9].

Resolving hop dominating sets in graphs was studied and presented with an example of its application in minimization problems in [5]. This paper introduces and characterizes the concept of strong resolving hop domination in graphs.

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We consider connected graphs that are finite, simple, and undirected. For elementary Graph Theory concepts, it is recommended that readers refer to [3].

Let $G = \big(V(G), E(G)\big)$ be a graph. $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ is a *neighborhood* of v. An element $u \in N_G(v)$ is called a *neighbor* of v. $N_G[v] = N_G(v) \cup \{v\}$ is a *closed neighborhood* of v. The degree of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{v\in S} N_G(v)$ and $N_G[S] = \bigcup_{v\in S} N_G[v]$.

The *distance* $d_G(u, v)$ of two vertices u, v in G is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

A set $S \subseteq V(G)$ of vertices of G is a *dominating set* if every $u \in V(G) \setminus S$ is adjacent to at least one vertex $v \in S$. The *domination number* of a graph G, denoted by $\gamma(G)$, is given by $\gamma(G) = min\{|S| : S \text{ is a dominating set of } G\}$.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

The *floor function* of a real number x, denoted by $\lfloor x\rfloor$, is a function that returns the highest integer less than or equal to x. Formally, for all real number x,

$$\lfloor x\rfloor = \max\{n \in \mathbb{Z} \,|\, n \leq x\}.$$

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G(u, 2) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G[u, 2] = N_G(u, 2) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G(X, 2) = \bigcup_{u\in X} N_G(u, 2)$. The *closed hop neighborhood* of X in G is the set $N_G[X, 2] = N_G(X, 2) \cup X$.

A vertex x of a graph G is said to *resolve two vertices* u and v of G if $d_G(x, u) \neq d_G(x, v)$. For an ordered set $W = \{x_1, ..., x_k\} \subseteq V(G)$ and a vertex v in G, the $k-vector$

$$r_G(v/W) = (d_G(v, x_1), d_G(v, x_2), \cdots, d_G(v, x_k))$$

is called the representation of v with respect to W. The set W is a *resolving set* for G if and only if no two vertices of G have the same representation with respect to W. The *metric dimension* of G, denoted by, $dim(G)$, is the minimum cardinality over all resolving sets of G. A resolving set of cardinality $dim(G)$ is called *basis*.

A set $S \subseteq V(G)$ is a *resolving hop dominating set* of G if S is both a resolving set and a hop dominating set. The minimum cardinality of a resolving hop dominating set of G, denoted by $\gamma_{Rh}(G)$, is called the *resolving hop domination number* of G. Any resolving hop dominating set with cardinality equal to $\gamma_{Rh}(G)$ is called a γ_{Rh}-set.

A *u-v* path of length $d_G(u, v)$ is called a *u-v geodesic*. The set $I_G[u, v]$ is defined as the set of all vertices of G lying on any *u-v* geodesic.

A vertex $w \in V(G)$ *strongly resolves* two different vertices $u, v \in V(G)$ if $v \in I_G[u, w]$ or if $u \in I_G[v, w]$. A set W of vertices in G is a *strong resolving set* of G if every two vertices of G are strongly resolved by some vertex of W. The smallest cardinality of a strong resolving set of G is called the *strong metric dimension* of G and is denoted by $sdim(G)$. A strong resolving set of cardinality $sdim(G)$ is called a *strong metric basis* of G.

A vertex u of G is *maximally distant* from vertex v of G, $u \neq v$, if for every vertex $w \in N_G(u)$, $d_G(v, w) \leq d_G(u, v)$. If u is maximally distant from v, and v is maximally distant from u, then we say that u and v are *mutually maximally distant*, denoted by uMMDv. The *boundary* of a graph G, denoted by $\partial(G)$, is defined as

$$\partial(G) = \{u \in V(G) : u\text{MMD}v \text{ for some } v \in V(G)\}.$$

A set $S \subseteq V(G)$ is a *strong resolving hop dominating set* of G if S is both a strong resolving set and a hop dominating set. The minimum cardinality of a strong resolving hop dominating set of G, denoted by $\gamma_{sRh}(G)$, is called the *strong resolving hop domination number* of G. Any resolving hop dominating set with cardinality equal to $\gamma_{sRh}(G)$ is called a γ_{sRh}-set.

A *clique* in a graph G is a complete induced subgraph of G. A clique C in a connected graph G is called a *superclique* if for every pair of distinct vertices $u, v \in C$, there exists $w \in V(G) \setminus C$ such that $w \in N_G(u) \setminus N_G(v)$ or $w \in N_G(v) \setminus N_G(u)$. A superclique C is *maximum* in G if $|C| \geq |C^*|$ for all supercliques C^* in G. The *superclique number*, $\omega_S(G)$, of G is the cardinality of a maximum superclique in G.

A superclique C in G is called a *hop dominated superclique* if for every $v \in C$ there exists $u \in V(G) \setminus C$ such that $d_G(u, v) = 2$. A hop dominated superclique C is *maximum* in G if $|C| \geq |C^*|$ for all hop dominated supercliques C^* in G. The *hop dominated superclique number*, $\omega_{hS}(G)$, of G is the cardinality of a maximum hop dominated superclique in G.

2. Preliminary Results

Remark 1. *Every strong resolving hop dominating set of G is a resolving hop dominating set. Thus,*

$$2 \leq \gamma_{Rh}(G) \leq \gamma_{sRh}(G).$$

Remark 2. *Every strong resolving hop dominating set of G is a hop dominating set. Thus,*

$$2 \leq \gamma_h(G) \leq \gamma_{sRh}(G).$$

Remark 3. *Every strong resolving hop dominating set of G is a resolving set. Thus,*

$$1 \leq dim(G) \leq \gamma_{sRh}(G).$$

Remark 4. *Every strong resolving hop dominating set of a connected graph G is hop dominating and every strong resolving hop dominating set of G is strong resolving. Thus,* $\gamma_h(G) \leq \gamma_{sRh}(G)$ *and* $sdim(G) \leq \gamma_{sRh}(G)$.

Remark 5. *For any connected graph of order* n*,* $2 \leq \gamma_{sRh}(G) \leq n$*. Moreover,* $\gamma_{sRh}(G) = 1$ *if and only if* G *is a trivial graph and* $\gamma_{sRh}(G) = n$ *if* $G = K_n$*.*

Proposition 1. *Let* S *be a hop dominating set of* P_n *for* $n \geq 3$*. Then* S *is a strong resolving hop dominating set of* P_n *if and only if there exists* $u \in S$ *with* $deg_{P_n}(u) = 1$*.*

Proof: Suppose S is a strong resolving hop dominating set of P_n. Let u and v be distinct vertices of P_n such that $deg(u) = deg(v) = 1$. Then the pair of vertices u and v is only strongly resolved by u or by v, since $u \in I[u, v]$ or $v \in I[u, v]$ and $u \in I[v, u]$ or $v \in I[v, u]$. Implying that $u, v \in S$.

For the converse, let $u \in S$ with $deg_{P_n}(u) = 1$. Then every pair of distinct vertices $x, y \in V(P_n)$ are strongly resolved by u, since $x \in I[u, y]$ or $y \in I[u, x]$. Hence, S is a strong resolving set of P_n. Since S is hop dominating by assumption, S is a strong resolving hop dominating set of P_n. □

Proposition 2. *Let* S *be a hop dominating set of* C_n *for* $n \geq 4$*. If* $d_{C_n}(u, v) < \lfloor n/2 \rfloor$ *for each pair* $u, v \in V(C_n) \setminus S$*, then* S *is a strong resolving hop dominating set of* C_n*.*

Proof: Let S be a hop dominating set of C_n and let $u, v \in V(C_n) \setminus S$ such that $d(u, v) < \lfloor n/2 \rfloor$. Note that $diam(C_n) = \lfloor n/2 \rfloor$. Then there exists, $s \in S$ such that $d_{C_n}(s, u) \leq d_{C_n}(s, v) \leq \lfloor n/2 \rfloor$ or $d_{C_n}(s, v) \leq d_{C_n}(s, u) \leq \lfloor n/2 \rfloor$. It follows that $u \in I[s, v]$ or $v \in I[s, u]$. Hence, s strongly resolves u and v. Therefore, S is a strong resolving hop dominating set of C_n. □

Remark 6. *Every strong resolving hop dominating set of a connected graph* G *is a resolving hop dominating set. Thus,* $\gamma_{Rh}(G) \leq \gamma_{sRh}(G)$*.*

Remark 7. *Any superset of a strong resolving hop dominating set is a strong resolving hop dominating set.*

Proposition 3. [1] *Let* G *be a connected graph of order* n *and let*

$$A = \{x \in V(G) : deg_G(x) = n - 1\}.$$

If $A \neq \varnothing$ *and* $\langle C \rangle$ *is a superclique in* G*, then* $|C \cap A| \leq 1$*. Moreover, if* $\langle C \rangle$ *is a maximum superclique of* G*, then* $|C \cap A| = 1$*.*

Proposition 4. [1] *Let* G *be a non-trivial connected graph with* $diam(G) \leq 2$*. Then* $S = V(G) \setminus C$ *is a strong resolving set of* G *if and only if* $C = \varnothing$ *or* $\langle C \rangle$ *is a superclique in* G*. In particular,* $sdim(G) = |V(G)| - \omega_S(G)$*.*

Proposition 5. *Let* G *be a connected graph of order* n*, and let* $A = \{x \in V(G) : deg_G(x) = n - 1\}$*. If* $A \neq \varnothing$ *and* C *is a hop dominated superclique in* G*, then* $C \cap A = \varnothing$*.*

Proof: Suppose C is a hop dominated superclique in G. By Proposition 3, $|C \cap A| \leq 1$. If $|C \cap A| = 1$ and $x \in C \cap A$, then $d_G(x, v) = 1$ for each $v \in V(G) \setminus \{x\}$ since $deg_G(x) = n - 1$. This is a contradiction to the definition of hop dominated superclique. It follows that $|C \cap A| = 0$. Therefore $C \cap A = \varnothing$. □

Lemma 1. *Let G be a nontrivial connected graph with $diam(G) \leq 2$. Then $S = V(G) \setminus C$ is a strong resolving hop dominating set of G if and only if $C = \varnothing$ or C is a hop dominated superclique in G. In particular, $\gamma_{sRh}(G) = |V(G)| - \omega_{hS}(G)$.*

Proof: Suppose that $S = V(G) \setminus C$ is a strong resolving hop dominating set of G. Then S is a strong resolving set. By Proposition 4, $C = \varnothing$ or C is a superclique in G. Let $x \in C$. Then $x \in V(G) \setminus S$. Since S is hop dominating, a vertex $y \in S \cap N_G(x, 2)$ exists. Hence, $y \in V(G) \setminus C$ and $d_G(x, y) = 2$. Thus, C is a hop dominated superclique in G.

Conversely, suppose $C = \varnothing$ or C is a hop dominated superclique in G. Then $C \neq \varnothing$ or C is a superclique in G. By Proposition 4, $S = V(G) \setminus C$ is a strong resolving set of G. Let $x \in V(G) \setminus S$. Then $x \in C$. Since C is a hop dominated superclique, there exists $y \in V(G) \setminus C \cap N_G(x, 2)$. Since $S = V(G) \setminus C$, $y \in S \cap N_G(x, 2)$. Therefore, S is a hop dominating set of G.

Accordingly, S is a strong resolving hop dominating set of G.

Suppose S is a γ_{sRh}-set of G. Then $S = V(G) \setminus C$, where C is a hop dominated superclique in G and $|C| = \omega_{hS}(G)$. Hence,

$$\gamma_{sRh}(G) = |S| = |V(G)| - |C| = |V(G)| - \omega_{hS}(G). \qquad \square$$

3. On Strong Resolving Hop Domination in the Join of Graphs

Let A and B be sets which are not necessarily disjoint. The *disjoint union* of A and B, denoted by $A \mathbin{\dot{\cup}} B$, is the set obtained by taking the union of A and B treating each element in A as distinct from each element in B. The *union* $G_1 \cup G_2$ of graphs G_1 and G_2 with disjoint vertex-sets $V(G_1)$ and $V(G_2)$, respectively, is the graph G with $V(G) = V(G_1) \mathbin{\dot{\cup}} V(G_2)$ and $E(G) = E(G_1) \mathbin{\dot{\cup}} E(G_2)$. The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \mathbin{\dot{\cup}} V(H)$ and edge-set $E(G + H) = E(G) \mathbin{\dot{\cup}} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Remark 8. *For the joins $\langle v \rangle + P_n$ and $\langle w \rangle + C_n$, $\gamma_{sRh}(\langle v \rangle + P_n) = n - 1$ for $n \geq 3$ and $\gamma_{sRh}(\langle w \rangle + C_n) = n - 1$ for $n \geq 4$.*

Theorem 1. [1] *Let G be a non-trivial connected graph of order n with $\gamma(G) \neq 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a strong resolving set of $K_1 + G$ if and only if $S = V(G)$, or $S = V(G) \setminus C$, or $S = V(K_1 + G) \setminus C$ where $\langle C \rangle$ is a superclique in G.*

Theorem 2. [1] *Let G be a non-trivial connected graph of order n with $\gamma(G) = 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a strong resolving set of $K_1 + G$ if and only if $S = V(G)$, or $S = V(K_1 + G) \setminus C^*$, or*

$$S = (V(G) \setminus C^*) \cup \{x \in C^* : deg_G(x) = n - 1\}$$

where $\langle C^* \rangle$ is a superclique in G.

Theorem 3. [1] *Let* $K_1 = \langle v \rangle$ *and* G *be a disconnected graph whose components are* G_i *for* $i = 1, 2, \ldots, m$*. A proper subset* S *of* $V(K_1 + G)$ *is a strong resolving set of* $K_1 + G$ *if and only if* $S = V(G)$ *or* $S = V(G) \setminus C_i$ *or* $S = V(K_1+G) \setminus C_i$ *where* $\langle C_i \rangle$ *is a superclique in* G_i*, for some* $i \in \{1, 2, \ldots, m\}$*.*

Lemma 2. *Let* G *be a connected graph of order* n *and let* $K_1 = \langle v \rangle$*. Then* $C \subseteq V(K_1+G)$ *is a hop dominated superclique in* $K_1 + G$ *if and only if* C *is a hop dominated superclique in* G*.*

Proof: Suppose C is a hop dominated superclique in $K_1 + G$. Since $deg_{K_1+G}(v) = n$, by Proposition 5, $v \notin C$. Thus, C is a hop dominated superclique in G. The converse follows immediately from Proposition 5 and definition of hop dominated superclique. □

Theorem 4. *Let* G *be a connected graph of order* $n > 1$*, and let* $K_1 = \langle v \rangle$*. Then* $S \subseteq V(K_1 + G)$ *is a strong resolving hop dominating set of* $K_1 + G$ *if and only if* $S = V(K_1 + G) \setminus C$ *where* $C = \varnothing$ *or* C *is a hop dominated superclique in* G*.*

Proof: Suppose S is a strong resolving hop dominating set of $K_1 + G$. By Theorem 1 and Theorem 2, $S = V(G)$, or $S = (V(G) \setminus C) \cup \{x \in C : deg_G(x) = n-1\}$, or $S = V(K_1 + G) \setminus C$ where C is a superclique in G. Since S is a hop dominating set of $K_1 + G$, $v \in S$. Thus, $S \neq V(G)$ and $S \neq (V(G) \setminus C) \cup \{x \in C : deg_G(x) = n-1\}$. Since $diam(K_1 + G) \leq 2$, by Lemma 1 and Proposition 5, $S = V(K_1 + G) \setminus C$ where $C = \varnothing$ or C is a hop dominated superclique in G.

The converse follows immediately from Lemma 1 and Proposition 5. □

The next result is a consequence of Theorem 4.

Corollary 1. *Let* G *be a connected graph of order* n*. Then*

$$\gamma_{sRh}(K_1 + G) = n + 1 - \omega_{hS}(G).$$

The next result follows immediately from Theorem 3, and definitions of strong resolving hop domination and $K_1 + G$.

Theorem 5. *Let* $K_1 = \langle v \rangle$ *and* G *be a disconnected graph whose components are* G_i *for* $i = 1, 2, \cdots, m$*. A proper subset* S *of* $V(K_1 + G)$ *is a strong resolving hop dominating set of* $K_1 + G$ *if and only if* $S = V(K_1 + G) \setminus C_i$ *where* C_i *is a superclique in* G_i *for some* $i \in \{1, 2, \cdots, m\}$*.*

Proof: Suppose S is a strong resolving hop dominating proper subset of $K_1 + G$. By Theorem 3, $S = V(G)$, or $S = V(G) \setminus C_i$ or $S = V(K_1 + G) \setminus C_i$ where C_i is a superclique in G_i for some $i \in \{1, 2, \cdots, m\}$. By definition of strong resolving hop domination, $v \in S$. Hence, $S \neq V(G)$ and $S \neq V(G) \setminus C_i$. Thus, $S = V(K_1+G) \setminus C_i$ where C_i is a superclique in G_i for some $i \in \{1, 2, \cdots, m\}$.

Conversely, suppose $S = V(K_1 + G) \setminus C_i$ where C_i is a superclique in G_i for some $i \in \{1, 2, \cdots, m\}$. Let $x \in V(K_1 + G) \setminus S$. Then $x \in C_i$ for some $i \in \{1, 2, \cdots, m\}$. Since $m \geq 2$, there exists G_j, $j \neq i$ and vertex $y \in V(G_j) \cap N_{K_1+G}(x, 2)$. Since

$S = V(K_1 + G) \setminus C_i$, $y \in S \cap N_{K_1+G}(x, 2)$. Therefore, S is a hop dominating set of $K_1 + G$. □

As a consequence of Theorem 5, the next result follows.

Corollary 2. *Let G_i be connected graphs of orders n_i and G be a disconnected graph whose components are G_i for $i = 1, 2, \cdots, m$. Then*

$$\gamma_{sRh}(K_1 + G) = \sum_{i=1}^{m} n_i - \max\{\omega_{hS}(G_i)(i = 1, 2, \cdots, m)\}.$$

Theorem 6. *Let G and H be nontrivial connected graphs. A proper subset S of $V(G+H)$ is a strong resolving hop dominating set of $G+H$ if and only if at least one of the following is satisfied:*

(i) $S = V(G + H) \setminus C_G$ *where C_G is a hop dominated superclique in G.*

(ii) $S = V(G + H) \setminus C_H$ *where C_H is a hop dominated superclique in H.*

(iii) $S = V(G + H) \setminus (C_G \cup C_H)$ *where C_G and C_H are hop dominated supercliques of G and H, respectively.*

Proof: Note that $diam(G + H) \leq 2$ and C_G, C_H, and $(C_G \cup C_H)$ are hop dominated supercliques in $G + H$. Then by Lemma 1, the theorem holds. □

The next result is a consequence of Theorem 6.

Corollary 3. *Let G and H be nontrivial connected graphs of orders m and n, respectively. Then*

$$\gamma_{sRh}(G + H) = m + n - (\omega_{hS}(G) + \omega_{hS}(H)).$$

4. On Strong Resolving Hop Domination in the Corona of Graphs

The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining every vertex of the ith copy of H to the ith vertex of G. For $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v, v \in V(G)$.

Theorem 7. [1] *Let G be a non-trivial connected graph and H a connected graph. A proper subset S of $V(G \circ H)$ is a strong resolving dominating set of $G \circ H$ if and only if one of the following holds:*

(i) $S = A \bigcup \left(\bigcup_{u \in V(G)} V(H^u) \right)$ *where $A \subseteq V(G)$;*

(ii) $S = A \bigcup \left(\bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right) \bigcup B_v$ *for a unique vertex v in G, where $A \subseteq V(G)$ and B_v is a strong resolving set of H^v if $\gamma(H) = 1$ or B_v is a resolving set of $\langle v \rangle + H^v$ if $\gamma(H) \neq 1$.*

Theorem 8. *Let G be a nontrivial connected graph and H a connected graph. A proper subset S of $V(G \circ H)$ is a strong resolving hop dominating set of $G \circ H$ if and only if one of the following holds:*

(i) $S = A \cup \left(\bigcup_{v \in V(G)} V(H^v) \right)$ *where* $A \subseteq V(G)$;

(ii) $S = A \cup \left(\bigcup_{v \in V(G) \setminus \{u\}} V(H^v) \right) \cup B_u$ *for a unique vertex u in G, where $A \subseteq V(G)$, B_u is a strong resolving set of H^u and B_u is strong resolving hop dominating of H^u if $N_G(u) \cap A = \varnothing$.*

Proof: Suppose S is a strong resolving hop dominating set of $G \circ H$. By Theorem 7, (i) holds or $S = A \cup \left(\bigcup_{v \in V(G) \setminus \{u\}} V(H^v) \right) \cup B_u$ for a unique vertex u in G, where $A \subseteq V(G)$, B_u is a strong resolving set of H^u if $\gamma(H) = 1$ or B_u is a strong resolving set of $\{u\} + H^u$ if $\gamma(H) \neq 1$. Let $N_G(u) \cap A = \varnothing$ and $x \in V(H^u) \setminus B_u$. Since S is hop dominating and $x \in V(G \circ H) \setminus S$, there exists $y \in N_{G \circ H}(x, 2) \cap S$. Thus, $y \in N_{H^u}(x, 2) \cap B_u$. It follows that B_u is strong resolving hop dominating set of H^u. Hence, (ii) holds.

For the converse, suppose (i) and (ii) hold. Then by Theorem 7, S is a strong resolving set of $G \circ H$. Let $x \in V(G \circ H) \setminus S$. Consider the following cases:

Case 1. $x \in V(G) \setminus A$

Since G is nontrivial connected, there exists a vertex $y \in V(G) \cap N_G(x)$. By (i) or (ii), vertex $z \in N_{G \circ H}(x, 2) \cap V(H^y)$ or $z \in N_{G \circ H}(x, 2) \cap B_y$ exists.

Case 2. $x \in V(H^u) \setminus B_u$ for a unique vertex u in G.

If $N_G(u) \cap A \neq \varnothing$, then $w \in ((N_G(u) \cap A) \cap N_{G \circ H}(x, 2)$ exists. On the other hand, if $N_G(u) \cap A = \varnothing$, by (ii), B_u is a strong resolving hop dominating set of H^u. Hence, there exists $p \in N_{H^u}(x, 2) \cap B_u$. This implies that $p \in N_{G \circ H}(x, 2) \cap S$.

In any case, S is a hop dominating set of $G \circ H$.

Accordingly, S is a strong resolving hop dominating set of $G \circ H$. □

Corollary 4. *Let G and H be connected graphs of orders $m > 1$ and n, respectively. Then*

$$\gamma_{sRh}(G \circ H) = (m-1)n + \gamma_{sRh}(H).$$

Moreover, if $diam(H) \leq 2$, then

$$\gamma_{sRh}(G \circ H) = (m-1)n + |V(H)| - \omega_{hS}(H) = mn - \omega_{hS}(H).$$

Proof: Let S be a γ_{sRh}-set of $G \circ H$. Then, $S = A \cup \left(\bigcup_{v \in V(G) \setminus \{u\}} V(H^v) \right) \cup B_u$ for a unique vertex u in G, where $A \subseteq V(G)$, and B_u satisfies (ii) of Theorem 8. Let $A = \varnothing$

and B_u be γ_{sRh}-set of H^u. Then

$$\begin{aligned}\gamma_{sRh}(G \circ H) =&|S|\\ =&|A| + \left|\bigcup_{v\in V(G)\setminus\{u\}} V(H^v)\right| + |B_u|\\ =&0 + (m-1)n + \gamma_{sRh}(H)\\ =&(m-1)n + \gamma_{sRh}(H).\end{aligned}$$

Suppose that $diam(H) \leq 2$. Then by Lemma 1,

$$\gamma_{sRh}(H) = |V(H)| - \omega_{hS}(H) = n - \omega_{hS}(H).$$

Hence,

$$\begin{aligned}\gamma_{sRh}(G \circ H) =&(m-1)n + n - \omega_{hS}(H)\\ =&mn - n + n - \omega_{hS}(H)\\ =&mn - \omega_{hS}(H).\end{aligned}$$

Therefore,

$$\gamma_{sRh}(G \circ H) = (m-1)n + |V(H)| - \omega_{hS}(H) = mn - \omega_{hS}(H).$$ □

5. On Strong Resolving Hop Domination in the Lexicographic Product of Graphs

The *lexicographic product* of two graphs G and H, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ such that $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

Lemma 3. [1] *Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph with $\gamma(H) \neq 1$. Then $A \times C \subseteq V(G[H])$ induces a superclique in $G[H]$ if and only if A is a nonempty subset of $V(G)$ and $\langle C\rangle$ is a superclique in H.*

Lemma 4. [1] *Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph with $\gamma(H) = 1$. Then $A \times C \subseteq V\big(G[H]\big)$ is a superclique in $G[H]$ if and only if A is a nonempty subset of $V(G)$ and C is a superclique in H such that $|A| = 1$ whenever $C \cap C^* \neq \varnothing$ for some γ-set of H.*

Lemma 5. *Let $G = K_n$ for $n > 1$ and H a connected graph with $|V(H)| \geq 3$. Then $A \times C \subseteq V(G[H])$ is a hop dominated superclique in $G[H]$ if and only if A is a nonempty subset of $V(G)$ and C is a hop dominated superclique in H.*

Proof: Suppose that $A \times C \subseteq V(G[H])$ is a hop dominated superclique in $G[H]$. By Lemma 3 and Lemma 4, A is a nonempty subset of $V(G)$ and C is a superclique in H. Let $x \in C$. Then $(a, x) \in A \times C$ for any $a \in A$. Since $A \times C$ is a hop dominated superclique, there exists $(b, y) \in [V(G[H]) \setminus (A \times C)] \cap N_{G[H]}((a, x), 2)$. Suppose $\gamma(H) \neq 1$. Since $G = K_n$ for $n > 1$, $a = b$ and $y \in (V(H) \setminus C) \cap N_H(x, 2)$. If $\gamma(H) = 1$, then by Proposition 5, $C \cap C^* = \varnothing$ for all γ-sets C^* of H. Thus, $x \in C \setminus C^*$ and $y \in N_H(x, 2)$ exists. Hence, C is a hop dominated superclique in H.

For the converse, suppose that A is a nonempty subset of $V(G)$ and C is a hop dominated superclique in H. By Lemma 3, Lemma 4 and Proposition 5, $A \times C$ is a superclique in $G[H]$. Let $(a, x) \in A \times C$ and $\gamma(H) \neq 1$. Since C is a hop dominated superclique in H, there exists $y \in (V(H) \setminus C) \cap N_H(x, 2)$. Hence,

$$(a, y) \in [V(G[H]) \setminus (A \times C)] \cap N_{G[H]}((a, x), 2).$$

Suppose $\gamma(H) = 1$. Then by Proposition 5, $C \cap C^* = \varnothing$ for all γ-sets C^* of H. Thus, $x \in C \setminus C^*$. This implies that a vertex $z \in N_H(x, 2)$ exists. Since C is a superclique, $z \in V(H) \setminus C$. Hence, $(a, z) \in [V(G[H]) \setminus (A \times C)] \cap N_G[H]((a, x), 2)$.

Therefore $A \times C$ is a hop dominated superclique in $G[H]$. □

Theorem 9. *Let $G = K_n$ for $n > 1$ and H a connected graph with $|V(H)| \geq 3$. A subset S of $V(G[H])$ is a strong resolving hop dominating set of $G[H]$ if and only if $S = V(G[H]) \setminus (A \times C)$, where A is a subset of $V(G)$ and $C = \varnothing$ or A is a nonempty subset of $V(G)$ and C is a hop dominated superclique in H.*

Proof: Let S be a strong resolving hop dominating set of $G[H]$. Since $diam(G[H]) = 2$, by Lemma 1, $S = V(G[H]) \setminus (A \times C)$ where $A \times C = \varnothing$ or $A \times C$ is a hop dominated superclique in $G[H]$. If $A \times C = \varnothing$, then $A = \varnothing$ or $C = \varnothing$. On the other hand, if $A \times C$ is a hop dominated superclique in $G[H]$, A is a nonempty subset of $V(G)$ and C is a hop dominated superclique in H, by Lemma 5.

For the converse, suppose $S = V(G[H]) \setminus (A \times C)$, where $A \subseteq V(G)$ and $C = \varnothing$ or A is a nonempty subset of $V(G)$ and C is a hop dominated superclique in H. If $C = \varnothing$, then $A \times C = \varnothing$ and $S = V(G[H])$ is a strong resolving hop dominating set of $G[H]$. On the other hand, if A is a nonempty subset of $V(G)$ and C is a hop dominated superclique in H, then by Lemma 5 and Lemma 1, S is a strong resolving hop dominating set of $G[H]$. □

Corollary 5. *Let $G = K_n$ for $n > 1$ and H a connected graph of order $m \geq 3$. Then*

$$\gamma_{sRh}(G[H]) = mn - \omega_{hS}(G[H]).$$

Proof: Let S be a γ_{sRh}-set of $G[H]$. Then by Theorem 9, $S = V(G[H]) \setminus (A \times C)$ where $A \times C$ is a hop dominated superclique in $G[H]$. Since S is a minimum strong resolving hop dominating set of $G[H]$. Hence,

$$\gamma_{sRh}(G[H]) = |S| = |V(G[H])| - |A \times C| = mn - \omega_{hS}(G[H]). \quad \square$$

By Lemma 5 and Corollary 5 the next result follows.

Corollary 6. *Let $G = K_n$ for $n > 1$ and H a connected graph of order m. Then*

$$\gamma_{sRh}(G[H]) = n(m - \omega_{hS}(H)).$$

Theorem 10. [1] *Let G be a non-trivial connected graph and H be non-trivial complete graph. A subset*

$$C = \left(\bigcup_{x\in S}\Big\{\,\{x\}\times T_x\Big\}\right)\bigcup\left(\bigcup_{x\in W_G}\,\Big\{\,\{x\}\times (V(H)\setminus T_x)\Big\}\right)$$

of $V\big(G[H]\big)$ where $W_G \subseteq S$ and $T_x \subseteq V(H)$, $\forall x \in S$, is a strong resolving set of $G[H]$ if and only if

(i) $S = V(G)$.

(ii) $V(H) \setminus T_x$ *is a superclique of* H.

(iii) W_G *is a strong resolving set of* G.

Theorem 11. *Let G be a nontrivial connected graph and H be a nontrivial complete graph. A subset $C = \bigcup_{x\in S}(\{x\}\times T_x)$ of $V(G[H])$ where $S \subset V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a strong resolving hop dominating set of $G[H]$ if and only if*

(i) $S = V(G)$;

(ii) $0 \le |V(H)\setminus T_x| \le 1$ *for every* $x \in S$;

(iii) $T_x = V(H)$ *or* $T_y = V(H)$ *for every pair of vertices* $x, y \in V(G)$ *such that* $x \in W_G$ *where* W_G *is a strong resolving set of* G;

(iv) $T_x = V(H)$ *for every* $x \in S$ *with* $deg_G(x) = |V(G)| - 1$.

Proof: Suppose C is a strong resolving hop dominating set of $G[H]$. By Theorem 10 (i) and (ii) hold. Now, let $x \in S$ with $deg_G(x) = |V(G)| - 1$. Suppose $T_x \neq V(H)$. Let $a \in V(H)\setminus T_x$. Then $(x, a) \notin C$. Since C is a hop dominating set of $G[H]$, there exists $(y, b) \in C \cap N_{G[H]}((x, a), 2)$. Since H is complete, $y \in S \cap N_G(x, 2)$. This implies that $y \in (V(G)\setminus\{x\})\setminus N_G(x)$, showing that $deg_G(x) < |V(G)| - 1$, a contradiction. Hence, $T_x = V(H)$ and (iv) is true.

For (iii), let $x, y \in V(G)$ such that $xMMDy$. Suppose $T_x \neq V(H)$ and $T_y \neq V(H)$. Let $p \in V(H)\setminus T_x$ and $q \in V(H)\setminus T_y$. Then $(x, p), (y, q) \notin C$. Since C is a strong resolving set of $G[H]$, there exists $(z, w) \in C$ such that (z, w) strongly resolves (x, p) and (y, q). Suppose $(x, p) \in I_{G[H]}[(y, q), (z, w)]$. Then

$$d_{G[H]}((z, w), (x, p)) + d_{G[H]}((x, p), (y, q)) = d_{G[H]}((z, w), (y, q)).$$

If $z = x$, then $w \in T_x$ and $d_{G[H]}((z, w), (x, p)) = 1$. Thus

$$1 + d_{G[H]}((x, p), (y, q)) = d_{G[H]}((z, w), (y, q)),$$

that is, $1 + d_{G[H]}((z,p),(y,q)) = d_{G[H]}((z,w),(y,q))$ or $1 + d_G(z,y) = d_G(z,y)$, which is a contradiction. Thus, $z \neq x$. Consequently, $d_G(z,x) + d_G(x,y) = d_G(z,y)$ implying that $x \in I_G[z,y]$. This is a contradiction to the assumption that $xMMDy$. Similarly, if $(y,q) \in I_{G[H]}[(x,p),(z,w)]$, then a contradiction can be obtained. Hence $T_x = V(H)$ or $T_y = V(H)$.

For the converse, suppose C satisfies conditions (i) to (iv). Let $((x_1,y_1),(x_2,y_2)) \notin C$ with $(x_1,y_1) \neq (x_2,y_2)$. Then $y_1 \notin T_{x_1}$ and $y_2 \notin T_{x_2}$. Hence, x_1 is not mutually maximally distant with x_2. Thus, there exists $u \in N_G(x_1)$ such that $d_G(u,x_2) > d_G(x_1,x_2)$, that is, $x_1 \in I_G[x_2,u]$. Let $v \in T_u$. Then $(u,v) \in C$ strongly resolves (x_1,y_1) and (x_2,y_2). Hence, C is a strong resolving set of $G[H]$. Now, we show that C is a hop dominating set of $G[H]$. Let $(x,y) \in C$. Then $y \in V(H) \setminus T_x$. By (iv), $deg_G(x) < |V(G)| - 1$. Thus, there exists $z \in V(G \setminus \{x\}) \setminus N_G(x)$. Let $z \in N_G(x,2)$ and $w \in T_z$. Then, $(z,w) \in C \cap N_{G[H]}((x,y),2)$. This shows that C is a hop dominating set of $G[H]$.

Accordingly, C is a strong resolving hop dominating set of $G[H]$. □

Corollary 7. *Let G be a connected graph of order $m > 1$ and $\gamma(G) \neq 1$ and $H = K_n$ for $n > 1$. Then*

$$\gamma_{sRh}(G[H]) = m(n-1) + |\partial(G)| - 1.$$

Proof: Let C be a γ_{sRh}-set of $G[H]$. Then $C = \bigcup_{x\in S}(\{x\} \times T_x)$ satisfying conditions of Theorem 11. Thus, $S = V(G) = [V(G) \setminus \partial(G)] \cup (\partial(G))$. Hence,

$$\begin{aligned}
\gamma_{sRh}(G[H]) =& |C| \\
=& \left|\bigcup_{x\in V(G)\setminus\partial(G)} (\{x\} \times T_x)\right| + \left|\bigcup_{y\in\partial(G)} (\{y\} \times T_y)\right| \\
=& |V(G) \setminus \partial(G)||T_x| + |\partial(G) - 1||T_y| + n - 1 \\
=& (m - |\partial(G)|)(n-1) + |\partial(G) - 1|n + n - 1 \\
=& m(n-1) - |\partial(G)|n + |\partial(G)| + |\partial(G)|n - n + n - 1 \\
=& m(n-1) + |\partial(G)| - 1.
\end{aligned}$$

Therefore,

$$\gamma_{sRh}(G[H]) = m(n-1) + |\partial(G)| - 1.$$ □

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