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Outer-connected Hop Dominating Sets in Graphs

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Abstract. Let G be an undirected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A hop dominating set $S \subseteq V(G)$ is called an outer-connected hop dominating set if $S = V(G)$ or the subgraph $\langle V(G) \setminus S \rangle$ induced by $V(G) \setminus S$ is connected. The minimum size of an outer-connected hop dominating set is the outer-connected hop domination number $\widetilde{\gamma_{ch}}(G)$. A dominating set of size $\widetilde{\gamma_{ch}}(G)$ of G is called a $\widetilde{\gamma_{ch}}$-set. In this paper, we investigate the concept and study it for graphs resulting from some binary operations. Specifically, we characterize the outer-connected hop dominating sets in the join, corona and lexicographic products of graphs, and determine bounds of the outer-connected hop domination number of each of these graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Hop domination, outer-connected, join, corona, lexicographic

1. Introduction

As pointed out in [1] and [9], the domination concept has been one of the mainstreams of research in Graph Theory and it has numerous applications, interesting questions and results, and unsolved research questions. Moreover, the concept has already plenty of variations (see [3], [8], [13], [15], [17], [18]).

Outer-connected domination, a variation of domination, was first introduced by Cyman in 2007 [5]. A set $D \subseteq V(G)$ is said to be an outer-connected dominating set of G if D is dominating and either $D = V(G)$ or $\langle V(G) \setminus D \rangle$ is connected. This concept has been studied by several authors like Jiang and Shang [11] and Ahkbari et al. [2], and an outer-connected domination variant was introduced in [12].

In 2015, Natarajan and S. K. Ayyaswamy [16] introduced a new distance related domination parameter and called it the hop domination number of a graph. As defined in [16], a subset S of $V(G)$ is a hop dominating set of G if for every $v \in V(G) \setminus S$, there exists

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$u \in S$ such that $d_G(u, v) = 2$.The concept and some of its variants are also studied in [4], [10], [14], [19], and [20].

Motivated by the hop domination concept and the introduction of the outer-connected domination concept by Cyman, the authors will try to introduce and make an initial study of the concept of outer-connected hop domination. Since domination and hop domination (including their respective variations) have similar applications (e.g. in modeling facility location and protection strategy problems), these two variants can have similar applications as well.

2. Terminology and Notation

For any two vertices u and v in an undirected connected graph G, the distance $d_G(u, v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u, v)$ is called a *u-v geodesic*. The *open neighborhood* of a point u is the set $N_G(u)$ consisting of all points v which are adjacent to u. The *closed interval* $I[x, y]$ consists of x, y and all vertices lying on some x-y geodesic of G and, for $S \subseteq V(G)$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. The *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G(A) = \bigcup\limits_{v \in A} N_G(v)$ is called the *open neighborhood of A* and $N_G[A] = N_G(A) \cup A$ is called the *closed neighborhood of A*. The *open hop neighborhood* of a point u is the set $N_G^2(u) = \{v \in V(G) : d_G(v, u) = 2\}$. The *closed hop neighborhood* of u is $N_G^2[u] = N_G^2(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G^2(A) = \bigcup\limits_{v \in A} N_G^2(v)$ is called the *open hop neighborhood of A* and $N_G^2[A] = N_G^2(A) \cup A$ is called the *closed hop neighborhood of A*.

A set $S \subseteq V(G)$ is a *dominating* set of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The minimum cardinality of a dominating set of a graph G, denoted by $\gamma(G)$, is called the *domination number* of G. A set $S \subseteq V(G)$ is an *outer-connected dominating* set of G if S is dominating and either $S = V(G)$ or the subgraph $\langle V(G) \setminus S \rangle$ induced by $V(G) \setminus S$ is connected. The minimum cardinality of an outer-connected dominating set of a graph G, denoted by $\widetilde{\gamma_c}(G)$, is called the *outer-connected domination number* of G. A dominating set (resp. outer-connected dominating set) S of G with $|S| = \gamma(G)$ (resp. $|S| = \widetilde{\gamma_c}(G)$) is referred to as a γ-set (resp. $\widetilde{\gamma_c}$-set) of G.

A set $S \subseteq V(G)$ is a *hop dominating* set (resp. *total hop dominating* set) if $N_G^2[S] = V(G)$ (resp. $N_G^2(S) = V(G)$). The minimum cardinality of a hop dominating set (resp. total hop dominating set) of a graph G, denoted by $\gamma_h(G)$ (resp. $\gamma_{th}(G)$) is called the *hop domination number* (resp. *total hop domination number*) of G. A hop dominating set (resp. total hop dominating set) of G with cardinality equal to $\gamma_h(G)$ (resp. $\gamma_{th}(G)$) is referred to as a γ_h-set (resp. γ_{th}-set) of G.

A hop dominating set $S \subseteq V(G)$ is called an *outer-connected hop dominating* set if $S = V(G)$ or $\langle V(G) \setminus S \rangle$ is connected. The minimum cardinality of an outer-connected hop dominating set of a graph G, denoted by $\widetilde{\gamma_{ch}}(G)$, is called the *outer-connected hop domination number* of G. An outer-connected hop dominating set of size $\widetilde{\gamma_{ch}}(G)$ of G is called a $\widetilde{\gamma_{ch}}$-set.

A set $D \subseteq V(G)$ is *pointwise non-dominating* if for each $v \in V(G) \setminus D$, there exists $u \in D$ such that $v \notin N_G(u)$. The minimum cardinality of a pointwise non-dominating set of a graph G, denoted by $pnd(G)$, is called the *pointwise non-domination number* of G. A pointwise non-dominating set D of $V(G)$ is an *outer-connected pointwise non-dominating* set if $D = V(G)$ or $\langle V(G) \setminus D \rangle$ is connected. The minimum cardinality of an outer-connected pointwise non-dominating set of a graph G, denoted by $\widetilde{pnd}(G)$, is called the *outer-connected pointwise non-domination number* of G.

Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} \rangle + H^v$. The *lexicographic product* $G[H]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(v,a)(u,b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$. Any non-empty set $C \subseteq V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Specifically, $T_x = \{a \in V(H) : (x,a) \in C\}$ for each $x \in S$. Some parameters studied on these types of graphs can be found in [6] and [7].

3. Known Results

Proposition 1. *[13] Let G be a graph. Then $1 \leq pnd(G) \leq |V(G)|$. Moreover,*

(i) $pnd(G) = |V(G)|$ if and only if G is a complete graph,

(ii) $pnd(G) = 1$ if and only if G has an isolated vetex, and

(iii) $pnd(G) = 2$ if and only of G has no isolated vertex and there exist distinct vertices a and b such that $N_G(a) \cap N_G(b) = \varnothing$.

Theorem 1. *[14] Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets of G and H, respectively.*

Theorem 2. *[14] Let G and H be connected non-trivial graphs. A subset $C = \bigcup_{x \in S} [x \times T_x]$ of $V(G[H])$ is a hop dominating set of $G[H]$ if and only if the following conditions hold:*

(i) S is a hop dominating set of G;

(ii) T_x is a pointwise non-dominating set of H for each $x \in S$ with $|N_G^2(x) \cap S| = 0$.

Theorem 3. *[14] Let G be a connected graph with $\gamma(G) \neq 1$. If S is a hop dominating set of G, then $\gamma_{th}(G) \leq |S \cap N_G^2(S)| + 2|S \setminus N_G^2(S)|$. Moreover, $\gamma_{th}(G) \leq 2\gamma_h(G)$.*

4. Results

Theorem 4. *Let G be any graph on $n \geq 2$ vertices. Then $2 \leq \gamma_h(G) \leq \widetilde{\gamma_{ch}}(G) \leq n$. Moreover,*

- (i) *$\widetilde{\gamma_{ch}}(G) = 2$ if and only if there exist distinct vertices $x, y \in V(G)$ such that $\langle V(G) \setminus \{x, y\}\rangle$ is connected and the following hold:*
  - (a) *$|N_G(v) \setminus \{x, y\} \cap \{x, y\}| \leq 1$ for all $v \in V(G) \setminus \{x, y\}$.*
  - (b) *$N_G(v) \cap N_G(\{x, y\}) \neq \varnothing$ for all $v \in V(G) \setminus \{x, y\}$ such that $N_G(v) \cap \{x, y\} = \varnothing$.*
  - (c) *For all $v \in V(G) \setminus \{x, y\}$, $N_G(v) \cap N_G(x) \neq \varnothing$ if $v \in N_G(y)$ and $N_G(v) \cap N_G(y) \neq \varnothing$ if $v \in N_G(x)$.*
- (ii) *$\widetilde{\gamma_{ch}}(G) = n$ if and only if every component of G is complete.*

Proof. Since every outer-connected hop dominating set of G is a hop dominating set, $\gamma_h(G) \leq \widetilde{\gamma_{ch}}(G)$. Also, since $V(G)$ is an outer-connected hop dominating set of G and any connected graph G with at least two vertices satisfies $2 \leq \gamma_h(G)$, it follows that $2 \leq \gamma_h(G) \leq \widetilde{\gamma_{ch}}(G) \leq n$.

For (i), suppose that $\widetilde{\gamma_{ch}}(G) = 2$. Let $S = \{x, y\}$ be a $\widetilde{\gamma_{ch}}$-set of G. Since S is an outer-connected hop dominating set, $\langle V(G) \setminus \{x, y\}\rangle$ is connected. Let $v \in V(G) \setminus \{x, y\}$. Since S is a hop dominating set, $d_G(x, v) = 2$ or $d_G(y, v) = 2$. Hence, $|N_G(v) \cap \{x, y\}| \leq 1$, showing that (a) holds. If $d_G(x, v) = 2$ (or $d_G(y, v) = 2$), then there exists $z \in N_G(x) \cap N_G(v)$ (resp. there exists $w \in N_G(y) \cap N_G(v)$). Hence, $N_G(v) \cap N_G(\{x, y\}) \neq \varnothing$ whenever $N_G(v) \cap \{x, y\} = \varnothing$, showing (b) holds. Finally, suppose that $|N_G(v) \cap \{x, y\}| = 1$. If $v \in N_G(y)$, then $d_G(x, v) = 2$. Hence, there exist $p \in N_G(v) \cap N_G(x)$, that is, $N_G(v) \cap N_G(x) \neq \varnothing$. Similarly, $N_G(v) \cap N_G(y) \neq \varnothing$ if $v \in N_G(x)$. This shows that (c) holds.

Conversely, suppose there exist distinct vertices $x, y \in V(G)$ such that $\langle V(G) \setminus \{x, y\}\rangle$ is connected and satisfy (a), (b) and (c). Let $v \in \langle V(G) \setminus \{x, y\}\rangle$. By (a), $|N_G(v) \cap \{x, y\}| \leq 1$. If $|N_G(v) \cap \{x, y\}| = 0$, then $d_G(v, x) = 2$ or $d_G(v, y) = 2$ by (b). Suppose $|N_G(v) \cap \{x, y\}| = 1$, say $v \in N_G(y)$. By (c), $N_G(v) \cap N_G(x) \neq \varnothing$. This implies that $d_G(v, x) = 2$. Therefore S is an outer-connected hop dominating set of G. Since G is non-trivial, $\widetilde{\gamma_{ch}}(G) = 2$.

For (ii), suppose that $\widetilde{\gamma_{ch}}(G) = |V(G)|$ and suppose that one component of G, say G_1, is not complete. Then there exist distinct vertices $x_1, y_1 \in V(G_1)$ such that $d_G(x_1, y_1) = 2$. Consequently, $S = V(G) \setminus \{y_1\}$ is an outer-connected hop dominating set of G, contrary to our assumption that $\widetilde{\gamma_{ch}}(G) = |V(G)|$. Thus, every component of G is complete.

Suppose every component of G is complete. Suppose $\widetilde{\gamma_{ch}}(G) = r < |V(G)|$, say S is a $\widetilde{\gamma_{ch}}$-set of G. Suppose $\{G_1, G_2, ..., G_k\}$ contains all the components of G. Since $\langle V(G) \setminus S\rangle$ is connected, there exists $m \in \{1, 2, ..., k\}$ such that $V(G) \setminus S \subseteq V(G_m)$. Hence, $S = (V(G_m) \cap S) \cup [\cup_{j \neq m} V(G_j)]$. Let $v \in V(G_m) \setminus S$ (this v exists because $r < |V(G)|$). Since G_m is complete, $d_G(v, s) = 1$ for all $s \in V(G_m) \cap S$, contrary to the assumption that S is a hop dominating set of G. Therefore, $\widetilde{\gamma_{ch}}(G) = |V(G)|$. □

Corollary 1. *Let G be a graph with n vertices. Then $\widetilde{\gamma_{ch}}(K_n) = \widetilde{\gamma_{ch}}(\overline{K}_n) = n$.*

Proposition 2. *Let n be a positive integer.*

(i) *For path P_n on n vertices,* $\widetilde{\gamma_{ch}}(P_n) = \begin{cases} 2, & \text{if } n = 3 \\ n-2, & \text{if } n = 4,5,6 \\ n-3, & \text{if } n = 7 \\ n-4, & \text{if } n \geq 8. \end{cases}$

(ii) *For cycle C_n on n vertices,* $\widetilde{\gamma_{ch}}(C_n) = \begin{cases} 3, & \text{if } n = 3 \\ 2, & \text{if } n = 4,5 \\ n-4, & \text{if } n \geq 6. \end{cases}$

Proof.

(i) Let $P_n = [v_1, v_2, ..., v_n]$. Clearly, $\widetilde{\gamma_{ch}}(P_n) = 2$ for $n = 3, 4$. Let $n \geq 5$ and let S be a $\widetilde{\gamma_{ch}}$-set of P_n. Since $\langle V(P_n) \setminus S \rangle$ is connected and S is a hop dominating set, $2 \leq |V(P_n) \setminus S| \leq 4$. Clearly, at least one of v_1 and v_n is in S. Suppose that $v_1 \in S$. Suppose further that $|V(P_n) \setminus S| = 2$. Then $n = 5$ or $n = 6$. Hence, $\widetilde{\gamma_{ch}}(P_5) = 5 - 2 = 3$ and $\widetilde{\gamma_{ch}}(P_6) = 6 - 2 = 4$. Next, suppose that $|V(P_n) \setminus S| = 3$. If p is the smallest positive integer such that $v_p \notin S$, then $p \notin \{1, 2, n-3, n-2\}$. It follows that $v_1, v_2, v_{n-1}, v_n \in S$. In this case, it can easily be verified that $n = 7$, and so $\widetilde{\gamma_{ch}}(P_7) = 7 - 3 = 4$. For $n \geq 8$, the set $S' = V(P_n) \setminus \{v_3, v_4, v_5, v_6\}$ is clearly an outer-connected hop dominating set. Thus, $\widetilde{\gamma_{ch}}(P_n) = n - 4$ for all $.n \geq 8$.

(ii) Let $C_n = [v_1, v_2, ..., v_n, v_1]$. Clearly, $\widetilde{\gamma_{ch}}(C_3) = \widetilde{\gamma_{ch}}(K_3) = 3$. Let $n \geq 4$ and let S be a $\widetilde{\gamma_{ch}}$-set of C_n. Since $\langle V(C_n) \setminus S \rangle$ is connected and S is a hop dominating set, $2 \leq |V(C_n) \setminus S| \leq 4$. If $|V(C_n) \setminus S| = 2$, then $n = 4$. Thus, $\widetilde{\gamma_{ch}}(C_4) = 2$. If $|V(C_n) \setminus S| = 3$, then $n = 5$. Hence, $\widetilde{\gamma_{ch}}(C_5) = 2$. Suppose $n \geq 6$. Then $V(C_n) \setminus \{v_2, v_3, v_4, v_5\}$ is an outer-connected hop dominating set of C_n. Therefore, $\widetilde{\gamma_{ch}}(C_n) = n - 4$. □

Theorem 5. *Let a and b be positive integers such that $2 \leq a \leq b \leq n$. Then there exists a connected graph G such that $\gamma_h(G) = a$ and $\widetilde{\gamma_{ch}}(G) = b$.*

Proof. Suppose that $a = b$. Consider the graph $G = K_a$. Then $\gamma_h(G) = \widetilde{\gamma_{ch}}(G) = a$. Next, suppose that $a < b$. Consider the following cases:

Case 1. $a = 2$.
Let $m = b - a$ and consider the graph G in Figure 1. Let $S_1 = \{y_1, y_2\}$ and $S_2 = \{x_1, x_2, z_1, ..., z_m\}$. Then S_1 and S_2 are γ_h-set and $\widetilde{\gamma_{ch}}$-set, respectively, of G. Hence, $\gamma_h(G) = a$ and $\widetilde{\gamma_{ch}}(G) = a + m = b$.

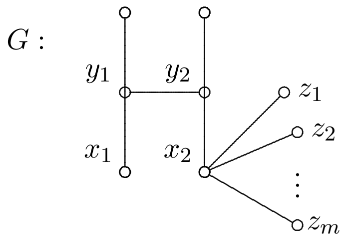

Figure 1

Case 2. $a \geq 3$.

Let $r = b - a + 1$ and consider the graph $G^{'}$ in Figure 2. Let $D_1 = \{x_1, x_2, ..., x_a\}$ and $D_2 = \{x_1, x_2, ..., x_{a-1}, z_1, ..., z_r\}$. Then D_1 and D_2 are γ_h-set and $\widetilde{\gamma_{ch}}$-set, respectively, of G'. Hence, $\gamma_h(G^{'}) = a$ and $\widetilde{\gamma_{ch}}(G^{'}) = r + a - 1 = b$.

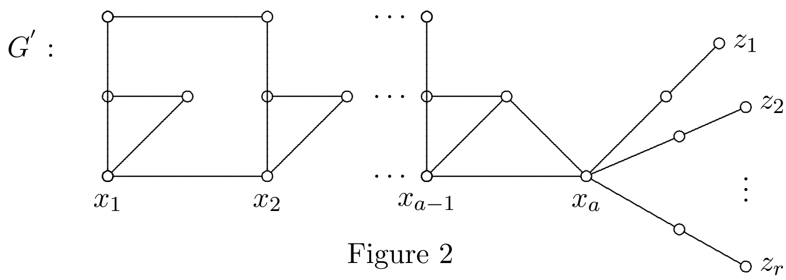

Figure 2

Corollary 2. *For each positive integer n, there exists a connected graph G such that $\widetilde{\gamma_{ch}}(G) - \gamma_h(G) = n$, that is, $\widetilde{\gamma_{ch}}$- γ_h can be made arbitrarily large.*

The next few results deal with the concept of outer-connected pointwise non-dominating sets.

Theorem 6. *Let G be a graph. Then $1 \leq pnd(G) \leq \widetilde{pnd}(G) \leq |V(G)|$. Moreover,*

(i) $\widetilde{pnd}(G) = |V(G)|$ *if and only if G is a complete graph,*

(ii) $\widetilde{pnd}(G) = 1$ *if and only if G has at most two components such that one of them is the trivial graph, and*

(iii) $\widetilde{pnd}(G) = 2$ *if and only if G satisfies one of the following conditions:*

- *(a) G has at most two non-trivial components such that one of them is K_2.*
- *(b) G has exactly three components such that at least two of them are trivial graphs.*
- *(c) G is connected non-complete graph and there exist $a, b \in V(G) (a \neq b)$ such that $N_G(a) \cap N_G(b) = \emptyset$ and $\langle V(G) \setminus \{a, b\} \rangle$ is connected.*

Proof. Since an empty set cannot be an outer-connected pointwise non-dominating set of G and $V(G)$ is an outer connected pointwise non-dominating set of G, it follows that $1 \leq \widetilde{pnd}(G) \leq |V(G)|$.

For (i), suppose first that $\widetilde{pnd}(G) = |V(G)|$ and suppose that G is not a complete graph. Then there exist non-adjacent vertices x and y of G. Consequently, $S = V(G) \setminus \{y\}$ is an outer-connected pointwise non-dominating set of G, contrary to our assumption that $\widetilde{pnd}(G) = |V(G)|$.Thus, G must be a complete graph.

Conversely, suppose G is a complete graph. Suppose that $\widetilde{pnd}(G) = k < |V(G)|$, say S is an $\widetilde{pnd}$-set. Choose any $w \in V(G) \setminus S$. Since S is a pointwise non-dominating set, there exists $u \in S$ such that $uw \notin E(G)$, a contradiction. Therefore, $\widetilde{pnd}(G) = |V(G)|$.

Next, suppose that $\widetilde{pnd}(G) = 1$, say $S = \{v\}$ is an outer-connected pointwise non-dominating set of G. Since $\langle V(G) \setminus S\rangle$ is connected,$V(G) \setminus S$ is contained in a component of G. Thus, G has at most two components and one of them is a trivial graph.

Conversely, if G has at most two components G_1 and G_2 where G_1 is a trivial graph, then $S = V(G_1)$ is an outer-connected point-wise non-dominating set of G. This shows that (ii) holds.

Finally, suppose that $\widetilde{pnd}(G) = 2$. Let $S_1 = \{a, b\}$ be an $\widetilde{pnd}$-set of G.

Case 1. Suppose $a, b \in V(G_1)$, where G_1 is a component of G. If G is connected, then $G = G_1$. If $S_1 = V(G)$, then $G = K_2$. Suppose $S_1 \neq V(G)$. Then by (i), G is a non-complete graph. Since S_1 is an outer-connected pointwise non-dominating set, $N_G(a) \cap N_G(b) = \emptyset$ and $\langle V(G) \setminus S_1\rangle$ is connected. Suppose G is disconnected. Since $\langle V(G) \setminus S_1\rangle$ is connected, $G_1 = K_2$ and G has exactly 2 components G_1 and G_2. Hence, (a) or (c) holds.

Case 2. Suppose $a \in V(G_1)$ and $b \in V(G_2)$, where G_1 and G_2 are distinct components of G. Suppose $G = G_1 \cup G_2$. Then G_1 and G_2 are non-trivial graphs by (ii) and the assumption that $\widetilde{pnd}(G) = 2$. Hence, $\langle V(G) \setminus S_1\rangle$ is disconnected, a contradiction. Therefore, G has more than 2 components. This would imply that $G_1 = G_2 = K_1$. Since $\langle V(G) \setminus S_1\rangle$ is connected, it follows that G has exactly 3 components. In particular, the G_1, G_2 and $\langle V(G) \setminus S_1\rangle$ are the components of G. Thus, (b) holds.

Conversely, suppose that (a) holds. If G has only one component, then $G = K_2$. Hence, $\widetilde{pnd}(G) = 2$. Suppose G has two non-trivial components, say G_1 and G_2, where $G_1 = K_2$. Let $S = V(G_1)$. Clearly, S is a pointwise non-dominating set and $\langle V(G) \setminus S\rangle = G_2$ is connected. Hence, $\widetilde{pnd}(G) = 2$. Suppose (b) holds. Let G_1, G_2 and G_3 be the components of G such that G_1 and G_2 are trivial graphs. Let $S' = V(G_1) \cup V(G_2)$. Clearly, S' is a pointwise non-dominating set and $\left\langle V(G) \setminus S' \right\rangle = G_3$ is connected. Hence, $\widetilde{pnd}(G) = 2$. Suppose (c) holds. Set $S'' = \{a, b\}$ and let $w \in V(G) \setminus S''$. Then by assumption, w is not adjacent to a or b. This implies that S'' is a pointwise non-dominating set of G. Since G is connected and non-complete, $\widetilde{pnd}(G) \neq 1$ by (ii). Hence, $\widetilde{pnd}(G) = 2$. □

The next result is a consequence of Theorem 6 $(iii)(c)$.

Corollary 3. *Let G be a graph on n vertices. Then $\widetilde{pnd}(P_n) = 2$ for all $n \geq 3$ and $\widetilde{pnd}(C_n) = 2$ for all $n \geq 4$.*

Proposition 3. *Let G be a graph with components $G_1, G_2, ..., G_k$ where $k \geq 2$. Then $\widetilde{pnd}(G) = |V(G)|$ - $max\{|V(G_j)| : j \in \{1, ..., k\}\}$.*

Proof. Let S be a $\widetilde{pnd}$-set of G. Since S is outer-connected, $\langle V(G) \setminus S\rangle$ is connected. This implies that $\langle V(G) \setminus S\rangle = G_j$ for some $j \in \{1, 2, ..., k\}$. Since S is a $\widetilde{pnd}$-set, it follows that $|V(G_j)| \geq |V(G_i|$ for all $i \in \{1, 2, ..., k\}$. Therefore,

$$\begin{aligned} \widetilde{pnd}(G) &= |S| \\ &= |\cup_{i \neq j} V(G_i)| \\ &= |V(G)| - |V(G_j| \\ &= |V(G)| - max\{|V(G_i)| : i \in \{1, 2, ..., k\}\}. \end{aligned}$$

This proves the assertion. □

Corollary 4. *Let G be a disconnected graph of order $n \geq 2$. Then $\widetilde{pnd}(G) = n - 1$ if and only if $G = \overline{K}_n$.*

Proof. Let $G_1, G_2, ..., G_k$ be components of G and suppose that $\widetilde{pnd}(G) = n-1$. Then, by Proposition 3,

$$max\{|V(G_j)| : j \in \{1, 2, ..., k\}\} = 1.$$

This implies that $G_j = K_1$ for every $j \in \{1, 2, ..., k\}$. Therefore, $G = \overline{K}_n$.

The converse also follows from Proposition 3. □

Given a complete graph K_n on $n \geq 2$ vertices and $E \subseteq E(K_n)$, we denote by $K_n \setminus E$ the graph obtained from K_n by deleting the edges in set E.

Theorem 7. *Let G be a connected graph on $n \geq 3$ vertices. Then $\widetilde{pnd}(G) = n - 1$ if and only if $G = K_n \setminus E_G$, where $E_G \subseteq E(K_n)$ and for some $2 \leq r \leq n-1$, $\langle\{x : xy \in E_G \text{ for some } y \in V(G)\}\rangle = \overline{K}_r$ in G.*

Proof. Construct a complete graph K_n with $V(K_n) = V(G)$. Then $G = K_n \setminus E_G$ where $E_G \subseteq E(K_n)$. Let $V_G = \{x : xy \in E_G$ for some $y \in V(G)\}$. Suppose $\widetilde{pnd}(G) = n - 1$, say $S = V(G) \setminus \{v\}$ is a $\widetilde{pnd}$-set of G. Since G is connected and S is pointwise non-dominating, there exists $w \in V(G)$ such that $d_G(v, w) = 2$. Hence, $vw \in E_G$. Let r be the largest index such that $v, w \in V(\overline{K}_r)$ and $V(\overline{K}_r) \subseteq V_G$. Since G is connected, $2 \leq r \leq n-1$. Let $z \in N_G(v) \cap N_G(w)$. Suppose that there exists $u \in V(G)$ such that $uz \notin E(G)$. Then $V(G) \setminus \{v, z\}$ is an outer-connected pointwise non-dominating set, contrary to our assumption that $\widetilde{pnd}(G) = n - 1$. Hence, $zy \in E(G)$ for all $y \in V(G) \setminus \{z\}$. Suppose now that $V_G \neq V(\overline{K}_r)$, say $q \in V_G \setminus V(\overline{K}_r)$. By our assumption of r, there exists $t \in V(\overline{K}_r)$ such that $qt \in E(G)$. Note that $tv, tw \notin E(G)$ because $t, v, w \in V(\overline{K}_r)$. Also, since $q \in V_G$, there exists $x \in V_G$ such that $xq \in E_G$, that is, $xq \notin E(G)$. Hence, $V(G) \setminus \{q, t\}$ is an outer-connected pointwise non-dominating set of G, a contradiction. Thus, $\langle V_G\rangle = \overline{K}_r$.

For the converse, suppose that G is obtained from K_n as described. Let S be a $\widetilde{pnd}$-set of G. Then $V(G) \setminus V_G$ contains all the dominating vertices of G. Consequently,

$V(G) \setminus V_G \subseteq S$. Since $\langle V(G) \setminus S\rangle$ is connected and $\langle V_G\rangle = \overline{K}_r$, S contains all but a single vertex of V_G. Thus, $\widetilde{pnd}(G) = |S| = n - 1$. □

Corollary 5. *For $n \geq 3$, $\widetilde{pnd}(K_{1,n-1}) = \widetilde{pnd}(K_n \setminus e) = n - 1$, where $e \in E(K_n)$.*

We now characterize the outer-connected hop dominating sets in some graphs under some binary operations.

Theorem 8. *Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is an outer-connected hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating subsets of G and H, respectively, such that*

(i) *$\langle V(H) \setminus S_H\rangle$ is connected whenever $S_H \neq V(H)$ and $S_G = V(G)$ and*

(ii) *$\langle V(G) \setminus S_G\rangle$ is connected whenever $S_G \neq V(G)$ and $S_H = V(H)$.*

Proof. Suppose S is an outer-connected hop dominating set of $G + H$. Since S is hop dominating, by Theorem 1, $S = S_G \cup S_H$ where S_G and S_H are pointwise non-dominating sets of G and H, respectively. Suppose $S_G = V(G)$ and $S_H \neq V(H)$. Since S is outer-connected, $\langle V(G + H) \setminus S\rangle = \langle V(H) \setminus S_H\rangle$ is connected. Therefore (i) holds. Similarly, (ii) holds.

For the converse, let $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating subsets of G and H, respectively. Then S is a hop dominating set by Theorem 1. If $S = V(G+H)$, then it is an outer-connected hop dominating set. Suppose $S \neq V(G+H)$. Consider the following cases:

Case 1. $S_G \neq V(G)$ and $S_H \neq V(H)$.

Then $\langle V(G + H) \setminus S\rangle = \langle V(G) \setminus S_G\rangle + \langle V(H) \setminus S_H\rangle$ is connected.

Case 2. $S_G = V(G)$ and $S_H \neq V(H)$.

Then $\langle V(G + H) \setminus S\rangle = \langle V(H) \setminus S_H\rangle$ is connected by (i).

Case 3. $S_H = V(H)$ and $S_G \neq V(G)$.

Then $\langle V(G + H) \setminus S\rangle = \langle V(G) \setminus S_G\rangle$ is connected by (ii).

Therefore S is an outer-connected hop dominating set of $G + H$. □

The next result is based from Proposition 1 (i), Theorem 6, Corollary 3, Corollary 4 and Theorem 8.

Corollary 6. *Let G and H be any two graphs of orders m and n, respectively. Then*

$$\widetilde{\gamma_{ch}}(G+H) = \begin{cases} pnd(G) + pnd(H), & G \text{ and } H \text{are non-complete,} \\ |V(G)| + \widetilde{pnd}(H), & G \text{ is complete and } H \text{is non-complete and} \\ |V(H)| + \widetilde{pnd}(G), & H \text{ is complete and } G \text{is non-complete} \end{cases}$$

In particular,

(i) *$\widetilde{\gamma_{ch}}(G + H) = m + n$, if G and H are complete;*

(*ii*) $\widetilde{\gamma_{ch}}(K_{1,n}) = \widetilde{\gamma_{ch}}(K_1 + \overline{K}_n) = 1 + \widetilde{pnd}(\overline{K}_n) = n$ *for* $n \geq 2$.

(*iii*) $\widetilde{\gamma_{ch}}(F_n) = 1 + \widetilde{pnd}(P_n) = 3$ *for* $n \geq 2$.

(*iv*) $\widetilde{\gamma_{ch}}(W_n) = 1 + \widetilde{pnd}(C_n) = 3$ *for* $n \geq 4$.

(*v*) $\widetilde{\gamma_{ch}}(K_{m,n}) = pnd(\overline{K}_m) + pnd(\overline{K}_n) = 2$ *for* $m, n \geq 2$.

Theorem 9. *Let G be a connected graph and let H be any graph. Then a subset C of $V(G \circ H)$ is an outer connected hop dominating set of $G \circ H$ if and only if*

$$C = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$$

where $S_v \subseteq V(H^v)$ for each $v \in V(G)$ and satisfies each of the following statements:

(*i*) *$A = V(G)$ or $\langle V(G) \setminus A \rangle$ is connected.*

(*ii*) *If $A = V(G)$, then $\langle V(H^v) \setminus S_v \rangle$ is a connected proper subgraph of H^v for at most one vertex $v \in A$. Otherwise, $S_v = V(H^v)$ for all $v \in A$.*

(*iii*) *For all $v \in (V(G) \setminus N_G^2[A])$, there exists $w \in N_G(v)$ such that $S_w \neq \varnothing$.*

(*iv*) *For all $v \in (V(G) \setminus N_G[A])$, S_v is a pointwise non-dominating set of H^v.*

Proof. Let C be an outer-connected hop dominating set of $G \circ H$, $S_v = V(H^v) \cap C$ for each $v \in V(G)$ and set $A = V(G) \cap C$. Suppose $A \neq V(G)$. If $|V(G) \setminus A| = 1$, then we are done. Suppose $|V(G) \setminus A| \geq 2$. Let $u, v \in V(G) \setminus A$ where $u \neq v$. Then $u, v \in V(G \circ H) \setminus C$. Since $\langle V(G \circ H) \setminus C \rangle$ is connected, there is a u-v geodesic P in $\langle V(G \circ H) \setminus C \rangle$. Hence, P is a u-v geodesic in $\langle V(G) \setminus A \rangle$. Thus, $\langle V(G) \setminus A \rangle$ is connected, showing that (i) holds. Suppose $A = V(G)$. If $S_v = V(H^v)$ for all $v \in A$, then we are done. Suppose there exists $v \in A$ such that $S_v \neq V(H^v)$. Suppose further there exists $w \in A \setminus \{v\}$ such that $S_w \neq V(H^w)$. Let $p \in V(H^v) \setminus S_v$ and $q \in V(H^w) \setminus S_w$. Since every p-q path contains vertices v and w, it follows that there is no p-q path in $\langle V(G \circ H) \setminus C \rangle$, contrary to the assumption that $\langle V(G \circ H) \setminus C \rangle$ is connected. Hence, there is at most a single vertex $v \in A$ such that $S_v \neq V(H^v)$. Suppose $A \neq V(G)$. Suppose further that there exists $v \in A$ such that $S_v \neq V(H^v)$. Choose any $z \in V(G) \setminus A$ and $y \in V(H^v) \setminus S_v$. Since any y-z path contains v as a vertex it follows that there is no y-z path in $\langle V(G \circ H) \setminus C \rangle$, a contradiction. Therefore $S_v = V(H^v)$ for all $v \in A$. Hence, (ii) holds.

Next, let $v \in (V(G) \setminus N_G^2[A])$. Then $v \notin C \cup N_G^2(A)$. Since C is a hop dominating set, there exists $y \in C \cap N_{G \circ H}^2(v)$. Since $v \notin N_G^2(A)$ and $d_{G \circ H}(y, v) = 2$, it follows that there exists $w \in N_G(v)$ such that $y \in S_w$. Hence, (iii) holds.

Finally, let $v \in V(G) \setminus N_G[A]$ and let $z \in V(H^v) \setminus S_v$. Then $z \notin C$. Since C is hop dominating, there exists $x \in C \cap N_{G \circ H}^2(z)$. Hence, $x \in S_v$ and $d_{G \circ H}(z, x) = d_{H^v}(z, x) = 2$. Therefore, S_v is a pointwise non-dominating set of H^v, showing that (iv) holds.

Conversely, suppose that C has the given form and satisfies properties (i), (ii), (iii) and (iv). Let $x \in V(G+H) \setminus C$ and let $v \in V(G)$ such that $x \in V(v+H^v)$. Suppose $x = v$. If $v \in N_G^2(A)$, then we are done. Suppose $v \in (V(G) \setminus N_G^2[A]$. By (iii), there exists $w \in N_G(v)$ such that $S_w \neq \varnothing$. Choose any $p \in S_w$ then $p \in C$ and $d_{G \circ H} = 2$. Therefore C is a hop dominating set.

Let $a, b \in V(G \circ H) \setminus C$ with $a \neq b$ and let $v, w \in V(G)$ such that $a \in V(v+H^v)$ and $b \in V(w+H^w)$. Consider the following cases:
Case 1. $v = w$.

If $a = v$ and $b \in V(H^v) \setminus S_v$, then $ab \in E(G \circ H)$ and we are done. Let $a, b \in V(H^v) \setminus S_v$. If $A = V(G)$, then $v \in A$. By (ii), $\langle V(H^v) \setminus S_v \rangle$ is connected, hence, there exists an a-b path in $\langle V(H^v) \setminus S_v \rangle$. This a-b path is also an a-b path in $\langle V(G \circ H) \setminus C \rangle$. If $A \neq V(G)$, then $v \notin A$ by the second part of (ii). Therefore, $[a, v, b]$ is an a-b path in $\langle V(G \circ H) \setminus C \rangle$.
Case 2. $v \neq w$.

If $a = v$ and $b = w$, then $v, w \in V(G) \setminus A$. By (i), there exists an a-b path P in $\langle V(G) \setminus A \rangle$. Hence, P is a path in $\langle V(G \circ H) \setminus C \rangle$. Suppose $a \in V(H^v) \setminus S_v$ and $b \in V(H^w) \setminus S_w$. By (ii), $v, w \notin A$. By (i), there exists a v-w path $P = [v_1, v_2, ..., v_k]$ where $v_1 = v$, $v_k = w$ in $\langle V(G) \setminus A \rangle$. Hence, $P' = [a, v_1, v_2, ..., v_k, b]$ is an a-b path in $\langle V(G \circ H) \setminus C \rangle$.

Suppose $a = v$ and $b \in (V(H^w) \setminus S_w)$. Since $v \notin A$, $A \neq V(G)$. Hence, by the second part of (ii), $w \notin A$. It follows from (i) that there exists a v-w path $P = [v_1, v_2, ..., v_k]$ where $v_1 = a$, $v_k = w$ in $\langle V(G) \setminus A \rangle$. Consequently, $P^* = [v_1, v_2, ..., v_k, b]$ is an a-b path in $\langle V(G \circ H) \setminus C \rangle$.

Therefore, $\langle V(G) \setminus C \rangle$ is connected. Accordingly, C is an outer-connected hop dominating set of $G \circ H$. □

Corollary 7. *Let G be a connected graph and let H be any graph of orders m and n, respectively. Then $\widetilde{\gamma_{ch}}(G \circ H) \leq min\{(n+1)\widetilde{\gamma_c}(G), m(pnd(H))\}$.*

Proof. Let A be a $\widetilde{\gamma_c}$-set in G. By Theorem 9,

$$C = A \cup \left(\bigcup_{v \in A} V(H^v) \right)$$

is an outer-connected hop dominating set in $G \circ H$. Thus,

$$\begin{aligned} \widetilde{\gamma_{ch}}(G \circ H) &\leq |C| \\ &= |A| + \sum_{v \in A} |V(H^v)| \\ &= (n+1)\widetilde{\gamma_c}(G). \end{aligned}$$

Next. let $A = \varnothing$ and S_v be a pnd-set of H for each $v \in V(G)$. By Theorem 9,

$$C^* = \bigcup_{v \in V(G)} S_v$$

is an outer-connected hop dominating set in $G \circ H$. Hence,

$$\begin{aligned}\widetilde{\gamma_{ch}}\,(G \circ H) &\leq |C^*| \\ &= \sum_{v \in V(G)} |S_v| \\ &= m(pnd(H)).\end{aligned}$$

Therefore, $\widetilde{\gamma_{ch}}(G \circ H) \leq min\,\{(n+1)\widetilde{\gamma_c}(G), m(pnd(H))\}$. □

Example 1. *Consider the corona $C_4 \circ P_3$ in Figure 1. It can be verified that*

$$\begin{aligned}\widetilde{\gamma_{ch}}\,(C_4 \circ P_3) &= 6 \\ &< 8 \\ &= |V(C_4)|\, pnd(P_3)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\gamma_{ch}}\,(C_4 \circ P_3) &= 6 \\ &< 8 \\ &= \widetilde{\gamma_c}(C_4)\,(|V(P_3)|+1)\,.\end{aligned}$$

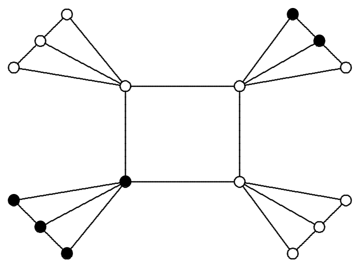

Figure 1: The corona $C_4 \circ P_3$

Example 2. *Consider the corona of $K_2 \circ P_3$. Then*

$$\begin{aligned}\widetilde{\gamma_{ch}}(K_2 \circ P_3) &= 4 \\ &= \widetilde{\gamma_c}(K_2)(|V(P_3)|+1) \\ &= |V(K_2)|\,(pnd(P_3))\end{aligned}$$

Figure 2: The corona $K_2 \circ P_3$

Note that Example 2 shows that the given bound in Corollary 7 cannot be improved. On the other hand, Example 1 shows that strict inequality given in Corollary 7 is attainable.

Theorem 10. *Let G and H be connected non-trivial graphs. A subset $C = \bigcup_{x\in S}[\{x\} \times T_x]$ of $V(G[H])$ is an outer-connected hop dominating set of $G[H]$ if and only if the following conditions hold:*

(i) S is a hop dominating set of G; and

(ii) T_x is a pointwise non-dominating set of H for each $x \in S$ with $\left|N_G^2(x) \cap S\right| = 0$.

(iii) $\langle (V(G) \setminus S) \cup \{v \in S : T_v \neq V(H)\}\rangle$ is a connected graph in G.

Proof. Suppose C is an outer-connected hop dominating set of $G[H]$ and let $W = (V(G) \setminus S) \cup \{v \in S : T_v \neq V(H)\}$. Since every outer-connected hop dominating set is a hop dominating set, (i) and (ii) hold by Theorem 2. Let $u, v \in W$ where $u \neq v$. Suppose $u, v \in V(G) \setminus S$. Let $a \in V(H)$. Then (u, a), $(v, a) \in V(G[H]) \setminus C$. Since $\langle V(G[H]) \setminus C\rangle$ is connected, there exists (u, a)-(v, a) geodesic $[(u_1, a_1), (u_2, a_2), ..., (u_k, a_k)]$, where $(u_1, a_1) = (u, a)$ and $(u_k, a_k) = (v, a)$, in $\langle V(G[H]) \setminus C\rangle$. Then $u_i \in W$ for all $i \in \{1, 2, ..., k\}$. Thus, $[u_1, ..., u_k]$ is a u-v path in $\langle W\rangle$.

Suppose $u, v \in S$ where $T_u \neq V(H)$ and $T_v \neq V(H)$. Let $a \in V(H) \setminus T_u$ and $b \in V(H) \setminus T_v$. Then $(u, a), (v, b) \in V(G[H]) \setminus C$. Since $\langle V(G[H]) \setminus C\rangle$ is connected, there exists (u, a)-(v, b) geodesic $[(u_1, a_1), ..., (u_m, a_m)]$, where $(u, a) = (u_1, a_1)$ and $(u_m, a_m) = (v, b)$, in $\langle V(G[H]) \setminus C\rangle$. Again, $u_i \in W$ for all $i \in \{1, 2, ..., k\}$. Thus, $[u_1, ..., u_k]$ is a u-v path in $\langle W\rangle$. Similarly if $u \in V(G) \setminus S$ and $v \in S$ and $T_v \neq V(H)$, then it can be shown that a u-v path in $\langle W\rangle$ exists. Therefore, $\langle W\rangle$ is connected.

Conversely, suppose that C has the given form and satisfies properties (i), (ii) and (iii). By (i), (ii) and Theorem 2, C is a hop dominating set. Let (u, a), $(v, b) \in V(G[H]) \setminus C$ with $(u, a) \neq (v, b)$. Consider the following cases.

Case 1. $u, v \in V(G) \setminus S$.

Subcase 1.1 $u = v$.

Then $a \neq b$. Since H is connected, there exists an a-b path $[a_1, a_2, ..., a_k]$ with $a_1 = a$ and $a_k = b$ in H. Hence, the path $[(u, a_1), (u, a_2), ..., (u, a_k)]$ is a (u, a)-(v, b) path in $\langle V(G[H]) \setminus C\rangle$.

Subcase 1.2 $u \neq v$.

Since $u, v \in V(G) \setminus S$, there exists a u-v path $[u_1, u_2, ..., u_k]$ where $u_1 = u$ and $u_k = v$, in $\langle W\rangle$ by (iii). For each $i \in \{2, 3, ..., k-1\}$, choose any $a_i \in V(H)$ if $u_i \in V(G) \setminus S$. Otherwise let $a_i \in V(H) \setminus T_{u_j}$ if $u_j \in S$. Then $[(u_1, a_1), (u_2, a_2), ..., (u_k, a_k)]$ where $a_1 = a$ and $a_k = b$ is a (u, a)-(v, b) path in $\langle V(G[H]) \setminus C\rangle$. Hence, the path $[(u, a_1), (u, a_2), ..., (u, a_k)]$ is a (u, a)-(v, b) path in $\langle V(G[H]) \setminus C\rangle$.

Case 2. $u \in V(G) \setminus S$ and $v \in S$.

Then $b \in V(H) \setminus T_v$. By (iii), a u-v path $[u_1, u_2, ..., u_p]$ where $u_1 = u$, $u_p = v$ exists in $\langle W \rangle$. For each $i \in \{2, 3, ..., p-1\}$, choose any $a_i \in V(H)$ if $u_i \in V(G) \setminus S$. Otherwise let $a_i \in V(H) \setminus T_{u_j}$ if $u_j \in S$. Hence, $[(u_1, a_1), (u_2, a_2), ..., (u_p, a_p)]$, where $a_1 = a$, $a_p = b$, is a (u, a)-(v, b) path in $\langle V(G[H]) \setminus C \rangle$.

Case 3. $u, v \in S$.

Then $a \in V(H) \setminus T_u$. and $b \in V(H) \setminus T_v$. Again, by (iii) and by using similar arguments used in Case 1 and Case 2, there exists (u, a)-(v, b) path in $\langle V(G[H] \setminus C \rangle$.

Accordingly, C is an outer-connected hop dominating set of $G[H]$. □

Corollary 8. *Let G and H be non-trivial connected graphs such that $\gamma(G) \neq 1$. Then*

$$\widetilde{\gamma_{ch}}(G[H]) = \gamma_{th}(G)$$

Proof. Let S be a γ_{th}-set of G and let $p \in V(H)$. Set $T_x = \{p\}$ for every $x \in S$. Then

$$C = \bigcup_{x \in S} (\{x\} \times T_x) = S \times \{p\}$$

is an outer-connected hop dominating set in $G[H]$ by Theorem 10. Hence

$$\begin{aligned} \widetilde{\gamma_{ch}}(G[H]) &\leq |C| \\ &= |S \times \{p\}| \\ &= \gamma_{th}(G). \end{aligned}$$

Next, let $C_0 = \cup_{x \in S_0} (\{x\} \times R_x)$ be $\widetilde{\gamma_{ch}}$-set of $G[H]$. Then S_0 is a hop dominating set and R_x is a pointwise non-dominating set of H for each $x \in S_0 \setminus N_G^2(S_0)$, by Theorem 10 Hence,

$$\begin{aligned} \widetilde{\gamma_{ch}}(G[H]) &= |C_0| \\ &= \sum_{x \in S_0} |R_x| \\ &= \sum_{x \in S_0 \cap N_G^2(S)} |R_x| + \sum_{x \in S \setminus N_G^2(S_0)} |R_x| \\ &\geq \left|S_0 \cap N_G^2(S_0)\right| + \left|S_0 \setminus N_G^2(S_0)\right| pnd(H). \end{aligned}$$

Since H is a non-trivial connected graph, $pnd(H) \geq 2$. Thus, by Theorem 3, $\widetilde{\gamma_{ch}}(G[H]) \geq \gamma_{th}(G)$. This establishes the desired equality. □

5. Conclusion

Outer-connected hop domination, a variant of hop domination, has been introduced and studied for some graphs and graphs resulting from the join, corona and lexicographic product of two graphs. In the case of the join of graphs, the concept of outer-connected pointwise non-domination plays a vital role. It is recommended that some bounds on the outer-connected hop domination be determined and that the parameter be studied for other graphs.

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