EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 819-832
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

$F-$open and $F-$closed sets in Topological Spaces

Mesfer H. Alqahtani

Department of Mathematics, University College of Umluj, University of Tabuk, Tabuk, Saudi Arabia

Abstract. An open (resp., closed) subset A of a topological space $(X, \mathcal{T})$ is called *F-open* (resp., F-closed) set if $cl(A) \setminus A$ (resp., $A \setminus int(A)$) is finite set. In this work, we study the main properties of these definitions and examine the relationships between F-open and F-closed sets with other kinds such as regularly open, regularly closed, closed, and open sets. Then, we establish some operators such as F-interior, F-closure, and F-derived...etc., using F-open and F-closed sets. At the end of this work, we introduce definitions of F-continuous function, F-compact space, and other related properties.

2020 Mathematics Subject Classifications: 05C99, 54F15
Key Words and Phrases: Topological space, F-open, open domain, F-interior, F-closure, F-derived, F-continuous, F-compact

1. Introduction

In the topological space X, a subset B of a space X is said to be a *regularly-closed*, called also *closed domain* if $B = cl(\text{int}(B))$. A subset B of X is said to be a *regularly-open*, called also *open domain* if $B = int(cl(B))$ [2]. In this work, we are interested in studying the concepts of F-open and F-closed sets in topological spaces in detail. We organize this paper as follows. In Section 2, we recall the basic concepts and findings that make this work self-contained. In Section 3, we introduce definitions of F-open and F-closed sets and examine the relationships between them and other types such as open domain, closed domain, closed, and open sets. We also present some important theorems. In Section 4, we establish some operators such as F-interior, F-closure, F-border, F-frontier, F-exterior, and F-derived using F-open and F-closed sets. Then we introduce some related theorems. In Section 5, we introduce definitions of F-continuous, F-open, F-closed, and F-homeomorphism functions. We also provide definitions of F-compact, F-Lindelöf, and F-countably compact spaces. In addition, we present some related properties. Throughout this paper, the subset B of a topological space $(X, \mathcal{T})$, we will denote the complement of B in $(X, \mathcal{T})$ by $X \setminus B$, the set of positive integers by $\mathbb{N}$, the set of integers numbers by $\mathbb{Z}$, the set of real numbers by $\mathbb{R}$, and usual topology in $\mathbb{R}$ by $\mathcal{U}$. Unless or otherwise mentioned, X stands for the topological space $(X, \mathcal{T})$.

DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4583

Email address: m_halqahtani@ut.edu.sa (M. H. Alqahtani)

2. Preliminaries

In this Section, we recall the basic concepts and findings that make this work self-contained.

Definition 1. *[1] Let K be a subset of the topological space $(X, \mathcal{T})$, then the interior of K is defined as the union of all open subsets of K (or the largest open set contained in K) and is denoted by $int(K)$.*

Definition 2. *[1] Let K be a subset of the topological space $(X, \mathcal{T})$, then the clouser of K is defined as the intersection of all closed sets containing K, and is denoted by $cl(K)$.*

Definition 3. *[1] Let K be a subset of the topological space $(X, \mathcal{T})$. A point $x \in X$ is said to be limit points (or an accumulation point, or a cluster point) of K if and only if every open set V containing x, contains at least one point of K different from x. The set of all limit points of K is called the derived set of K and denoted by $D(K)$.*

Definition 4. *[1] Let K be a subset of the topological space $(X, \mathcal{T})$, then the border of K is defined as $Bd(K) = K \setminus int(K)$.*

Definition 5. *[1] Let K be a subset of the topological space $(X, \mathcal{T})$, then the frontier of K is defined as $Fr(K) = cl(K) \setminus int(K)$.*

Definition 6. *[1] Let K be a subset of the topological space $(X, \mathcal{T})$, then the exterior of K is defined as $Ext(K) = int(X \setminus K)$.*

Definition 7. *[1] A function h from the topological space $(X, \mathcal{T})$ into the topological space $(Y, \mathcal{P})$ is said to be continuous if $h^{-1}(U)$ is an open subset in X for every open subsets U in Y.*

Definition 8. *[1] A function h from the topological space $(X, \mathcal{T})$ into the topological space $(Y, \mathcal{P})$ is said to be open (resp. closed) if $h(U)$ is an open (resp., closed) subset in Y for every open (resp., closed) subsets U in X.*

Definition 9. *[1] A bijection function h from the topological space $(X, \mathcal{T})$ into the topological space $(Y, \mathcal{P})$ is said to be homeomorphism if and only if h and h^{-1} are continuous.*

Definition 10. *[1] Let $(X, \mathcal{T})$ be a topological space, then $(X, \mathcal{T})$ is compact (resp., Lindelöf) if and only if any open cover of X has a finite (resp., countable) subcover of open sets.*

Definition 11. *[1] Let $(X, \mathcal{T})$ be a topological space, then $(X, \mathcal{T})$ is countably compact space if and only if any countable open cover of X has a finite subcover of open sets.*

3. F-open and F-closed sets

Definition 12. *An open subset A of a topological space $(X, \mathcal{T})$ is called F-open set if $cl(A) \setminus A$ is finite set. That is, A is an open set and the frontier of A is a finite set.*

Definition 13. *A closed subset A of a topological space $(X, \mathcal{T})$ is called F-closed set if $A \setminus int(A)$ is finite set. That is, A is a closed set and the frontier of A is a finite set.*

Theorem 1. *Let $(X, \mathcal{T})$ be a topological space, then*

(i) The complement of any F-open subset of X is a F-closed;

(ii) The complement of any F-closed subset of X is a F-open.

Proof. Let A be any F-open subset in X, then $X \setminus A$ is closed and $(X \setminus A) \setminus int(X \setminus A) = (X \setminus A) \setminus (X \setminus cl(A)) = (X \setminus A) \cap cl(A) = cl(A) \setminus A$ is finite. Therefore, $X \setminus A$ is a F-closed. Using the same way to prove the complement of any F-closed subset in X is a F-open.

Remark 1.

i) Any clopen (open-and-closed) subset of the topological space $(X, \mathcal{T})$ is F-open and F-closed sets;

ii) Any finite closed subset of the topological space $(X, \mathcal{T})$ is F-closed.

The collection of all F-open (resp., F-closed) subsets of the topological space $(X, \mathcal{T})$ is denoted by $FO(X)$ (resp., $FC(X)$).

Example 1. *Let $(\mathbb{R}, \mathcal{U})$ be a topological space, where $\mathcal{U}$ is denoted for the usual topology. Let $A = (2, 100)$ is an open interval in $(\mathbb{R}, \mathcal{U})$, then A is a F-open set, because, $(2, 100) \in \mathcal{U}$ and $cl(2, 100) \setminus (2, 100) = [2, 100] \setminus (2, 100) = \{2, 100\}$ is finite. Any open intervals of $(\mathbb{R}, \mathcal{U})$ is a F-open set. On the other hand, let $B = [1, 6]$ is a closed interval in $(\mathbb{R}, \mathcal{U})$, then B is a F-closed set, because, $[1, 6]$ is closed set and $[1, 6] \setminus int[1, 6] = [1, 6] \setminus (1, 6) = \{1, 6\}$ is finite. Any closed interval or finite set of $(\mathbb{R}, \mathcal{U})$ is a F-closed set.*

Definition 14. *If $(X, \mathcal{T})$ is a topological space, x is a point of X, a F-open neighbourhood of x is a F-open subset U of X, which is containing x.*

Lemma 1. *Let $(X, \mathcal{T})$ be a topological space and $\emptyset \neq V \subset X$ is F-open set, then for every $x \in V$ there exists a F-open neighborhood U_x of the point x contained in V.*

Proof. Assume that V is F-open and pick $x \in V$ arbitrary. Let $U_x = V$. Then V is a F-open neighborhood of x and $x \in V \subseteq V$.

The converse of the previous Lemma is not true in general. We have the following example:

Example 2. *Let* $A_n = (n, n+1)$ *be a subset of* $(\mathbb{R}, \mathcal{U})$*, for all* $n \in \mathbb{Z}$*. Then,* $A_n = (n, n+1) \in \mathcal{U}$ *and* $cl(n, n+1) \setminus (n, n+1) = \{n, n+1\}$ *is finite set for all* $n \in \mathbb{Z}$*. Then* $A_n = (n, n+1)$ *is* F*-open set for all* $n \in \mathbb{Z}$*. Let* $B = \bigcup_{n\in\mathbb{Z}}(n, n+1)$ *and let* $x \in B = \bigcup_{n\in\mathbb{Z}}(n, n+1)$ *be arbitrary, then there exists some* $n_1 \in \mathbb{Z}$ *such that* $x \in (n_1, n_1+1)$ *is a* F*-open neighborhood containing* x *and contained in* B*, that is mean for any* $x \in B$ *there exist a* F*-open neighborhood containing* x *and contained in* B*. However,* B *is not* F*-open set, because* $B = \bigcup_{n\in\mathbb{Z}}(n, n+1) = \mathbb{R} \setminus \mathbb{Z}$ *is open set and* $cl(B) \setminus (B) = \mathbb{R} \setminus (\mathbb{R} \setminus \mathbb{Z}) = \mathbb{Z}$ *is not finite set. Therefore,* $B = \bigcup_{n\in\mathbb{Z}}(n, n+1)$ *is not* F*-open set.*

It is clear by the definitions every F-open and F-closed sets are open and closed, respectively. However, the converse is not true in general. Here is an example of open (resp., closed) set which is not F-open (resp., F-closed).

Example 3. *There exist* $A = \mathbb{R}\setminus\mathbb{Z}$ *is an open subset of* $(\mathbb{R}, \mathcal{U})$*. However,* $cl(\mathbb{R}\setminus\mathbb{Z})\setminus(\mathbb{R}\setminus\mathbb{Z}) = \mathbb{R} \setminus (\mathbb{R} \setminus \mathbb{Z})) = \mathbb{Z}$ *is not finite. Hence* A *is not* F*-open set. Also,* $\mathbb{Z}$ *is a closed set in* $(\mathbb{R}, \mathcal{U})$*, but* $\mathbb{Z} \setminus int(\mathbb{Z}) = \mathbb{Z} \setminus \emptyset = \mathbb{Z}$ *is not finite. therefore,* $\mathbb{Z}$ *is not* F*-closed set.*

There is an example of F-open (resp., F-closed) set which is not an open domain (resp., closed domain).

Example 4. *Let* $A = (2, 5) \cup (5, 9)$ *is a* F*-open subset in* $(\mathbb{R}, \mathcal{U})$*, because,* $(2, 5) \cup (5, 9)$ *is open and* $cl((2, 5) \cup (5, 9)) \setminus ((2, 5) \cup ((5, 9)) = [2, 9] \setminus ((2, 5) \cup (5, 9)) = \{2, 5, 9\}$ *is finite. But* A *is not open domain, because* $(2, 5) \cup (5, 9) \neq int(cl((2, 5) \cup (5, 9))) = (2, 9)$*. Moreover,* $\mathbb{R} \setminus A$ *is* F*-closed set, because* $\mathbb{R} \setminus A$ *is closed and* $(\mathbb{R} \setminus A) \setminus int(\mathbb{R} \setminus A) = ((-\infty, 2] \cup [9, \infty) \cup \{5\}) \setminus ((-\infty, 2) \cup (9, \infty)) = \{2, 5, 9\}$ *is finite. But* $\mathbb{R} \setminus A$ *is not closed domain because the complement of closed domain is open domain or* $cl(int(\mathbb{R} \setminus A)) = cl(int((-\infty, 2] \cup [9, \infty) \cup \{5\})) = cl((-\infty, 2) \cup (9, \infty)) = ((-\infty, 2] \cup [9, \infty)) \neq (\mathbb{R} \setminus A)$*.*

Now, there is an example of an open domain (resp., closed domain) set which is not F-open (resp., F-closed) set.

Example 5. *Let* $(\mathbb{R}, \mathcal{T}_{\mathbb{Z}})$ *be the excluded set topological space on* $\mathbb{R}$ *by* $\mathbb{Z}$*. Let* $K = (0, 1)$*, then* $K \in \mathcal{T}_{\mathbb{Z}}$ *and* $cl(K) \setminus K = ((0, 1) \cup \mathbb{Z}) \setminus (0, 1) = \mathbb{Z}$ *is not finite, then* K *is not* F*-open set. But* $K = int(cl(K)) = int(cl(0, 1)) = int((0, 1) \cup \mathbb{Z}) = (0, 1) = K$*, then* K *is open domain. By Theorem 1, the complement of* F*-closed set is* F*-open set and the complement of open domain is closed domain, then* $X \setminus K = X \setminus (0, 1)$ *is an example of closed domain set which is not* F*-closed.*

Theorem 2. *Let* $(X, \mathcal{T})$ *be a topological space, then*

i) Finite union of F*-closed subsets in* X *is a* F*-closed;*

ii) Finite union of F*-open subsets in* X *is a* F*-open;*

iii) Finite intersection of F*-open subsets in* X *is a* F*-open;*

iv) Finite intersection of F*-closed subsets in* X *is a* F*-closed.*

Proof.

i) Suppose that K_i be F-closed set for all $i \in \{1,2,3,\dots,n\}$, then K_i is a closed set and $K_i \setminus int(K_i)$ is finite for all i. Since $\bigcup_{i=1}^n K_i$ is closed, then we need to show the other condition of F-closed set.
Claim:
$$\bigcup_{i=1}^{n} K_i \setminus int(\bigcup_{i=1}^{n} K_i) \subseteq \bigcup_{i=1}^{n}(K_i \setminus int(K_i)),$$
Let $x \in \bigcup_{i=1}^n K_i \setminus int(\bigcup_{i=1}^n K_i)$ be arbitrary. Since $\bigcup_{i=1}^n int(K_i) \subseteq int(\bigcup_{i=1}^n K_i)$, then there exists $i_1 \in \{1,2,3,\dots,n\}$ such that $x \in K_{i_1}$ and $x \notin int(K_i)$ for all $i \in \{1,2,3,\dots,n\}$. Then $x \in (K_{i_1}) \setminus int(K_{i_1})$, then $x \in \bigcup_{i=1}^n (K_i \setminus int(K_i))$. Claim is proved.
Since the finite union of finite sets is finite. Then, $\bigcup_{i=1}^n K_i \setminus int(\bigcup_{i=1}^n K_i)$ is finite. Therefore, $\bigcup_{i=1}^n K_i$ is F-closed.

ii) Suppose that K_i be F-open set for all $i \in \{1,2,3,\dots,n\}$, then K_i is an open set and $cl(K_i) \setminus K_i$ is finite for all i. Since $\bigcup_{i=1}^n K_i$ is open, then we need to show the other condition of F-open set.
Claim:
$$cl(\bigcup_{i=1}^{n} K_i) \setminus \bigcup_{i=1}^{n} K_i \subseteq \bigcup_{i=1}^{n}(cl(K_i) \setminus K_i),$$
Let $x \in cl(\bigcup_{i=1}^n K_i) \setminus \bigcup_{i=1}^n K_i$ be arbitrary. Since $cl(\bigcup_{i=1}^n K_i) = \bigcup_{i=1}^n cl(K_i)$, then $x \in \bigcup_{i=1}^n cl(K_i) \setminus \bigcup_{i=1}^n K_i$, that is mean there exist $i_1 \in \{1,2,3,\dots,n\}$ such that $x \in cl(K_{i_1})$ and $x \notin K_i$ for all $i \in \{1,2,3,\dots,n\}$. Then $x \in (cl(K_{i_1}) \setminus K_{i_1})$, so thus $x \in \bigcup_{i=1}^n (cl(K_i) \setminus K_i)$. Claim is proved.
Since the finite union of finite sets is finite. Then, $cl(\bigcup_{i=1}^n K_i) \setminus \bigcup_{i=1}^n K_i$ is finite. Therefore, $\bigcup_{i=1}^n K_i$ is F-open.

iii) Suppose that K_i be F-open set for all $i \in \{1,2,3,\dots,n\}$, then $\bigcap_{i=1}^n K_i$ is open and by Morgan's Laws, Theorem 1 and Theorem 2 part (i) we have $\bigcap_{i=1}^n K_i$ is F-open, because $X \setminus \bigcap_{i=1}^n K_i = \bigcup_{i=1}^n (X \setminus K_i)$ is F-closed set.

iv) Suppose that K_i be F-closed set for all $i \in \{1,2,3,\dots,n\}$, then $\bigcap_{i=1}^n K_i$ is closed and by Morgan's Laws, Theorem 1 and Theorem 2 part (ii) we have $\bigcap_{i=1}^n K_i$ is F-closed, because $X \setminus \bigcap_{i=1}^n K_i = \bigcup_{i=1}^n (X \setminus K_i)$ is F-open set.

In general, countable union of F-open sets may not be F-open set.

Example 6. *Let $A_n = (n, n+1)$ be a subset of $(\mathbb{R}, \mathcal{U})$, for all $n \in \mathbb{N}$, then, $A_n = (n, n+1) \in \mathcal{U}$ and $cl(n, n+1) \setminus (n, n+1) = \{n, n+1\}$ is finite for all $n \in \mathbb{N}$. Hence, $A_n = (n, n+1)$ is $F-$open set for all $n \in \mathbb{N}$. We have $\bigcup_{n\in\mathbb{N}}(n, n+1) = [1,\infty) \setminus \mathbb{N}$ is an open set. However, $cl(\bigcup_{n\in\mathbb{N}}(n,n+1)) \setminus \bigcup_{n\in\mathbb{N}}(n,n+1) = cl([1,\infty)\setminus\mathbb{N}) \setminus ([1,\infty)\setminus\mathbb{N}) = [1,\infty)\setminus([1,\infty)\setminus\mathbb{N}) = \mathbb{N}$ is not finite set. Therefore, $\bigcup_{n\in\mathbb{N}}(n, n+1)$ is not F-open set.*

In general countable intersection of F-closed sets may not be F-closed set.

Example 7. *By Theorem 1 and Example 6, we have* $\mathbb{R} \setminus (\bigcup_{n\in\mathbb{N}}(n, n+1)) = \bigcap_{n\in\mathbb{N}}(\mathbb{R} \setminus (n, n+1))$ *is countable intersection of* F*-closed sets which is not* F*-closed set or* $V = \mathbb{R} \setminus (n, n+1) = (-\infty, n] \cup [n+1, \infty)$ *is closed for all* $n \in \mathbb{N}$*, and* $((-\infty, n] \cup [n+1, \infty)) \setminus int((-\infty, n] \cup [n+1, \infty)) = \{n, n+1\}$ *is finite set for all* $n \in \mathbb{N}$*. Then* V *is a* F*-closed set for all* $n \in \mathbb{N}$*. Since* $\bigcap_{n\in\mathbb{N}}((-\infty, n] \cup [n+1, \infty)) = (-\infty, 1) \cup \mathbb{N}$ *is closed, then we need to show the other condition of* F*-closed set. Let* $\bigcap_{n\in\mathbb{N}}((-\infty, n] \cup [n+1, \infty)) \setminus int(\bigcap_{n\in\mathbb{N}}((-\infty, n] \cup [n+1, \infty))) = ((-\infty, 1) \cup \mathbb{N}) \setminus int((-\infty, 1) \cup \mathbb{N}) = ((-\infty, 1) \cup \mathbb{N}) \setminus (-\infty, 1) = \mathbb{N}$ *is not finite set. Therefore,* $\bigcap_{n\in\mathbb{N}}(\mathbb{R} \setminus (n, n+1))$ *is not* F*-closed set.*

In general countable union of F-closed sets may not be F-closed set.

Example 8. *Let* $K_n = \{\frac{1}{n}\}$ *is a* $F-$*closed set in* $(\mathbb{R},\mathcal{U})$ *for all* $n \in \mathbb{N}$*, because* $\{\frac{1}{n}\}$ *is closed for all* $n \in \mathbb{N}$*, and* $\{\frac{1}{n}\} \setminus int(\{\frac{1}{n}\}) = \{\frac{1}{n}\} \setminus \emptyset = \{\frac{1}{n}\}$ *is finite for all* $n \in \mathbb{N}$*. However,* $\bigcup_{n\in\mathbb{N}}\{\frac{1}{n}\}$ *is not a closed set. Hence is not a* F*-closed set.*

In general countable intersection of F-open sets may not be F-open set.

Example 9. *We have* $(\frac{-1}{n}, \frac{1}{n})$ *is a* F*-open set in* $(\mathbb{R},\mathcal{U})$ *for all* $n \in \mathbb{N}$*. But* $\bigcap_{n\in\mathbb{N}}(\frac{-1}{n}, \frac{1}{n}) = \{0\}$ *is not open set, then is not* F*-open set.*

4. Some topological properties

Definition 15. *Let* K *be a subset of the topological space* $(X, \mathcal{T})$*. Then,*

i) The F*-interior of* K *is defined as the union of all* F*-open subsets of* K *(or the largest* F*-open set contained in* K*) and is denoted by* $int^F(K)$*.*

ii) The F*-clouser of* K *is defined as the intersection of all* F*-closed sets containing* K*, and is denoted by* $cl^F(K)$*.*

Definition 16. *Let* $x \in int^F(K)$ *if and only if there exist a* F*-open set* U *such that* $x \in U \subseteq K$*.*

Theorem 3. *Let* K *be a subset of the topological space* $(X, \mathcal{T})$*. Then,*

i) $int^F(K) \subseteq int(K) \subseteq K$.

ii) $K \subseteq cl(K) \subseteq cl^F(K)$.

Proof.

i) Let $x \in int^F(K)$ be arbitrary, then there exist a F-open set U such that $x \in U \subseteq K$. Since any F-open set is open, then $x \in int(K)$. Hence, $int^F(K) \subseteq int(K)$. Also by the interior definition of K, we have $int(K) \subseteq K$. Therefore, $int^F(K) \subseteq int(K) \subseteq K$.

ii) By the closure definition, we have $K \subseteq cl(K)$. Let $cl(K) = F$ be the smallest closed set such that $K \subseteq F$. Since any F-closed set is closed, then $F \subseteq cl^F(K)$. Therefore, $K \subseteq cl(K) \subseteq cl^F(K)$.

In general $int(K) \nsubseteq int^F(K)$ and $cl^F(K) \nsubseteq cl(K)$. For example:

Example 10. *Let $K = \mathbb{R} \setminus \mathbb{Z}$ be a subset of the usual topological space $(\mathbb{R},\mathcal{U})$. Then $int(K) = \mathbb{R} \setminus \mathbb{Z}$. However, $\mathbb{R} \setminus \mathbb{Z}$ is not F-open, then the largest F-open subset of $\mathbb{R} \setminus \mathbb{Z}$ is not equal $\mathbb{R} \setminus \mathbb{Z}$. Therefore, $int(K) \nsubseteq int^F(K)$. Let $H = \mathbb{Z}$ be a subset of the usual topological space $(\mathbb{R},\mathcal{U})$, then $H = \mathbb{Z}$ is closed in $(\mathbb{R},\mathcal{U})$, so thus $cl(\mathbb{Z}) = \mathbb{Z}$. However, $\mathbb{Z}$ is not F-closed, then $\mathbb{Z} \subset cl^F(\mathbb{Z})$. Therefore, $cl^F(H) \nsubseteq cl(H)$.*

Corollary 1. *If U is F-open set and $U \cap V = \emptyset$, then $U \cap cl^F(V) = \emptyset$. In particular, If U and V are disjoint F-open sets, then $U \cap cl^F(V) = \emptyset = cl^F(U) \cap V$.*

Proof. Since $U \cap V = \emptyset$, then $V \subseteq X \setminus U$, so thus $cl^F(V) \subseteq cl^F(X \setminus U)$. However, U is F-open, then $X \setminus U$ is F-closed. Hence, $X \setminus U = cl^F(X \setminus U)$, so thus $cl^F(V) \subseteq X \setminus U$. Therefore, $U \cap cl^F(V) = \emptyset$.

Definition 17. *Let K be a subset of the topological space $(X, \mathcal{T})$. A point $x \in X$ is said to be F-limit points (or an F-accumulation point, or a F-cluster point) of K if and only if for any F-open set V containing x, we have $(V \setminus \{x\}) \cap K \neq \emptyset$. The set of all F-limit points of K is called the F-derived set of K and denoted by $D^F(K)$.*

Theorem 4. *Let K and H be subsets of a topological space $(X, \mathcal{T})$. Then we have the following properties:*

(i) $D(K) \subset D^F(K)$, *where $D(K)$ is the derived set of K.*

(ii) *If $K \subseteq H$, then $D^F(K) \subseteq D^F(H)$.*

(iii) $D^F(K) \cup D^F(H) = D^F(K \cup H)$ *and* $D^F(K \cap H) \subset D^F(K) \cap D^F(H)$.

Proof. For (i) it suffices to observe that every F-open is open. For (ii) follow from Definition 17. For (iii) is a modification of the standard proof for D.

In general $D^F(K) \nsubseteq D(K)$. For example:

Example 11. *Let $(\mathbb{R}, \mathcal{T}_{\frac{1}{2}})$ be a topological space, where $\mathcal{T}_{\frac{1}{2}}$ is the particular point topology at $\frac{1}{2}$. Let $K = \mathbb{Z}$, then $D(\mathbb{Z}) = \emptyset$, because $\{\frac{1}{2}, x\}$ for any $x \in \mathbb{R}$ is an open set containing x and $(\{\frac{1}{2}, x\} \setminus \{x\}) \cap \mathbb{Z} = \emptyset$. Let V be any F-open subset of $\mathbb{R}$, then we have $\frac{1}{2} \in V$, and $cl(V) \setminus V$ is finite. Hence, $V = \mathbb{R} \setminus H$ where H is finite. Suppose not, H is infinite, then $cl(V) \setminus V = \mathbb{R} \setminus \mathbb{R} \setminus H = H$ is infinite. Hence, V is not F-open set. let $x \in \mathbb{R}$ be arbitrary, then for any F-open set V containing x we have $V \cap \mathbb{Z} \neq \emptyset$. Hence, $D^F(\mathbb{Z}) = \mathbb{R}$, so thus $D^F(\mathbb{Z}) \nsubseteq D(\mathbb{Z})$. Therefore, $D^F(K) \nsubseteq D(K)$.*

In general $D^F(K \cap H) \nsupseteq D^F(K) \cap D^F(H)$. For example:

Example 12. *There exists $K = (1, 2)$ and $H = (2, 3)$ are F-open subsets of the usual topological space $(\mathbb{R},\mathcal{U})$ such that $D^F(K \cap H) = D^F((1, 2) \cap (2, 3)) = D^F(\emptyset) = \emptyset$ and $D^F(K) \cap D^F(H) = D^F((1, 2)) \cap D^F((2, 3)) = [1, 2] \cap [2, 3] = \{2\}$. Therefore, $D^F(K \cap H) \not\supseteq D^F(K) \cap D^F(H)$.*

Theorem 5. *Let K and H be subsets of a topological space $(X, \mathcal{T})$. Then we have the following properties:*

(i) $int^F(X) = X$.

(ii) $int^F(K) \subseteq K$.

(iii) *If* $K \subseteq H$, *then* $int^F(K) \subseteq int^F(H)$.

(iv) $int^F(int^F(K)) = int^F(K)$.

(v) $int^F(K \cap H) = int^F(K) \cap int^F(H)$.

(vi) $int^F(K) \cup int^F(H) \subseteq int^F(K \cup H)$.

Proof. The properties $(i), (ii), (iii)$ and (iv) follow from Definitions 12 and Definition 15. To prove (v), by property (ii) we have $int^F(K) \subseteq K$ and $int^F(H) \subseteq H$, then $int^F(K) \cap int^F(H) \subseteq K \cap H$. As $int^F(K) \cap int^F(H)$ is F-open, then we have $int^F(K) \cap int^F(H) \subseteq int^F(K \cap H)$, because $int^F(K) \cap int^F(H)$ is F-open and $int^F(K \cap H)$ is the largest F-open set contained in $K \cap H$. Conversely, $(K \cap H) \subseteq K$ and $(K \cap H) \subseteq H$, by property (iii) we have $int^F(K \cap H) \subseteq int^F(K)$ and $int^F(K \cap H) \subseteq int^F(H)$. Hence, $int^F(K \cap H) \subseteq int^F(K) \cap int^F(H)$. Therefore, $int^F(K \cap H) = int^F(K) \cap int^F(H)$. To prove (vi), since $K \subseteq (K \cup H)$ and $H \subseteq (K \cup H)$, from property (iii) we have $int^F(K) \subseteq int^F(K \cup H)$ and $int^F(H) \subseteq int^F(K \cup H)$. Therefore, $int^F(K) \cup int^F(H) \subseteq int^F(K \cup H)$.

In general, $int^F(K \cup H) \nsubseteq int^F(K) \cup int^F(H)$. For example:

Example 13. *Let $K = \mathbb{R} \setminus \mathbb{N}$ and $H = \mathbb{N}$ be a subsets of the usual topological space $(\mathbb{R},\mathcal{U})$, where $\mathbb{N} = \{1, 2, 3, \dots\}$. Then $int^F(K \cup H) = \mathbb{R}$. However, $\mathbb{R} \setminus \mathbb{N}$ is not F-open set, then $int^F(\mathbb{R} \setminus \mathbb{N}) \subset \mathbb{R} \setminus \mathbb{N}$ and $int^F(H) = \emptyset$. Therefore, $int^F(K \cup H) \nsubseteq int^F(K) \cup int^F(H)$.*

Theorem 6. *Let K and H be subsets of a topological space $(X, \mathcal{T})$. Then we have the following properties:*

(i) $cl^F(\emptyset) = \emptyset$.

(ii) $K \subseteq cl^F(K)$.

(iii) *If* $K \subseteq H$, *then* $cl^F(K) \subseteq cl^F(H)$.

(iv) $cl^F(K \cup H) = cl^F(K) \cup cl^F(H)$.

(v) $cl^F(cl^F(K)) = cl^F(K)$.

Proof. The properties $(i),(ii),(iii)$ and (v) follow from Definition 13 and Definition 15. The property (iv), follow from property (iii), Definition 12, Definition 15 and using set theoretic properties.

Theorem 7. *Let K be a subset of the topological space $(X,\mathcal{T})$. Then,*

(i) $int^F(K) = X \setminus cl^F(X \setminus K)$.

(ii) $cl^F(K) = X \setminus int^F(X \setminus K)$.

Proof.

(i) We have $int^F(K) = X \setminus cl^F(X \setminus K)$, then $X \setminus K \subseteq cl^F(X \setminus K)$, thus $X \setminus cl^F(X \setminus K) \subseteq K$. Since $X \setminus cl^F(X \setminus K)$ is F-open, then $X \setminus cl^F(X \setminus K) \subseteq int^F(K)$...(1). Now, let V be any F-open set contained in K, i.e., $V \subseteq K$ and V is F-open, then $X \setminus K \subseteq X \setminus V = cl^F(X \setminus V)$. Then $cl^F(X \setminus K) \subseteq X \setminus V$, hence $V \subseteq X \setminus cl^F(X \setminus K)$. That is, any F-open set contained in K is contained in $X \setminus cl^F(X \setminus K)$, which means that $int^F(K) \subseteq X \setminus cl^F(X \setminus K)$ ...(2). From (1) and (2) equality holds.

(ii) Can be proved by replacing K and $X \setminus K$ by $X \setminus K$ and K, respectively in (i) and using set theoretic properties.

Definition 18. *Let K be a subset of the topological space X. Then F-border of K is defined as $Bd^F(K) = K \setminus int^F(K)$.*

Theorem 8. *Let K be a subset of the topological space $(X,\mathcal{T})$. Then we have the following properties:*

(i) $Bd(K) \subset Bd^F(K)$, *where $Bd(K)$ denotes the border of K.*

(ii) $K = int^F(K) \cup Bd^F(K)$.

(iii) $int^F(K) \cap Bd^F(K) = \emptyset$.

(iv) *K is a $F-$open set if and only if* $Bd^F(K) = \emptyset$.

(v) $int^F(Bd^F(K)) = \emptyset$.

(vi) $Bd^F(Bd^F(K)) = Bd^F(K)$.

(vii) $Bd^F(K) = K \cap cl^F(X \setminus K)$.

Proof. To prove (i), let $x \in Bd(K)$ be arbitrary, then $x \in K \setminus int(K)$, so thus $x \notin int(K)$. By Theorem 3 part (i) we have $int^F(K) \subseteq int(K) \subseteq K$, then $x \notin int^F(K)$. Hence $x \in (K \setminus int^F(K)) = Bd^F(K)$. Therefore, $Bd(K) \subseteq Bd^F(K)$. The properties $(ii),(iii)$ and (iv) follow from Definition 18. To prove (v), let $x \in int^F(Bd^F(K))$ be arbitrary, then $x \in Bd^F(K)$. Since $Bd^F(K) \subseteq K$, then $x \in int^F(Bd^F(K)) \subseteq int^F(K)$, so thus $x \in int^F(K) \cap Bd^F(K)$ which contradicts (iii). For (vi) from Definition 18 and property (v) we have $Bd^F(Bd^F(K)) = Bd^F(K)\backslash int^F(Bd^F(K)) = Bd^F(K)\backslash\emptyset = Bd^F(K)$. The property (vii) follow from Theorem 7 part (i) and Definition 18.

In general $Bd^F(K) \nsubseteq Bd(K)$. For example:

Example 14. *Let $K = \mathbb{R} \setminus \mathbb{N}$ be a subset of the usual topological space $(\mathbb{R},\mathcal{U})$, where $\mathbb{N} = \{1, 2, 3, \dots\}$. Then $Bd(K) = \emptyset$. Since $\mathbb{R}\backslash\mathbb{N}$ is not F-open set, then $int^F(\mathbb{R}\backslash\mathbb{N}) \subset \mathbb{R}\backslash\mathbb{N}$. Hence, $((\mathbb{R} \setminus \mathbb{N}) \setminus int^F(\mathbb{R} \setminus \mathbb{N})) \neq \emptyset$. Therefore, $Bd^F(K) \nsubseteq Bd(K)$.*

Definition 19. *Let K be a subset of the topological space X. Then F-frontier of K is defined as $Fr^F(K) = cl^F(K) \setminus int^F(K)$.*

Theorem 9. *Let K be a subset of the topological space $(X, \mathcal{T})$. Then we have the following properties:*

- *(i)* $Fr(K) \subset Fr^F(K)$, *where $Fr(K)$ denotes the frontier of K.*
- *(ii)* $cl^F(K) = int^F(K) \cup Fr^F(K)$.
- *(iii)* $int^F(K) \cap Fr^F(K) = \emptyset$.
- *(iv)* $Bd^F(K) \subset Fr^F(K)$.
- *(v)* $Fr^F(K) = cl^F(K) \cap cl^F(X \setminus K)$.
- *(vi)* $Fr^F(K) = Fr^F(X \setminus K)$.
- *(viii)* $int^F(K) = K \setminus Fr^F(K)$.

Proof. The property (i), by Theorem 3 and Definition 19, we have $int^F(K) \subseteq int(K)$ and $cl(K) \subseteq cl^F(K)$, then $Fr(K) = (cl(K) \setminus int(K)) \subseteq cl^F(K) \setminus int^F(K) = Fr^F(K)$. The properties (ii) and (iii) follow from Definition 19. For (iv), since $K \subseteq cl^F(K)$, then $Bd^F(K) = (K \setminus int^F(K)) \subseteq (cl^F(K) \setminus int^F(K)) = Fr^F(K)$. The property (v) follow from Theorem 7 and Definition 19. The property (vi) follow from property (v) and Definition 19. The property (vii) follow from Definition 19.

In general $Fr^F(K) \nsubseteq Fr(K)$ and $Fr^F(K) \nsubseteq Bd^F(K)$. For example:

Example 15. *Let $K = \mathbb{N}$ be a subset of the usual topological space $(\mathbb{R},\mathcal{U})$, where $\mathbb{N} = \{1, 2, 3, \dots\}$. Then $Fr(K) = \mathbb{N}$. Without loss of generality, we assume that, $cl^F(K) = \{1, 2, 3, \dots, n\} \cup [n, \infty)$, for some $n \in \mathbb{N}$, then $Fr^F(K) = cl^F(K)\backslash int^F(K) = \{1, 2, 3, \dots, n\} \cup [n, \infty) \setminus \emptyset = \{1, 2, 3, \dots, n\} \cup [n, \infty)$. Hence, $Fr^F(K) \nsubseteq Fr(K)$. For $Fr^F(K) \nsubseteq Bd^F(K)$, let $K = (1, 2]$ be a subsets in $(\mathbb{R},\mathcal{U})$, then $Bd^F(K) = \{2\}$ and $Fr^F(K) = \{1, 2\}$. Hence, $Fr^F(K) \nsubseteq Bd^F(K)$.*

Definition 20. *Let K be a subset of the topological space X. Then F-exterior of K is defined as $Ext^F(K) = int^F(X \setminus K)$.*

Theorem 10. *Let K be a subset of the topological space $(X,\mathcal{T})$. Then we have the following properties:*

(i) $Ext^F(K) \subset Ext(K)$*, where* $Ext(K)$ *denotes the exterior of* K*.*

(ii) $Ext^F(K)$ *is open.*

(iii) $Ext^F(K) = X \setminus cl^F(K)$*.*

(iv) $Ext^F(Ext^F(K)) = int^F(cl^F(K))$*.*

(v) If $K \subseteq H$*, then* $Ext^F(H) \subseteq Ext^F(K)$*.*

(vi) $Ext^F(X \setminus Ext^F(K)) = Ext^F(K)$*.*

(vii) $int^F(K) \subset Ext^F(Ext^F(K))$*.*

(viii) $Ext^F(K \cup H) \subset Ext^F(K) \cup Ext^F(H)$*.*

(ix) $Ext^F(K \cap H) \supset Ext^F(K) \cap Ext^F(H)$*.*

(x) $X = int^F(K) \cup Ext^F(K) \cup Fr^F(K)$*.*

Proof. The property (i), follow from Definition 20 and Theorem 3. The property (ii), follow from Definition 15. The property (iii), follow from Theorem 7 and Definition 20. To prove (iv) $Ext^F(Ext^F(K)) = Ext^F(int^F(X \setminus K)) = Ext^F(X \setminus cl^F(K)) = int^F(X \setminus (X \setminus cl^F(K))) = int^F(cl^F(K))$. To prove (v) since $K \subseteq H$, then $X \setminus H \subseteq X \setminus K$, then $int^F(X\setminus H) \subseteq int^F(X\setminus K)$, hence, $Ext^F(H) \subseteq Ext^F(K)$. For (vi), $Ext^F(X\setminus Ext^F(K)) = Ext^F(X\setminus int^F(X\setminus K)) = int^F(X\setminus(X\setminus int^F(X\setminus K))) = int^F(int^F(X\setminus K)) = int^F(X\setminus K) = Ext^F(K)$. For (vii), $int^F(K) \subseteq int^F(cl^F(K)) \subseteq int^F(X \setminus int^F(X \setminus K)) = int^F(X \setminus Ext^F(K)) = Ext^F(Ext^F(K))$. The property $(viii)$ $Ext^F(K \cup H) = int^F(X \setminus (K \cup H)) = (X\setminus cl^F(K\cup H)) = (X\setminus(cl^F(K)\cup cl^F(H))) \subset X\setminus cl^F(K)\cup X\setminus cl^F(H) = (X\setminus cl^F(K))\cap(X\setminus cl^F(H))) = Ext^F(K) \cap Ext^F(H) \subset Ext^F(K) \cup Ext^F(H)$. The property (ix): $Ext^F(K \cap H) = int^F(X \setminus (K \cap H)) = (X \setminus cl^F(K \cap H)) \supset (X \setminus (cl^F(K) \cap cl^F(H))) = (X \setminus cl^F(K)) \cup (X \setminus cl^F(H))) = Ext^F(K) \cup Ext^F(H) \supset Ext^F(K) \cap Ext^F(H)$. The property (x) follow from the Definitions 19 and 20.

In general $Ext(K) \nsubseteq Ext^F(K)$, $Ext^F(K \cup H) \nsupseteq Ext^F(K) \cup Ext^F(H)$ and $Ext^F(K \cap H) \nsubseteq Ext^F(K) \cap Ext^F(H)$. For example:

Example 16. *Let* $K = \mathbb{Z}$ *be a subset of the usual topological space* $(\mathbb{R},\mathcal{U})$*. Then* $Ext(K) = int(\mathbb{R}\setminus\mathbb{Z}) = \mathbb{R}\setminus\mathbb{Z}$*. But* $\mathbb{R}\setminus\mathbb{Z}$ *is not F-open, then* $Ext^F((K) = int^F((\mathbb{R}\setminus\mathbb{Z}) \subset \mathbb{R}\setminus\mathbb{Z}$*. Hence,* $Ext(K) \nsubseteq Ext^F(K)$*. For* $Ext^F(K\cup H) \nsupseteq Ext^F(K)\cup Ext^F(H)$ *, let* $K = (-\infty, 2)$ *and* $H = (0,\infty)$*, then* $Ext^F(K \cup H) = Ext^F(\mathbb{R}) = \emptyset$ *and* $Ext^F(K) \cup Ext^F(H) = (2,\infty) \cup (-\infty, 0)$*. Hence,* $Ext^F(K \cup H) \nsupseteq Ext^F(K) \cup Ext^F(H)$*. For* $Ext^F(K \cap H) \nsubseteq Ext^F(K) \cap Ext^F(H)$*, let* $K = (-\infty, 2]$ *and* $H = [2,\infty)$*, then* $Ext^F(K \cap H) = Ext^F(\{2\}) = \mathbb{R} \setminus \{2\}$ *and* $Ext^F(K)\cap Ext^F(H) = (2,\infty)\cap(-\infty, 2) = \emptyset$*. Hence,* $Ext^F(K\cap H) \nsubseteq Ext^F(K)\cap Ext^F(H)$*.*

5. F-continuity and F-compactness

Definition 21. *A function $h : (X, \mathcal{T}) \to (Y, \mathcal{P})$ is said to be F-continuous if $h^{-1}(U)$ is a F-open set in X for every open sets U in Y.*

Definition 22. *A function $h : (X, \mathcal{T}) \to (Y, \mathcal{P})$ is said to be $F-$open if $h(U)$ is $F-$open in Y for every open sets U in X.*

Definition 23. *A function $h : (X, \mathcal{T}) \to (Y, \mathcal{P})$ is said to be F-closed if $h(U)$ is F-closed in Y for every closed sets U in X.*

Definition 24. *A bijection function $h : (X, \mathcal{T}) \to (Y, \mathcal{P})$ is said to be $F-$hmoeomrphism if and only if h and h^{-1} are F-continuous.*

Theorem 11. *Let $h : (X, \mathcal{T}) \to (Y, \mathcal{P})$ be a F-continuous function, then h is continuous function.*

Proof. Let U be any open set in Y, then by F-continuity $h^{-1}(U)$ is F-open set in X. Since any F-open set is open, then $h^{-1}(U)$ is open set in X.

In general, the converse of the previous theorem is not true. There is an example of continuous function which is not $F-$continuous function.

Example 17. *The identity function $id : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ is a continuous function. However, there exist $(\mathbb{R} \setminus \mathbb{Z}) \in \mathcal{U}$ and $id^{-1}(\mathbb{R} \setminus \mathbb{Z}) = \mathbb{R} \setminus \mathbb{Z}$ is not $F-$open set, because $cl(\mathbb{R} \setminus \mathbb{Z}) \setminus (\mathbb{R} \setminus \mathbb{Z}) = \mathbb{Z}$ is not finite.*

Definition 25. *Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is a F-compact (resp., F-Lindelöf) space if and only if any open cover of X has a finite (resp., countable) subcover of F-open sets.*

Definition 26. *Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is a F-countably compact space if and only if any countable open cover of X has a finite subcover of F-open sets.*

Theorem 12. *Any F-compact space is compact.*

Proof. Obvious, because any F-open set is open.

Here an example of compact space which is not F-compact space.

Example 18. *Overlapping Interval Topology [3]. On the set $X = [-1, 1]$ we generate a topology from sets of the form $[-1, b)$ for $b > 0$ and $(a, 1]$ for $a < 0$. Then all sets of the form (a, b) are also open. We have X is a compact space, since in any open covering, the two sets which include 1 and -1 will cover X. The space X is not F-compact space, because there exists $\{[-1, 0.5), (-0.5, 1]\}$ open cover for X has no finite subcover of F-open sets, because $[-1, 0.5)$ and $(-0.5, 1]$ are not $F-$open sets ($cl[-1, 0.5) \setminus [-1, 0.5) = [-1, 1] \setminus [-1, 0.5) = [0.5, 1]$ is not finite set and $cl(-0.5, 1] \setminus (-0.5, 1] = [-1, 1] \setminus (-0.5, 1] = [-1, -0.5]$ is not finite set).*

Corollary 2. *Any F-Lindelöf (resp., F-countably compact) space is Lindelöf (resp., countably compact).*

Theorem 13. *If there exists a F-open F-continuous function h from F-compact space $(X, \mathcal{T})$ onto a topological space $(Y, \mathcal{P})$, then $(Y, \mathcal{P})$ is a F-compact space.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be any open cover of Y. Since h is F-continuous, then $h^{-1}(V_\alpha)$ is F-open in X for each $\alpha \in \Lambda$. Since $Y \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$, then $X = h^{-1}(Y) \subseteq h^{-1}(\bigcup_{\alpha\in\Lambda} V_\alpha) = \bigcup_{\alpha\in\Lambda} h^{-1}(V_\alpha)$), that is means $\{h^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an open cover of X (because any F-open set is open). Then, by the F-compactness of X, there exist $\alpha_1, \alpha_2, \cdots \alpha_n \in \Lambda$ such that $h^{-1}(V_{\alpha_1}) \cup h^{-1}(V_{\alpha_2}) \cup \cdots \cup h^{-1}(V_{\alpha_n}) = X$, then $h[h^{-1}(V_{\alpha_1}) \cup h^{-1}(V_{\alpha_2}) \cup \cdots \cup h^{-1}(V_{\alpha_n})] = h(X)$, then $h(h^{-1}(V_{\alpha_1})) \cup h(h^{-1}(V_{\alpha_2})) \cup \cdots \cup h(h^{-1}(V_{\alpha_n})) = Y$, then $V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n} = Y$. Since h is F-open, then $\{V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n}\}$ is a finite subcover of F-open sets for Y. Therefore, $(Y, \mathcal{P})$ is a F-compact space.

A subset B of a space X is F-compact if and only if B is a F-compact topological space with the subspace topology.

Theorem 14. *If there exists a F-continuous function h from F-compact space $(X, \mathcal{T})$ onto a topological space $(Y, \mathcal{P})$, then $(Y, \mathcal{P})$ is compact.*

Proof. Using the same proof of Theorem 13.

Theorem 15. *If there exists a F-open F-continuous function h from a F-Lindelöf (resp., F-countably compact) space $(X, \mathcal{T})$ onto a topological space $(Y, \mathcal{P})$, then $(Y, \mathcal{P})$ is F-Lindelöf (resp., F-countably compact).*

Proof. Using the same proof of Theorem 13.

Theorem 16. *If there exists a F-continuous function h from a F-Lindelöf space $(X, \mathcal{T})$ onto a topological space $(Y, \mathcal{P})$, then $(Y, \mathcal{P})$ is Lindelöf.*

Proof.

Let $\{V_\alpha : \alpha \in \Lambda\}$ be any open cover of Y. Since h is F-continuous, then $h^{-1}(V_\alpha)$ is F-open subsets in X for each $\alpha \in \Lambda$. Since $Y \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$, then $X = h^{-1}(Y) \subseteq h^{-1}(\bigcup_{\alpha\in\Lambda} V_\alpha) = \bigcup_{\alpha\in\Lambda} h^{-1}(V_\alpha)$), that is means $\{h^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an open cover of X (because any F-open set is open). Then, by the F-Lindelöfness of X, there exists $\alpha_1, \alpha_2, \cdots$ such that $h^{-1}(V_{\alpha_1}) \cup h^{-1}(V_{\alpha_2}) \cup \cdots = X$, then $h[h^{-1}(V_{\alpha_1}) \cup h^{-1}(V_{\alpha_2}) \cup \ldots] = h(X)$, then $h(h^{-1}(V_{\alpha_1})) \cup h(h^{-1}(V_{\alpha_2})) \cup \cdots = Y$, then $V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots = Y$. Hence, $\{V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots\}$ is a countable subcover of open sets for Y. Therefore, $(Y, \mathcal{P})$ is a Lindelöf space.

Theorem 17. *If there exists a F-continuous function h from a F-countably compact space $(X, \mathcal{T})$ onto a topological space $(Y, \mathcal{P})$, then $(Y, \mathcal{P})$ is countably compact.*

Proof. Using the same proof of Theorem 16.

References

[1] R. Engelking. *General Topology.* PWN, Warszawa, 1977.

[2] C. Kuratowski. *Topology I.* 4th. ed., in French, Hafner, New york, 1958.

[3] L. Steen and J. Seebach. *GCounterexamples in Topology.* Dover Publications, INC, 1995.