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Special Sequences On Generalized Metric Spaces

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Abstract. In this article, in a generalized metric space, we will focus on new types of sequences. We introduce three new kinds of Cauchy sequences and study their significance in generalized metric spaces. Also, we give several interesting properties of these sequences.

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1. Introduction

The notion of completeness plays in the theory of metric spaces governs the importance upgrade in topological metrics. Few researchers had examined the significance of complete metric spaces e.g. [4, 5, 8, 10–13, 15–19]. Inspiring the earlier literature, Korczak-Kubiak et.al introduced the concept of new metric space namely, generalized metric space during 2013 and then defined two new things in the name of kernel and perfect kernel in a generalized metric space. Using these tools, they generated three types of complete spaces which has been defined weakly complete, complete and strongly complete spaces. In addition,they have analyzed the nature of these three types of complete spaces in generalized metric spaces.

Similarly, the nature of Bourbaki-Cauchy sequences e.g. [2, 3, 9] has been obtained which influenced some new types of sequences namely Bourbaki-Cauchy, co-finally-Cauchy and pseudo-Cauchy in a metric space given by Aggarwal et.al [1].

Motivated by this, in a generalized metric space, we redefined three types of sequences namely, Bourbaki-Cauchy, cofinally-Cauchy and pseudo-Cauchy sequences as well as complete metric spaces. Also, we investigate their significance and give some relationships between these sequences in a generalized metric space.

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Throughout this paper, $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real and natural numbers, respectively.

2. Preliminaries

First, we recall some basic things defined in a generalized metric space which are useful to the development of the next sections.

Let $X \neq \emptyset$. The symbol Ω to denote the family consisting of metrics defined on subsets of X, that is $\sigma \in \Omega$, then there exists a non-null set $B_\sigma \subset X$ such that σ is a metric on B_σ where B_σ is a domain space of σ and it will be denoted by $dom(\sigma)$. Then (X, Ω) is called as a *generalized metric space* (GMS) [7].

If σ is a metric on X and C is a non-null subset X, then we write $\sigma|_C$ for the restriction of the metric σ to $C \times C$. Moreover, put $\Omega|_C = \{\sigma|_C \mid \sigma \in \Omega\}$ for any $\Omega \subset \Omega_X$ where Ω_X is the collection of all metrics defined on X [7].

Obviously, if (X, Ω_X) is a generalized metric space, $\Omega \subset \Omega_X$ and C is a non-null subset of X, then $(C, \Omega|_C)$ is a generalized metric space [7].

Denote μ_Ω is the family of Ω-*open sets* in a generalized metric space (X, Ω), more precisely, $H \in \mu_\Omega$ if and only if for each $c \in H$, there exist $\sigma \in \Omega$ and $\varepsilon > 0$ such that $B_\sigma(c, \varepsilon) \subset H$ where $B_\sigma(c, \varepsilon) = \{d \in dom(\sigma) \mid \sigma(c, d) < \varepsilon\}$ [7].

Let X be any non-null set. A collection μ of subsets of X is a *generalized topology* [7] in X if it contains the empty set and it closed under arbitrary union. Then the pair (X, μ) is called as a *generalized topological space* (GTS) [7].The elements of μ are called μ-open set of X.

Moreover, if (X, Ω) is a GMS, then (X, μ_Ω) is a GTS [7].

Now we remember some definitions and lemmas that are found in [7].

Definition 1. *A subset Q of X is said to be μ-dense if $c_\mu(Q) = X$.*

Definition 2. *Let (X, Ω) be a GMS. A finite family $\Omega_0 \subset \Omega$ is called a kernel of the space X if for any $J \in \tilde{\mu}_\Omega$, there exists $\sigma \in \Omega_0$ such that $i_\sigma(J) \neq \emptyset$ where $\tilde{\mu}_\Omega = \{K \in \tilde{\mu}_\Omega | K \neq \emptyset\}$.*

Definition 3. *Let (X, Ω) be a GMS and $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then $\{z_n\}$ is said to be :*
(i) *Cauchy* if there is a metric $\sigma \in \Omega$ and for every $\varepsilon > 0$, there exists a positive integer N_0 such that $\sigma(z_n, z_m) < \varepsilon$ for all $n, m \geq N_0$.
(ii) *convergent* to $z \in X$ if there is a metric $\sigma \in \Omega$ and for every $\varepsilon > 0$, there exists a positive integer N_1 such that $\sigma(z_n, z) < \varepsilon$ for all $n, m \geq N_1$.

Then z is called the *limit point* of the sequence $\{z_n\}_{n\in\mathbb{N}}$.

Definition 4. *A generalized metric space (X, Ω) is a weakly complete metric space if there exists a kernel $\Omega_0 \subset \Omega$ consisting of complete metrics.*

Definition 5. *A space X is called as a hyperconnected space [6] if every non-null μ-open subset H of X is a μ-dense set.*

3. Relationship among sequences

Here, we degenerate three kinds of sequences namely, Bourbaki-Cauchy, pseudo-Cauchy and cofinally-Cauchy sequences in generalized metric spaces. In a generalized metric space, the nature of these sequences is analyzed. In view of complete metric spaces founded in [7], we define three types of complete metric spaces, namely Bourbaki-complete, pseudo-complete and cofinally-complete metric spaces. The relationship between these complete spaces is explored.

Definition 6. Let (X, Ω) be a generalized metric space and ε be a positive number. An ordered set of points $\{c_0, c_1, ..., c_m\}$ is said to be *ε-chain of length of m* from c_0 to c_m if there is a metric $\sigma \in \Omega$ satisfying $\sigma(c_{i-1}, c_i) < \varepsilon$ where $i = 1$ to m.

Example 7. Consider the generalized metric space (X, Ω) where $X = \mathbb{R}, \Omega = \{\sigma_1, \sigma_2, \sigma_3\}$ and the metrics are defined by $\sigma_1(c, d) = |c - d|$;

$$\sigma_2(c, d) = \begin{cases} 0 & \text{if} \quad c = d, \\ 1 & \text{if} \quad c \neq d. \end{cases}$$

and $\sigma_3(c, d) = \frac{\sigma_1(c,d)}{1+\sigma_1(c,d)}$ for all $c, d \in X$.

(a). Let $\varepsilon = 0.5$ and $\{c_m\}_{m=0}^{m=13}$ be a set of points where $c_0 = 1; c_m = c_{m-1} + 0.01$ for $m = 1$ to 13. Then the set of points $\{c_m\}_{m=0}^{m=13}$ have a ε-chain of length of m from c_0 to c_m with respect to σ_1, σ_2 and σ_3.

(b). Let $\varepsilon = 0.6$ and $\{c_n\}_{n\in\mathbb{N}} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ be a sequence in X. Take $\{c_m\}_{m=0}^{m=13}$ be a set of points where $c_0 = 1; c_m = \frac{c_0}{m+1}$ for $m = 1$ to 13. Then the set of points $\{c_m\}_{m=0}^{m=13}$ have a ε-chain of length of m from c_0 to c_m with respect to σ_1.

Definition 8. Let (X, Ω) be a generalized metric space. A sequence $\{z_n\}_{n\in\mathbb{N}}$ is said to be:

(a) *Bourbaki-Cauchy* with respect to $\sigma \in \Omega$ in X if for every $\varepsilon > 0$, there exist $r \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that whenever $n > j \geq n_0$ the points z_j and z_n can be joined by an ε-chain of length r with respect to the same metric.

(b) *pseudo-Cauchy* with respect to $\sigma \in \Omega$ in X if for every $\varepsilon > 0$ and for every $n \in \mathbb{N}$, there exist $s, t \in \mathbb{N}, s \neq t$ such that $s, t > n$ and $\sigma(z_s, z_t) < \varepsilon$.

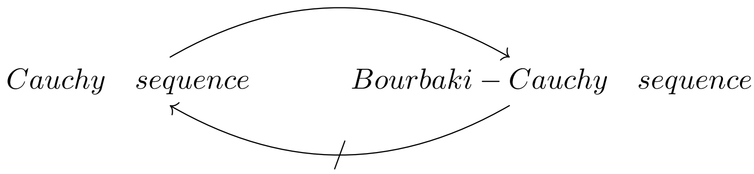
(c) *cofinally-Cauchy* with respect to $\sigma \in \Omega$ in X if for every $\varepsilon > 0$, there exists an infinite subset N_ε of $\mathbb{N}$ such that for every $l, m \in N_\varepsilon$ we have $\sigma(z_l, z_m) < \varepsilon$.

We notated by,
$\mathfrak{B}(\sigma) = \{\{d_n\} \mid \{d_n\}$ is a Bourbaki-Cauchy sequence with respect to σ in $X\}$;
$\mathfrak{P}(\sigma) = \{\{l_n\} \mid \{l_n\}$ is a pseudo-Cauchy sequence with respect to σ in $X\}$;
$\mathfrak{C}(\sigma) = \{\{l_n\} \mid \{l_n\}$ is a cofinally-Cauchy sequence with respect to σ in $X\}$;
$\mathcal{C}(\sigma) = \{\{u_n\} \mid \{u_n\}$ is a Cauchy sequence with respect to σ in $X\}$ where $\sigma \in \Omega$.

Definition 9. Let (X,Ω) be a GMS. Then $\sigma \in \Omega$ is called *Bourbaki* (resp. *pseudo*, *cofinally*) complete metric if every Bourbaki-Cauchy (resp. pseudo-Cauchy, cofinally-Cauchy) sequence with respect to σ is a convergent sequence with respect to σ.

Definition 10. A GMS (X,Ω) is said to be a *Bourbaki* (resp. *pseudo*, *cofinally*) complete metric space if there exists a kernel $\Omega_0 \subset \Omega$ consisting of all Bourbaki (resp. pseudo, cofinally) complete metrics.

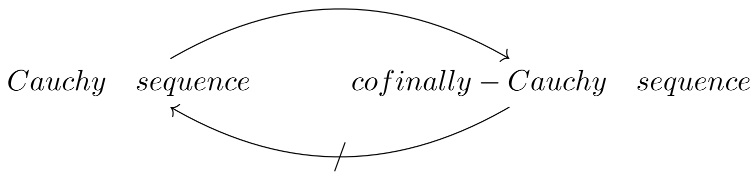
Theorem 11 and Example 12 are describe the below diagram.



Theorem 11 provides an easy way to check, in a generalized metric space, whether a given sequence is Bourbaki-Cauchy or not. This theorem is direct impact of the definitions (Definition 3 and Definition 8) so the trivial proof is neglected.

Theorem 11. *Let (X,Ω) be a generalized metric space. Then the followings are true.*
(a) Every Cauchy sequence is a Bourbaki-Cauchy sequence with the same metric.
(b) Every Bourbaki -complete metric space is a weakly complete metric space.

Example 12. Consider the generalized metric space (X,Ω) where $X = \mathbb{R}, \Omega = \{\sigma_m \mid m \in \mathbb{N}\} \subset \Omega_X$ with $\sigma_m(c,d) = min\{\sigma_E(c,d), \frac{1}{m}\}$ where $\sigma_E(c,d) = |c-d|$. Let $\{c_n\}_{n\in\mathbb{N}} = \{n + \frac{0.1}{8}\}$ and $\varepsilon > 0$ be given. Then $\{c_n\} \in \mathfrak{B}(\sigma_1)$. But $\{c_n\} \notin \mathcal{C}(\sigma_1)$.



The following Theorem 13 and Example 14 are describe the above diagram.

The following Theorem 13 gives a shortcut for finding the nature of a given Cauchy sequence in a generalized metric space.

Theorem 13. *Let (X,Ω) be a generalized metric space. Then the followings are true.*
(a) Every Cauchy sequence is a cofinally-Cauchy sequence with the same metric.
(b) Every cofinally-complete metric space is a weakly complete metric space.

Proof. (a). Let $\sigma \in \Omega$ and $\{c_n\}_{n\in\mathbb{N}} \in \mathcal{C}(\sigma)$. Let $\varepsilon > 0$ be given. Then there exists a positive integer N_0 such that $\sigma(c_n, c_m) < \varepsilon$ for all $n, m \geq N_0$. Take $N_\varepsilon = \{j : j \geq N_0\}$. Then N_ε is an infinite subset of $\mathbb{N}$ and $\sigma(c_k, c_l) < \varepsilon$ for all $k, l \in N_\varepsilon$. Hence $\{c_n\} \in \mathfrak{C}(\sigma)$.
(b). Directly follows from (a) and the definition of weakly complete metric space.

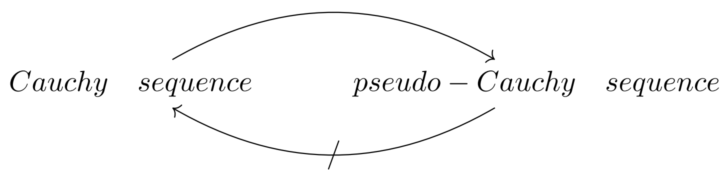
Example 14. Consider the generalized metric space (X, Ω_1) where $X = \mathbb{R}^+ \cup \{0\}, \Omega_1 = \{\sigma_1, \sigma_2, \sigma_3\} \subset \Omega_X$ and the metrics are defined by $\sigma_1(c, d) = |c - d|$;

$$\sigma_2(c, d) = \begin{cases} 0 & \text{if} \quad c = d, \\ c + d & \text{if } c \neq d. \end{cases}$$

and $\sigma_3(c, d) = \frac{\sigma_1(c,d)}{1+\sigma_1(c,d)}$ for all $c, d \in X$. Define a sequence $\{c_n\}_{n\in\mathbb{N}}$ by

$$\{c_n\} = \begin{cases} log(n) & \text{if} \quad n \text{ is prime}, \\ 0 & \text{otherwise}. \end{cases}$$

in X. Then $\{c_n\} \in \mathfrak{C}(\sigma_2)$. But for every $\varepsilon > 0$, there is no $N_0 \in \mathbb{N}$ such that $\sigma_2(c_n, c_m) < \varepsilon$ for $n, m \geq N_0$. Thus, $\{c_n\} \notin \mathcal{C}(\sigma_2)$.



The following Theorem 15 and Example 16 are describe the above diagram. Theorem 15 is reduce the complexity for finding, in a generalized metric space, whether a given Cauchy sequnce is pseudo-Cauchy sequence or not.

Theorem 15. *Let (X, Ω) be a generalized metric space. Then the followings are true.*
(a) Every Cauchy sequence is a pseudo-Cauchy sequence in X with the same metric.
(b) Every pseudo-complete metric space is a weakly complete metric space.

Proof. (a). Let $\{c_n\}_{n\in\mathbb{N}} \in \mathcal{C}(\sigma)$. Then there is $N_0 \in \mathbb{N}$ such that $\sigma(c_n, c_m) < \varepsilon$ for all $n, m \geq N_0$. Let $\varepsilon > 0, k \in \mathbb{N}$ be given. If $k \geq N_0$, then the result is obvious. Suppose that $k < N_0$. Since $N_0 \in \mathbb{N}$, we get elements $l, j \in \mathbb{N}, l \neq j$ such that $l, j > N_0 > k$. Also, $\sigma(c_l, c_j) < \varepsilon$. Therefore, $\{c_n\} \in \mathfrak{P}(\sigma)$.
(b). Follows from (a).

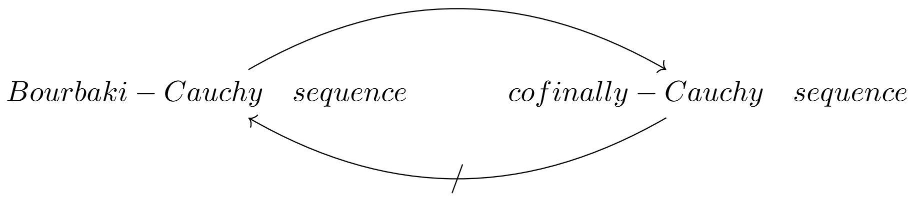
Example 16. Consider the generalized metric space (X, Ω_1) where $X = \mathbb{R}, \Omega_1 = \{\sigma_1, \sigma_2, \sigma_3\} \subset \Omega_X$ and the metrics are defined by

$$\sigma_1(c, d) = \begin{cases} 0 & \text{if} \quad c = d, \\ 1 & \text{if} \quad c \neq d. \end{cases}$$

$$\sigma_2(c, d) = |c - d|$$

and $\sigma_3(c, d) = min\{1, \sigma_1(c, d)\}$ for all $c, d \in X$. Define a sequence $\{c_n\}_{n\in\mathbb{N}}$ by $\{c_n\} = \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, ...\}$ in X. Then $\{c_n\} \in \mathfrak{P}(\sigma_2)$. But for every $\varepsilon > 0$, there exists no $N_0 \in \mathbb{N}$ such that $\sigma_2(c_n, c_m) < \varepsilon$ for $n, m \geq N_0$. Thus, $\{c_n\} \notin \mathcal{C}(\sigma_2)$.

Theorem 17 and Example 18 are describe the below diagram. The below Theorem 17 gives the relations between Bourbaki-Cauchy sequence and cofinally-Cauchy sequence in a GMS.



Theorem 17. *Let (X, Ω) be a GMS. Then the followings are true.*
(a) Every Bourbaki-Cauchy sequence is a cofinally-Cauchy sequence in X with the same metric.
(b) Every cofinally-complete metric space is a Bourbaki -complete metric space.

Proof. Let $\{c_n\}_{n\in\mathbb{N}} \in \mathfrak{B}(\sigma)$ and $\varepsilon > 0$ be given. Then there is $m, n_0 \in \mathbb{N}$ such that whenever $n > j \geq n_0$, the points c_j and c_n can be joined by an ε-chain of length m. Let $N_\varepsilon = \{n \mid n \geq n_0$ and c_n, c_j can be joined by $\frac{\varepsilon}{m}$-chain of length m whenever $m - 1$ points lie between c_n and c_j where $n \neq j; j \geq n_0; m, j \in \mathbb{N}\}$. Then N_ε is an infinite subset of $\mathbb{N}$. Let $k, l \in N_\varepsilon$. If k and l are consecutive numbers, then there is nothing to prove. Suppose that there is some p points lies between k and l. Then c_k and c_l can be joined by $\frac{\varepsilon}{p+1}$-chain of length $p + 1$ and so the distance between the consecutive numbers is less than $\frac{\varepsilon}{p+1}$. Thus, $\sigma(c_k, c_l) < \varepsilon$. Hence $\{c_n\} \in \mathfrak{C}(\sigma)$.
(b). Directly follows from the definition of Bourbaki-complete metric space and by (a).

Example 18. Consider the generalized metric space (X, Ω_1) where $X = \mathbb{R}, \Omega_1 = \{\sigma_1, \sigma_2, \sigma_3\} \subset \Omega_X$ and then the metrics are defined by $\sigma_1(c, d) = min\{|c - d|, 1\}$;

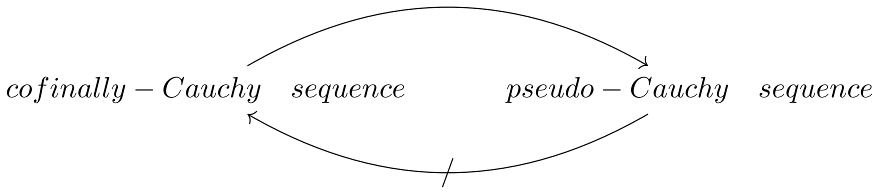
$$\sigma_2(c, d) = \begin{cases} 0 & \text{if} \quad c = d, \\ 1 & \text{if } c \neq d \end{cases}$$

and hence $\sigma_3(c, d) = \sigma_1(c, d) + \sigma_2(c, d)$ for all $c, d \in X$. Define a sequence $\{c_n\}_{n\in\mathbb{N}}$ by $\{c_n\} = \{1, -1, 2, -1, 3, -1, 4, -1, ...\}$ in X. Then $\{c_n\} \in \mathfrak{C}(\sigma_1)$. But $\{c_n\} \notin \mathfrak{B}(\sigma_1)$. Because, for every $\varepsilon > 0$, there is no $m, n_0 \in \mathbb{N}$ such that whenever $n > j \geq n_0$, the points c_j and c_n can be joined by an ε-chain of length m.

Theorem 19. *Let (X, Ω) be a GMS. Then every cofinally-Cauchy sequence has a Cauchy subsequence with the same metric.*

Proof. Let $\sigma \in \Omega$ and $\{c_n\}_{n\in\mathbb{N}} \in \mathfrak{C}(\sigma)$. Given $\varepsilon > 0$. Then there is an infinite subset N_ε of $\mathbb{N}$ such that $\sigma(c_l, c_k) < \varepsilon$ for each $l, k \in N_\varepsilon$. Let A be a partially ordered subset of N_ε with $<$ ordering and $\{c_{n_k}\}$ be a subsequence of $\{c_n\}$ where $n_k \in A$. Choose $M = min\{n_i \mid n_i \in A\}$. For $n_p, n_q > M$, we get $\sigma(c_{n_p}, c_{n_q}) < \varepsilon$, by the definition of A. Thus, $\sigma(c_{n_p}, c_{n_q}) < \varepsilon$ for every $n_p, n_q > M$. Thus, $\{c_{n_k}\} \in \mathcal{C}(\sigma)$. Therefore, $\{c_n\}$ has a subsequence in $\mathcal{C}(\sigma)$.

Theorem 20 and Example 21 are describe the below diagram.



Theorem 20 provides easier way to explore, in a generalized metric space, a given sequence is pseudo-Cauchy or not.

Theorem 20. *Let (X, Ω) be a GMS. Then every cofinally-Cauchy sequence is a pseudo-Cauchy sequence with the same metric.*

Proof. *Let $\{c_n\} \in \mathfrak{C}(\sigma)$ and $\varepsilon > 0$ be given. Then there exists an infinite subset N_ε of $\mathbb{N}$ such that $\sigma(c_k, c_l) < \varepsilon$ for every $k, l \in N_\varepsilon$. Let $n \in \mathbb{N}$. Since N_ε is an infinite subset of $\mathbb{N}$, there exist $p, q \in N_\varepsilon$ with $p \neq q$ and $p, q > n$. Also, $\sigma(c_p, c_q) < \varepsilon$. Hence $\{c_n\} \in \mathfrak{P}(\sigma)$.*

Example 21 shows that the reverse part of Theorem 20 is need not be true.

Example 21. Consider the generalized metric space (X, Ω_1) where $X = \mathbb{R}, \Omega_1 = \{\sigma_1, \sigma_2, \sigma_3\} \subset \Omega_X$ such that

$$\sigma_1(c,d) = \begin{cases} 0 & \text{if} \quad c = d, \\ 1 & \text{if} \quad c \neq d \end{cases}$$

$\sigma_2(c,d) = \sigma_E(c,d)$ where $\sigma_E(c,d) = |c - d|$ and then $\sigma_3(c,d) = \frac{\sigma_1(c,d)}{1+\sigma_1(c,d)}$ for all $c, d \in X$. Define a sequence $\{c_n\}_{n\in\mathbb{N}} = \{1,1,1,2,2,2,2,3,3,3,3,3,....\}$. Then $\{c_n\} \in \mathfrak{P}(\sigma_2)$. But there is no infinite subset N_ε of $\mathbb{N}$ such that $\sigma(c_k, c_l) < \varepsilon$ for every $\varepsilon > 0$ and $k, l \in N_\varepsilon$. Thus, $\{c_n\} \notin \mathfrak{C}(\sigma_2)$.

Next, in the rest of this section, in a generalized metric space, we define a rearrangement of a sequence and give some interesting results based on this definition.

Definition 22. Let (X, Ω) be a generalized metric space. Then the sequence $\{d_n\}$ is a *rearrangement* of a sequence $\{c_n\}$ if there is a 1-1 correspondence $h : \mathbb{N} \to \mathbb{N}$ such that $n \in \mathbb{N}, d_n = c_{h(n)}$.

Theorem 23. *Let (X, Ω) be a GMS. If $\{c_n\} \in \mathfrak{C}(\sigma)$, then every rearrangement sequence of $\{c_n\}$ is cofinally-Cauchy with the same metric.*

Proof. *Suppose $\{c_n\} \in \mathfrak{C}(\sigma)$. Let $\varepsilon > 0$ be given. Then there exists an infinite subset N_ε of $\mathbb{N}$ such that $\sigma(c_k, c_l) < \varepsilon$ for every $k, l \in N_\varepsilon$. Let $\{d_n\}$ be a rearrangement of a sequence $\{c_n\}$. Then there is a 1-1 correspondence $h : \mathbb{N} \to \mathbb{N}$ such that $n \in \mathbb{N}, d_n = c_{h(n)}$. Take $N_{\varepsilon_0} = \{h^{-1}(n) | n \in N_\varepsilon\}$. Then N_{ε_0} is an infinite subset of $\mathbb{N}$. Let $p, q \in N_{\varepsilon_0}$. Then $p = h^{-1}(s)$ and $q = h^{-1}(t)$ where $s, t \in N_\varepsilon$. Now $\sigma(d_p, d_q) = \sigma(d_{h^{-1}(s)}, d_{h^{-1}(t)}) = \sigma(c_{h(h^{-1}(s))}, c_{h(h^{-1}(t))})$ which implies $\sigma(d_p, d_q) = \sigma(c_s, c_t)$, since h is bijective. Thus, $\sigma(d_p, d_q) < \varepsilon$, since $s, t \in N_\varepsilon$. Therefore, $\{d_n\} \in \mathfrak{C}(\sigma)$. Since $\{d_n\}$ is an arbitrary rearrangement sequence of $\{c_n\}$ we have every rearrangement sequence of $\{c_n\}$ is cofinally-Cauchy with the same metric.*

Theorem 24. *Let (X, Ω) be a GMS. If the rearrangement sequence of $\{c_n\}$ is cofinally-Cauchy, then $\{c_n\}$ is cofinally-Cauchy with the same metric.*

Proof. *Assume that, the rearrangement sequence $\{d_n\}$ of $\{c_n\}$ is in $\mathfrak{C}(\sigma)$. Then there exists an infinite subset N_ε of $\mathbb{N}$ such that $\sigma(d_l, d_m) < \varepsilon$ for every $l, m \in N_\varepsilon$. By assumption, there is a 1-1 correspondence $h : \mathbb{N} \to \mathbb{N}$ such that $n \in \mathbb{N}, d_n = c_{h(n)}$. Take $N_{\varepsilon_1} = \{h(n) | n \in N_\varepsilon\}$. Then N_{ε_1} is an infinite subset of $\mathbb{N}$. Let $n, m \in N_{\varepsilon_1}$. Then $n = h(i)$ and $m = h(j)$ where $i, j \in N_\varepsilon$. Now $\sigma(c_n, c_m) = \sigma(c_{h(i)}, c_{h(j)})$ which implies $\sigma(c_n, c_m) = \sigma(d_i, d_j)$. Thus, $\sigma(c_n, c_m) < \varepsilon$, since $i, j \in N_\varepsilon$. Therefore, $\{c_n\} \in \mathfrak{C}(\sigma)$.*

Theorem 25. *Let (X, Ω) be a GMS, $h : \mathbb{N} \to \mathbb{N}$ be a strictly increasing, bijective function. If the rearrangement sequence of $\{c_n\}$ under h is pseudo-Cauchy, then $\{c_n\}$ is pseudo-Cauchy with the same metric.*

Proof. *Let $\{d_n\}$ be a rearrangement sequence of $\{c_n\}$ under h. Suppose $\{d_n\} \in \mathfrak{P}(\sigma)$. Let $\varepsilon > 0$ be given and $m \in \mathbb{N}$ where $m = h(n); n \in \mathbb{N}$. Since $n \in \mathbb{N}$ and by assumption, there exist $k, l \in \mathbb{N}, k \neq l$ such that $k > n; l > n$ and $\sigma(d_k, d_l) < \varepsilon$. Since $k, l \in \mathbb{N}, k \neq l$ we have $h(k), h(l) \in \mathbb{N}, h(k) \neq h(l)$, by h is one-one. Since h is strictly increasing and $k > n; l > n$ we have $h(k) > h(n); h(l) > h(n)$. Now $\sigma(c_{h(k)}, c_{h(l)}) = \sigma(d_k, d_l) < \varepsilon$. Thus, there exist $h(k), h(l) \in \mathbb{N}, h(k) \neq h(l)$ such that $h(k) > h(n) = m; h(l) > h(n) = m$ and $\sigma(c_{h(k)}, c_{h(l)}) < \varepsilon$. Hence $\{c_n\}$ is pseudo-Cauchy with the same metric.*

4. Asymptotic sequences

Here, in a generalized metric space, two kinds of new sequences are defined and explore their nature.

We begin with a definition of asymptotic sequence in a generalized metric space.

Definition 26. Let (X, Ω) be a generalized metric space. A pair of sequence $\{s_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ in X is said be:

(a) *asymptotic* with respect to $\sigma \in \Omega$ if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\sigma(s_n, t_n) < \varepsilon$ for all $n \geq n_0$.

(b) *uniformly asymptotic* with respect to $\sigma \in \Omega$ if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\sigma(s_m, t_n) < \varepsilon$ for all $m, n \geq n_0$.

Theorem 27 gives new tricks to check whether, in a generalized metric space, a given pair of sequence is in $\mathcal{C}(\sigma)$ or not.

Theorem 27. *Let (X, Ω) be a GMS. If the pair of sequence $\{c_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ is asymptotic with respect to $\sigma \in \Omega$, then the followings are true.*

(a) If $\{d_n\} \in \mathcal{C}(\sigma)$, then $\{c_n\} \in \mathcal{C}(\sigma)$.

(b) If $\{d_n\}$ is a convergent sequence with respect to $\sigma \in \Omega$, then $\{c_n\}$ is a convergent sequence with the same metric.

Proof. Assume that, $\{c_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ is asymptotic with respect to $\sigma \in \Omega$.

(a) Given that $\{d_n\} \in \mathcal{C}(\sigma)$. Let $\varepsilon > 0$. Then there is $N_0 \in \mathbb{N}$ such that $\sigma(d_n, d_m) < \frac{\varepsilon}{3}$ for all $n, m \geq N_0$. By hypothesis, there exists a positive integer N_1 such that $\sigma(c_n, d_n) < \frac{\varepsilon}{3}$

for all $n \geq N_1$. Take $N = max\{N_0, N_1 + 1\}$. Then for $n, m \geq N, \sigma(c_n, c_m) \leq \sigma(c_n, d_n) + \sigma(d_n, d_m) + \sigma(d_m, c_m)$ and so $\sigma(c_n, c_m) < \varepsilon$ for all $n, m \geq N$. Hence $\{c_n\} \in \mathfrak{C}(\sigma)$.
(b) Suppose that $\{d_n\}$ is a convergent sequence with respect to $\sigma \in \Omega$. Let $\varepsilon > 0$ be given and $y \in X$ be a limit point of $\{d_n\}$. Then there is $n_0 \in \mathbb{N}$ such that $\sigma(d_n, y) < \frac{\varepsilon}{2}$ for all $n \geq n_0$. By hypothesis, there is $n_1 \in \mathbb{N}$ such that $\sigma(c_n, d_n) < \frac{\varepsilon}{2}$ for all $n > n_1$. Take $M_0 = max\{n_0, n_1 + 1\}$. Then $\sigma(c_n, y) \leq \sigma(c_n, d_n) + \sigma(d_n, y)$ and so $\sigma(c_n, y) < \varepsilon$ for all $n \geq M_0$. Thus, $\{c_n\}$ is convergent to y with respect to σ. Hence $\{c_n\}$ is a convergent sequence with respect to $\sigma \in \Omega$.

Theorem 28. *Let (X, Ω) be a GMS, the pair of sequence $\{c_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ be asymptotic with respect to the metric $\sigma \in \Omega$. If $\{d_n\} \in \mathfrak{P}(\sigma)$, then $\{c_n\} \in \mathfrak{P}(\sigma)$.*

Proof. Assume that, $\{c_n\}$ and $\{d_n\}$ is asymptotic with respect to the metric $\sigma \in \Omega$ and $\{d_n\} \in \mathfrak{P}(\sigma)$. Let $\varepsilon > 0$ and $m \in \mathbb{N}$. Then there exists $N_1 \in \mathbb{N}$ such that $\sigma(c_n, d_n) < \frac{\varepsilon}{3}$ for all $n \geq N_1$ and there exist $k_1, j_1 \in \mathbb{N}, k_1 \neq j_1$ such that $k_1, j_1 > m$ and $\sigma(d_{k_1}, d_{j_1}) < \frac{\varepsilon}{3}$.
Case 1: Suppose that, $N_1 < m$. Take $k = k_1, j = j_1$. Then $k \neq j$ and $k, j > m$. Now $\sigma(c_k, c_j) \leq \sigma(c_k, d_k) + \sigma(d_k, d_j) + \sigma(d_j, c_j) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Thus, $\sigma(c_k, c_j) < \varepsilon$.
Case 2: Assume that, $N_1 > m$. If $k_1, j_1 > N_1$, then we take $k = k_1, j = j_1$. By same argument in Case 1, we get $\sigma(c_k, c_j) < \varepsilon$.
Suppose that, $k_1, j_1 < N_1$. Since $\{d_n\} \in \mathfrak{P}(\sigma)$ and $N_1 \in \mathbb{N}$, there is $k_2, j_2 \in \mathbb{N}, k_2 \neq j_2$ such that $k_2, j_2 > N_1$ and $\sigma(d_{k_2}, d_{j_2}) < \frac{\varepsilon}{3}$. Take $k = k_2, j = j_2$. Then $\sigma(c_k, c_j) \leq \sigma(c_k, d_k) + \sigma(d_k, d_j) + \sigma(d_j, c_j)$ and so $\sigma(c_k, c_j) < \varepsilon$.
If $k_1 < N_1 < j_1$, then there is $k_2, j_2 \in \mathbb{N}, k_2 \neq j_2$ such that $k_2, j_2 > j_1$ and $\sigma(d_{k_2}, d_{j_2}) < \frac{\varepsilon}{3}$, since $\{d_n\}$ is a pseudo-Cauchy sequence and $j_1 \in \mathbb{N}$. Take $k = k_2, j = j_2$. Then $\sigma(c_k, c_j) \leq \sigma(c_k, d_k) + \sigma(d_k, d_j) + \sigma(d_j, c_j)$ and so $\sigma(c_k, c_j) < \varepsilon$. From all the cases, we get for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exist $k, j \in \mathbb{N}, k \neq j$ such that $k, j > m$ and $\sigma(x_k, x_j) < \varepsilon$. Therefore, $\{c_n\} \in \mathfrak{P}(\sigma)$.

Theorem 29 provides an easier way to check the nature of a given pair of sequence using the tool namely, asymptotic.

Theorem 29. *Let (X, Ω) be a generalized metric space, the pair of sequence $\{c_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ be asymptotic with respect to the metric $\sigma \in \Omega$. If $\{d_n\} \in \mathfrak{B}(\sigma)$, then $\{c_n\} \in \mathfrak{B}(\sigma)$.*

Proof. Let $\varepsilon > 0$ be given. Then there exists $N_1 \in \mathbb{N}$ such that $\sigma(c_n, d_n) < \varepsilon$ for all $n \geq N_1$ and there exist $m_1, n_1 \in \mathbb{N}$ such that whenever $n > j \geq n_1$, the points d_j and d_n can be joined by an ε-chain of length m_1.
Case 1: If $N_1 > n_1$, then take $k = m_1 + 2$ and $l = N_1$. Then $k, l \in \mathbb{N}$. If $n > i \geq l$, then we can choose elements between c_i and c_n, namely, $c_i, d_i, d_{i+1}, ..., d_{n-1}, d_n, c_n$. Since $n_1 \leq i < n$, the points d_i and d_n can be joined by an ε-chain of length m_1, by assumption. Thus, $\sigma(d_i, d_{i+1}) < \varepsilon, \sigma(d_{i+1}, d_{i+2}) < \varepsilon, ..., \sigma(d_{n-1}, d_n) < \varepsilon$. Also, $\sigma(c_n, d_n) < \varepsilon$ and $\sigma(c_i, d_i) < \varepsilon$. Therefore, the points c_i and c_n can be joined by an ε-chain of length k.
Case 2: Suppose that $N_1 < n_1$. Choose $k = m_1 + 2$ and $l = n_1$. Then by same argument as in Case 1, we get the points c_i, c_n such that c_i and c_n can be joined by an ε-chain of length k.
Hence in both cases, we get that the sequence $\{c_n\} \in \mathfrak{B}(\sigma)$.

The following Theorem 30 is reduce the complexity for finding whether, in a generalized metric space, a given pair of sequence is uniformly asymptotic or not using asymptotic.

Theorem 30. *Let* (X, Ω) *be a generalized metric space, the pair of sequence* $\{c_n\}_{n\in\mathbb{N}}$ *and* $\{d_n\}_{n\in\mathbb{N}}$ *be asymptotic with respect to* $\sigma \in \Omega$. *If either* $\{c_n\}$ *or* $\{d_n\}$ *is Cauchy sequence with respect to* $\sigma \in \Omega$, *then the pair of sequence* $\{c_n\}$ *and* $\{d_n\}$ *is uniformly asymptotic with respect to* σ.

Proof. Given that the pair of sequence $\{c_n\}$ *and* $\{d_n\}$ *is asymptotic with respect to* $\sigma \in \Omega$. *Let* $\varepsilon > 0$. *Then there is* $N_1 \in \mathbb{N}$ *such that* $\sigma(c_n, d_n) < \frac{\varepsilon}{2}$ *for all* $n \geq N_1$. *If* $\{c_n\} \in \mathcal{C}(\sigma)$, *then there is* $N_0 \in \mathbb{N}$ *such that* $\sigma(c_n, c_m) < \frac{\varepsilon}{2}$ *for all* $n, m \geq N_0$. *Take* $N = max\{N_0, N_1 + 1\}$. *Then for* $n, m \geq N, \sigma(c_m, d_n) \leq \sigma(c_m, c_n) + \sigma(c_n, d_n)$ *and so* $\sigma(c_m, d_n) < \varepsilon$ *for all* $n, m \geq N$. *Hence the pair of sequence* $\{c_n\}$ *and* $\{d_n\}$ *is uniformly asymptotic with respect to* σ.
Similarly, we can prove that the pair of sequence $\{c_n\}$ *and* $\{d_n\}$ *is uniformly asymptotic if* $\{d_n\} \in \mathcal{C}(\sigma)$.

Theorem 31 provides the necessary condition for a given pair of a convergent sequence is asymptotic in a generalized metric space.

Theorem 31. *Let* (X, Ω) *be a generalized metric space and* $\{c_n\}, \{d_n\}$ *be convergent sequences with respect to* $\sigma \in \Omega$. *If they have the limit points joined by an* ε*-chain of length* 1 *for every* $\varepsilon > 0$, *then the pair of sequence is asymptotic with the same metric.*

Proof. Let $c, d \in X$ *be the limit points of the sequences* $\{c_n\}$ *and* $\{d_n\}$, *respectively with respect to* σ. *Let* $\varepsilon > 0$ *be given. Then there exists* $N_0, N_1 \in \mathbb{N}$ *such that* $\sigma(c_n, c) < \frac{\varepsilon}{3}$ *for every* $n \geq N_0$ *and* $\sigma(d_n, d) < \frac{\varepsilon}{3}$ *for every* $n \geq N_1$. *Take* $M = max\{N_0, N_1\}$. *For* $n \geq M, \sigma(c_n, d_n) \leq \sigma(c_n, c) + \sigma(c, d) + \sigma(y, d_n)$ *and so* $\sigma(c_n, d_n) < \frac{\varepsilon}{3} + \sigma(c, d) + \frac{\varepsilon}{3}$. *By assumption, the points* c *and* d *can be joined by an* $\frac{\varepsilon}{3}$*-chain of length* 1. *Thus,* $\sigma(c, d) < \frac{\varepsilon}{3}$ *and hence* $\sigma(c_n, d_n) < \varepsilon$ *for* $n \geq M$. *Hence the pair of sequence* $\{c_n\}$ *and* $\{d_n\}$ *is asymptotic with respect to the metric* σ.

The following Example 32 shows that the necessary condition in Theorem 31 can not be dropped.

Example 32. Consider the generalized metric space (X, Ω_1) where $X = \mathbb{N} \cup \{0\}, \Omega_1 = \{\sigma_1, \sigma_2, \sigma_3\} \subset \Omega_X$ and the metrics are defined by $\sigma_1(c, d) = |c-d|; \sigma_2(c, d) = min\{\sigma_1(c, d), 1\}$ and $\sigma_3(c, d) = \frac{\sigma_1(c,d)}{1+\sigma_1(c,d)}$ for all $c, d \in X$. Define the sequence $\{c_n\}$ and $\{d_n\}$ by $\{c_n\} = \{\frac{1}{n}\}$ and $\{d_n\} = \{1 + \frac{1}{n}\}$. Then $\{c_n\}$ converges to 0 and $\{d_n\}$ converges to 1 with respect to σ_1. Here the points 0 and 1 can not be joined by an ε-chain of length m where $\varepsilon > 0, m$ is a positive integer. Also, for every $\varepsilon > 0$, there is no $n_0 \in \mathbb{N}$ such that $\sigma_1(c_n, d_n) < \varepsilon$ for all $n > n_0$

Theorem 33. *Let* (X, Ω) *be a generalized metric space. Then every convergent sequence and its convergent subsequence are asymptotic with the same metric.*

Proof. Let $\{c_n\}$ *be a convergent sequence with respect to the metric* $\sigma \in \Omega$ *and* $\{z_n\}$ *be a convergent subsequence of* $\{c_n\}$ *with the metric* σ. *Let* x *be a limit point of the sequence*

$\{c_n\}$. Then x is also a limit point of the subsequence $\{z_n\}$. Let $\varepsilon > 0$ be given. Then there exists $N_0, N_1 \in \mathbb{N}$ such that $\sigma(c_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N_0$ and $\sigma(z_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N_1$. Choose $M = max\{N_0, N_1\}$. For $n \geq M, \sigma(c_n, z_n) \leq \sigma(c_n, x) + \sigma(x, z_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and so $\sigma(c_n, z_n) < \varepsilon$ for $n \geq M$. Hence the pair of sequence $\{c_n\}$ and $\{z_n\}$ is asymptotic with respect to the metric σ.

In a generalized metric space, the existence of a convergent sequence using convergent subsequence can be found straightforwardly by Theorem 34.

Theorem 34. *Let (X, Ω) be a generalized metric space and $\{c_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{z_n\}_{n\in\mathbb{N}}$ with respect to $\sigma \in \Omega$. If the pair of sequence $\{c_n\}$ and $\{z_n\}$ is asymptotic with respect to the metric σ, then $\{c_n\}$ is a convergent sequence with the same metric.*

Proof. Given that $\{z_n\}$ is a convergent subsequence of $\{c_n\}$ and z is a limit point of the sequence $\{z_n\}$ with respect to $\sigma \in \Omega$. Let $\varepsilon > 0$ be given. Then there exists $N_0 \in \mathbb{N}$ such that $\sigma(z_n, z) < \frac{\varepsilon}{2}$ for every $n \geq N_0$. Since the pair of sequence $\{c_n\}$ and $\{z_n\}$ is asymptotic with respect to σ, there exists $N_1 \in \mathbb{N}$ such that $\sigma(c_n, z_n) < \frac{\varepsilon}{2}$ for every $n \geq N_1$. Take $M = max\{N_0, N_1 + 1\}$. For $n \geq M$, $\sigma(c_n, z) \leq \sigma(c_n, z_n) + \sigma(z_n, z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus, $\sigma(c_n, z) < \varepsilon$ for every $n \geq M$. Hence $\{c_n\}$ is a convergent sequence with respect to σ.

The condition that convergence on the sequence $\{c_n\}$ can not be dropped in Theorem 33 as shown by the following Example 35. Also, it shows that the condition asymptotic is necessary in Theorem 34.

Example 35. Consider the generalized metric space (X, Ω_1) where $X = \mathbb{R}, \Omega_1 = \{\sigma_1, \sigma_2, \sigma_3\} \subset \Omega_X$ and the metrics are defined by $\sigma_1(c, d) = \frac{|c-d|}{1+|c-d|}; \sigma_2(c, d) = min\{\sigma_1(c, d), 1\}$ and

$$\sigma_3(c, d) = \begin{cases} 0 & \text{if} \quad c = d, \\ 1 & \text{if} \quad c \neq d \end{cases}$$

for all $c, d \in X$. Let $\{c_n\} = \{(-1)^n\}_{n\in\mathbb{N}}$ be a sequence in X and $\{d_n\}$ be a subsequence of $\{c_n\}$ defined by $d_n = 1$ for all $n \in \mathbb{N}$. Then $\{d_n\}$ is a convergent subsequence of $\{c_n\}$ with respect to the metric $\sigma_1 \in \Omega_1$. Here $\{c_n\}$ is not a convergent sequence. Also, $\{c_n\}$ and $\{d_n\}$ are not asymptotic with respect to the metric σ_1.

The following Example 36 shows that the condition convergent on $\{z_n\}$ can not be dropped in Theorem 34.

Example 36. Consider the generalized metric space (X, Ω_1) where $X = \mathbb{R}, \Omega_1 = \{\sigma_1, \sigma_2, \sigma_3\} \subset \Omega_X$ and the metrics are defined by $\sigma_1(c, d) = |c - d|; \sigma_2(c, d) = min\{\sigma_1(c, d), 1\}$ and

$$\sigma_3(c, d) = \begin{cases} 0 & \text{if} \quad c = d, \\ 1 & \text{if} \quad c \neq d \end{cases}$$

for all $c, d \in X$. Let

$$\{c_n\} = \begin{cases} 1 & \text{if} \quad n \text{ is odd} \\ 2 & \text{if} \quad n \text{ is even} \end{cases}$$

be a sequence in X and $\{z_n\}$ be a subsequence of $\{c_n\}$ defined by $z_n = c_{n+2}$ for all $n \in \mathbb{N}$. Then the pair of sequence $\{c_n\}$ and $\{z_n\}$ is asymptotic with respect to the metric σ_1. But $\{c_n\}$ is not a convergent sequence with respect to the metric σ_1. Because, $\{z_n\}$ is not a convergent subsequence of $\{c_n\}$ with respect to the metric $\sigma_1 \in \Omega_1$.

$$\begin{array}{ccccc} \{d_n\} \in \mathfrak{B}(\sigma) & \longrightarrow & \{c_n\} \in \mathcal{C}(\sigma) & \longleftarrow & \{d_n\} \in \mathfrak{P}(\sigma) \\ & & \uparrow & & \\ & & \{d_n\} \in \mathfrak{C}(\sigma) & & \end{array}$$

The following Theorem 37 describe the above diagram.

Theorem 37. *Let (X, Ω) be a generalized metric space, the pair of sequence $\{c_n\}$ and $\{d_n\}$ where $c_n \neq d_n$ for all $n \in \mathbb{N}$ be uniformly asymptotic with respect to $\sigma \in \Omega$. Then $\{c_n\} \in \mathcal{C}(\sigma)$ if any one of the following hold.*
(a) $\{d_n\} \in \mathfrak{B}(\sigma)$.
(b) $\{d_n\} \in \mathfrak{C}(\sigma)$.
(c) $\{d_n\} \in \mathfrak{P}(\sigma)$.

Proof. Let $\varepsilon > 0$ be given. Then there exists $N_0 \in \mathbb{N}$ such that $\sigma(c_m, d_n) < \frac{\varepsilon}{3}$ for every $m, n \geq N_0$, since the pair of sequence $\{c_n\}$ and $\{d_n\}$ is uniformly asymptotic.
(a) Given that $\{d_n\} \in \mathfrak{B}(\sigma)$. Then there exist $l, k_0 \in \mathbb{N}$ such that whenever $n > j \geq k_0$, the points d_j and d_n are joined by an $\frac{\varepsilon}{3}$-chain of length l. Choose $M = max\{N_0 + 1, k_0\}$. For $n, m \geq M$, $\sigma(c_n, c_m) \leq \sigma(c_n, d_{n+1}) + \sigma(d_{n+1}, d_{n+2}) + \sigma(d_{n+2}, c_m) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Thus, $\sigma(c_n, c_m) < \varepsilon$ for every $n, m \geq M$. Therefore, $\{c_n\} \in \mathcal{C}(\sigma)$.
(b) Let $\{d_n\} \in \mathfrak{C}(\sigma)$. Then there exists an infinite subset N_ε of $\mathbb{N}$ such that $\sigma(d_k, d_j) < \frac{\varepsilon}{3}$ for all $k, j \in N_\varepsilon$. Let $N_k = min\{N_i : N_i \in N_\varepsilon\}$ and take $M = max\{N_0 + 1, N_k\}$. For $n, m \geq M, \sigma(c_n, c_m) \leq \sigma(c_n, d_j) + \sigma(d_j, d_k) + \sigma(d_k, c_m)$, since N_ε is an infinite subset of $\mathbb{N}$, there exist $j, k \in N_\varepsilon$ such that $j, k > M$. Thus, $\sigma(c_n, c_m) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Hence $\sigma(c_n, c_m) < \varepsilon$ for all $n, m \geq M$. Therefore, $\{c_n\} \in \mathcal{C}(\sigma)$.
(c) Suppose that $\{d_n\} \in \mathfrak{P}(\sigma)$. Then there exists $k, l \in \mathbb{N}$ with $k \neq l$ and $k, l > N_0$ such that $\sigma(d_k, d_l) < \frac{\varepsilon}{3}$. Take $M = N_0+1$. For $n, m \geq M, \sigma(c_n, c_m) \leq \sigma(c_n, d_k)+\sigma(d_k, d_l)+\sigma(d_k, c_m)$. Thus, $\sigma(c_n, c_m) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Therefore, $\sigma(c_n, c_m) < \varepsilon$ for all $n, m \geq M$. Hence $\{c_n\} \in \mathcal{C}(\sigma)$.

Theorem 38. *Let (X, Ω) be a GMS. Then (X, Ω) is a weakly complete space if and only if there is a kernel $\Omega_0 \subset \Omega$ and if the pair of sequence $\{c_n\}$ and $\{d_n\}$ is asymptotic with respect to $\sigma \in \Omega_0$ and $\{d_n\}$ is Cauchy with respect to σ, then $\{c_n\}$ is a convergent sequence with the same metric.*

Proof. Given that (X, Ω) is a weakly complete space. Then there exists a kernel $\Omega_0 \subset \Omega$ consisting of complete metrics. Let the pair of sequence $\{c_n\}$ and $\{d_n\}$ be asymptotic with *respect to $\sigma \in \Omega_0$ and $\{d_n\}$ is Cauchy with the same metric. Then $\{d_n\}$ is convergent with respect to σ. By Theorem 27 (b), $\{c_n\}$ is a convergent sequence with the same metric. Conversely, let $\{z_n\}$ be a Cauchy sequence with respect to the metric $\sigma \in \Omega_0$. Define the sequence $\{t_n\}$ in X by $t_1 = 2, t_2 = 1$ and $t_n = z_{n-2}$ for $n \geq 3$. Since $\{z_n\}$ is Cauchy, for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\sigma(z_n, t_n) < \varepsilon$ for every $n \geq N_0$. Thus, the pair of sequence $\{z_n\}$ and $\{t_n\}$ is asymptotic with respect to $\sigma \in \Omega_0$. By assumption, $\{t_n\}$ is a convergent sequence with the same metric. By Theorem 27 (b), $\{z_n\}$ is a convergent sequence with respect to $\sigma \in \Omega_0$. Since σ is arbitrary, Ω_0 consisting of complete metrics. Therefore, (X, Ω) is a weakly complete space.*

Definition 39. Let (X, Ω_1) and (Y, Ω_2) be two generalized metric spaces. A function $h : (X, \Omega_1) \to (Y, \Omega_2)$ is said to be *Cauchy-continuous* if $\{h(z_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (Y, Ω_2) for any Cauchy sequence $\{z_n\}_{n\in\mathbb{N}}$ in (X, Ω_1).

That is, h is *Cauchy-continuous* if $\{z_n\}$ is Cauchy with respect to $\sigma \in \Omega_1$, then there is $\sigma \in \Omega_2$ such that $\{h(z_n)\}$ is Cauchy with respect to σ.

Theorem 40. *Let (X, Ω_1) and (Y, Ω_2) be two GMSs. If $h : X \to Y$ is a Cauchy continuous map and if $\{c_n\}$ is a cofinally-Cauchy sequence in X, then $\{h(c_n)\}$ has a cofinally-Cauchy subsequence in Y.*

Proof. Let $\{c_n\}$ be a cofinally-Cauchy sequence in X. Then there exists a metric $\sigma \in \Omega_1$ such that $\{c_n\}$ is a cofinally-Cauchy sequence with respect to this metric in X. By Theorem 19, $\{c_n\}$ has a Cauchy subsequence $\{c_{n_k}\}$ with respect to the same metric in X. Since h is Cauchy continuous, there exists a metric $d \in \Omega_2$ such that $\{h(c_{n_k})\}$ is Cauchy with respect to d. By Theorem 13 (a), $\{h(c_{n_k})\}$ is a cofinally-Cauchy sequence with respect to the metric d. Hence $\{h(c_n)\}$ has a cofinally-Cauchy subsequence in Y.

Definition 41. Let (X, Ω_1) and (Y, Ω_2) be two generalized metric spaces. A function $h : X \to Y$ is said to be:
i). *Bourbaki-Cauchy regular* (in short, BC-regular) if $\{h(z_n)\}_{n\in\mathbb{N}}$ is a Bourbaki Cauchy sequence in (Y, Ω_2) whenever $\{z_n\}_{n\in\mathbb{N}}$ is Bourbaki Cauchy in (X, Ω_1).

ii). *cofinally-Cauchy regular* (briefly, CC-regular) if $\{h(z_n)\}_{n\in\mathbb{N}}$ is a cofinally Cauchy sequence in (Y, Ω_2) whenever $\{z_n\}_{n\in\mathbb{N}}$ is cofinally Cauchy in (X, Ω_1).

iii). *pseudo-Cauchy regular* (in short, PC-regular) if $\{h(z_n)\}_{n\in\mathbb{N}}$ is a pseudo Cauchy sequence in (Y, Ω_2) whenever $\{z_n\}_{n\in\mathbb{N}}$ is pseudo Cauchy in (X, Ω_1).

That is, h is Bourbaki-Cauchy regular (resp. CC-regular, PC-regular) if $\{z_n\}_{n\in\mathbb{N}}$ is Bourbaki (resp. cofinally, pseudo) Cauchy with respect to $\sigma_1 \in \Omega_1$, then there is $d \in \Omega_2$ such that $\{h(z_n)\}_{n\in\mathbb{N}}$ is Bourbaki (resp. cofinally, pseudo) Cauchy with respect to d.

The following three theorems (Theorem 42, Theorem 43 and Theorem 44) are give some relations between the functions defined in above Definition 41.

Theorem 42. *Let (X, Ω_1) and (Y, Ω_2) be two GMSs, $h : X \to Y$ be a BC-regular map. If every cofinally-Cauchy sequence $\{c_n\}$ in X, there exists a sequence $\{d_n\}$ in X with $c_n \neq d_n$ for every $n \in \mathbb{N}$ such that the pair of sequence $\{c_n\}$ and $\{d_n\}$ is uniformly asymptotic with respect to the same metric, then h is a CC-regular map.*

Proof. Let $\{a_n\} \in \mathfrak{C}(\sigma)$ where $\sigma \in \Omega_1$. Then by hypothesis, there exists a sequence $\{b_n\}$ in X with $a_n \neq b_n$ for every $n \in \mathbb{N}$ such that the pair of sequence $\{a_n\}$ and $\{b_n\}$ is uniformly asymptotic with respect to σ. By Theorem 37, $\{b_n\} \in \mathcal{C}(\sigma)$ and so $\{b_n\} \in \mathfrak{B}(\sigma)$, by Theorem 11(a). Then $\{a_n\} \in \mathfrak{B}(\sigma)$, by Theorem 29. Since h is a BC-regular map, there is a metric $d \in \Omega_2$ such that $\{h(a_n)\} \in \mathfrak{B}(d)$ where $d \in \Omega_2$. By Theorem 17, $\{h(a_n)\} \in \mathfrak{C}(d)$ where $d \in \Omega_2$. Hence h is a CC-regular map.

Theorem 43. *Let (X, Ω_1) and (Y, Ω_2) be two generalized metric spaces, $h : X \to Y$ be a Cauchy-continuous map. If for every Bourbaki-Cauchy sequence $\{c_n\}$ in X, there exists a sequence $\{d_n\}$ in X with $c_n \neq d_n$ for every $n \in \mathbb{N}$ such that the pair of sequence $\{c_n\}$ and $\{d_n\}$ is uniformly asymptotic with the same metric, then h is a BC-regular map.*

Proof. Let $\{a_n\} \in \mathfrak{B}(\sigma)$ where $\sigma \in \Omega_1$. Then by hypothesis, there exists a sequence $\{b_n\}$ in X with $a_n \neq b_n$ for every $n \in \mathbb{N}$ such that the pair of sequence $\{a_n\}$ and $\{b_n\}$ is uniformly asymptotic with the same metric. By Theorem 37, $\{b_n\} \in \mathcal{C}(\sigma)$. Then $\{a_n\} \in \mathcal{C}(\sigma)$, by Theorem 27. Since h is a Cauchy-continuous map, there is a metric $d \in \Omega_2$ such that $\{h(a_n)\} \in \mathcal{C}(d)$ where $d \in \Omega_2$. By Theorem 11(a), $\{h(a_n)\} \in \mathfrak{B}(d)$ where $d \in \Omega_2$. Hence h is a BC-regular map.

Theorem 44. *Let (X, Ω_1) and (Y, Ω_2) be two generalized metric spaces, $h : X \to Y$ be a CC-regular map. If for every cofinally-Cauchy sequence $\{s_n\}$ in Y, there exists a sequence $\{t_n\}$ in Y with $s_n \neq t_n$ for every $n \in \mathbb{N}$ such that the pair of sequence $\{s_n\}$ and $\{t_n\}$ is uniformly asymptotic with the same metric, then h is a Cauchy-continuous map.*

Proof. Let $\{a_n\} \in \mathcal{C}(\sigma)$ for $\sigma \in \Omega_1$. By Theorem 13, $\{a_n\} \in \mathfrak{C}(\sigma)$.. Since h is a CC-regular map, there is a metric $d \in \Omega_2$ such that $\{h(a_n)\} \in \mathfrak{C}(d)$ where $d \in \Omega_2$. By hypothesis, there exists a sequence $\{b_n\}$ in Y with $h(a_n) \neq b_n$ for every $n \in \mathbb{N}$ such that the pair of sequence $\{h(a_n)\}$ and $\{b_n\}$ is uniformly asymptotic with the same metric. By Theorem 37, $\{b_n\} \in \mathcal{C}(d)$ for $d \in \Omega_2$. Then $\{h(a_n)\} \in \mathcal{C}(d)$ where $d \in \Omega_2$, by Theorem 27. Hence h is a Cauchy-continuous map.

The following Theorem 45 gives a necessary condition for a weakly complete metric space to be a cofinally-complete metric space.

Theorem 45. *Let (X, Ω) be a weakly complete space. If for every cofinally-Cauchy sequence $\{s_n\}$ in X, there exists a sequence $\{t_n\}$ in X with $s_n \neq t_n$ for every $n \in \mathbb{N}$ such that the pair of sequence $\{s_n\}$ and $\{t_n\}$ is uniformly asymptotic with the same metric, then (X, Ω) is a cofinally-complete space.*

Proof. Given that (X, Ω) is a weakly complete space. Then there exists a kernel $\Omega_0 \subset \Omega$ consisting of complete metrics. Let $\sigma_0 \in \Omega_0$ and $\{a_n\} \in \mathfrak{C}(\sigma_0)$. By hypothesis, there exists a sequence $\{t_n\}$ in X with $a_n \neq t_n$ for every $n \in \mathbb{N}$ such that the pair of sequence $\{a_n\}$

and $\{t_n\}$ is uniformly asymptotic with the same metric. By Theorem 37, $\{t_n\} \in \mathcal{C}(\sigma_0)$ and hence it is a convergent sequence, by assumption. Hence $\{a_n\}$ is a convergent sequence with respect to σ_0, by Theorem 27(b). Thus, Ω_0 consisting of complete metrics. Therefore, (X, Ω) is a cofinally-complete space.

In Theorem 45, if we replace "cofinally-Cauchy" by "pseudo-Cauchy" we get that (X, Ω) is a pseudo-complete space.

Theorem 46. *Let (X, Ω) be a GMS, the pair of sequence $\{c_n\}$ and $\{d_n\}$ is asymptotic, $\{d_n\} \in \mathfrak{B}(\sigma)$. If every $\frac{\varepsilon}{k}$-chain have length $k-1$ where $k \in \mathbb{N}, \varepsilon > 0$, then the pair of sequence $\{c_n\}$ and $\{d_n\}$ is uniformly asymptotic with respect to the same metric.*

Proof. *Let $\varepsilon > 0$ be given and $m \in \mathbb{N}$. Since $\frac{\varepsilon}{m} > 0$ and $\{d_n\} \in \mathfrak{B}(\sigma)$, there exist $n_1, m_1 \in \mathbb{N}$ whenever $n > j \geq n_1$ such that the points d_j and d_n can be joined by an $\frac{\varepsilon}{m}$-chain of length m_1. By hypothesis, $m_1 = m - 1$. Since $\frac{\varepsilon}{m} > 0$ and the pair of sequence $\{c_n\}, \{d_n\}$ is asymptotic, there exists $N_0 \in \mathbb{N}$ such that $\sigma(c_n, d_n) < \frac{\varepsilon}{m}$ for all $n \geq N_0$. Take $M = max\{n_1, N_0\}$. For $n, m \geq M$,*
Case 1: If $m < n$, then $M \geq m < n$. By assumption, the points d_m and d_n can be joined by an $\frac{\varepsilon}{m}$-chain of length $m-1$. Thus, $\sigma(d_m, d_n) < \frac{(m-1)\varepsilon}{m}$. Now $\sigma(c_m, d_n) \leq \sigma(c_m, d_m) + \sigma(d_m, d_n) < \frac{\varepsilon}{m} + \frac{(m-1)\varepsilon}{m} = \varepsilon$.
Case 2: Suppose that $m > n$. Then by similar argument as in Case-1, we get $\sigma(c_m, d_n) < \varepsilon$. From Case 1 and Case 2, we get, $\sigma(c_m, d_n) < \varepsilon$ for all $m, n \geq M$. Hence the pair of sequence $\{c_n\}$ and $\{d_n\}$ is uniformly asymptotic with respect to the metric $\sigma \in \Omega$.

Next, the rest of this section, in a hyperconnected space, some interesting results for uniformly asymptotic sequence are proven.

Theorem 47. *Let (X, Ω) be a GMS and $\{c_n\}, \{d_n\}$ be Cauchy, cofinally-Cauchy sequences with respect to $\sigma \in \Omega$ in X, respectively. If (X, μ_Ω) is a hyperconnected space, then the pair of sequence $\{c_n\}$ and $\{d_n\}$ is uniformly asymptotic with the same metric.*

Proof. *Let $\varepsilon > 0$ be given. Then there is $n_0 \in \mathbb{N}$ such that $\sigma(c_n, c_m) < \frac{\varepsilon}{4}$ for all $n, m \geq n_0$ and there is an infinite subset N_ε of $\mathbb{N}$ such that $\sigma(d_k, d_l) < \frac{\varepsilon}{4}$ for every $k, l \in N_\varepsilon$. Take $M_0 = max\{n_0, n_1 \mid n_1 = inf\{N_\varepsilon\}\}$. If $B_\sigma(c_n, \frac{\varepsilon}{4}) \cap B_\sigma(d_l, \frac{\varepsilon}{4}) = \emptyset$, then $B_\sigma(c_n, \frac{\varepsilon}{4}) \subset X - B_\sigma(d_l, \frac{\varepsilon}{4})$. This implies $i_{\mu_\Omega}(X - B_\sigma(d_l, \frac{\varepsilon}{4})) \neq \emptyset$ which implies that $c_{\mu_\Omega}(B_\sigma(d_l, \frac{\varepsilon}{4})) \neq X$. But $B_\sigma(d_l, \frac{\varepsilon}{4})$ is μ_Ω-dense, since (X, μ_Ω) is a hyperconnected space. Therefore, $B_\sigma(c_n, \frac{\varepsilon}{4}) \cap B_\sigma(d_l, \frac{\varepsilon}{4}) \neq \emptyset$. Let $t \in B_\sigma(c_n, \frac{\varepsilon}{4}) \cap B_\sigma(d_l, \frac{\varepsilon}{4})$. Then $\sigma(c_n, t) < \frac{\varepsilon}{4}$ and $\sigma(d_l, t) < \frac{\varepsilon}{4}$. For $l_1, m_1 \geq M_0$, $\sigma(c_{l_1}, d_{m_1}) \leq \sigma(c_{l_1}, c_n) + \sigma(c_n, t) + \sigma(t, d_l) + \sigma(d_l, d_{m_1})$. If $m_1 \in N_\varepsilon$, then we get $\sigma(d_l, d_{m_1}) < \frac{\varepsilon}{4}$. Therefore, $\sigma(c_{l_1}, d_{m_1}) < \varepsilon$ for all $l_1, m_1 \geq M_0$. Assume that, $m_1 \notin N_\varepsilon$. Since (X, μ_Ω) is a hyperconnected space, $B_\sigma(d_l, \frac{\varepsilon}{8})$ is μ_Ω-dense. Then $B_\sigma(d_l, \frac{\varepsilon}{8}) \cap B_\sigma(d_{m_1}, \frac{\varepsilon}{8}) \neq \emptyset$, since $B_\sigma(d_{m_1}, \frac{\varepsilon}{8}) \in \tilde{\mu}_\Omega$. Let $z \in B_\sigma(d_l, \frac{\varepsilon}{8}) \cap B_\sigma(d_{m_1}, \frac{\varepsilon}{8})$. Then $\sigma(d_l, d_{m_1}) \leq \sigma(d_l, z) + \sigma(z, d_{m_1}) < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}$. Therefore, $\sigma(c_{l_1}, d_{m_1}) < \varepsilon$ for all $l_1, m_1 \geq M_0$. Hence the pair of sequence $\{c_n\}$ and $\{d_n\}$ is uniformly asymptotic sequence with respect to $\sigma \in \Omega$ in X.*

Corollary 48. *Let (X, Ω) be a GMS and $\{c_n\}, \{d_n\}$ be Cauchy sequences with respect to $\sigma \in \Omega$ in X. If (X, μ_Ω) is a hyperconnected space, then the pair of sequence $\{c_n\}$ and $\{d_n\}$ is uniformly asymptotic sequence with respect to the same metric.*

Theorem 49. *Let (X, Ω) be a GMS and $\{c_n\}, \{d_n\}$ be Cauchy, convergent sequences with respect to $\sigma \in \Omega$ in X, respectively. If (X, μ_Ω) is a hyperconnected space, then $\{c_n\}$ is a convergent sequence with respect to σ.*

Proof. *Let $\varepsilon > 0$ be given and y be a limit point of $\{d_n\}$. Then there exist positive integers $n_0, n_1 \in \mathbb{N}$ such that $\sigma(c_n, c_m) < \frac{\varepsilon}{2}$ for all $n, m \geq n_0$ and $\sigma(d_n, y) < \frac{\varepsilon}{2}$ for all $n \geq n_1$. Take $M = max\{n_0, n_1\}$. Fix $n \geq M$. Since (X, μ_Ω) is a hyperconnected space, $B_\sigma(c_n, \frac{\varepsilon}{2}) \cap B_\sigma(y, \frac{\varepsilon}{2}) \neq \emptyset$. Let $t \in B_\sigma(c_n, \frac{\varepsilon}{2}) \cap B_\sigma(y, \frac{\varepsilon}{2})$. Then $\sigma(c_n, y) \leq \sigma(c_n, t) + \sigma(t, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore, $\sigma(c_n, y) < \varepsilon$ for all $n \geq M$. Hence $\{c_n\}$ is a convergent sequence with respect to σ.*

Definition 50. Let (X, Ω) be a GMS. Then Ω is said to satisfy the $\mathcal{S}$-property if $\sigma_i(c, d) \leq \sigma_j(c, d)$ or $\sigma_j(c, d) \leq \sigma_i(c, d)$ for any $\sigma_i, \sigma_j \in \Omega$ and $c, d \in X$.

Theorem 51. *Let (X, Ω) be a GMS, Ω satisfies the $\mathcal{S}$-property and the pair of sequences $\{c_n\}$ and $\{d_n\}$ be asymptotic with respect to $\sigma_j \in \Omega$. Then the following hold.*
(a) If $\{c_n\} \in \mathcal{C}(\sigma_i)$, then $\{d_n\} \in \mathcal{C}(\sigma_i)$.
(b) If $\{c_n\}$ is convergent with respect to $\sigma_i \in \Omega$ in X, then $\{d_n\}$ is convergent with respect to σ_i.

Proof. (a). Suppose that $\{c_n\} \in \mathcal{C}(\sigma_i)$. Let $\varepsilon > 0$ be given. Then there is positive integers n_0, n_1 such that $\sigma_j(c_n, d_n) < \frac{\varepsilon}{3}$ for all $n > n_0$ and $\sigma_i(c_n, c_m) < \frac{\varepsilon}{3}$ for all $n, m \geq n_1$. By hypothesis, $\sigma_i(c, d) \leq \sigma_j(c, d)$ or $\sigma_j(c, d) \leq \sigma_i(c, d)$.
Case 1: If $\sigma_i(c, d) < \sigma_j(c, d)$, then $\sigma_i(c_n, d_n) < \frac{\varepsilon}{3}$ for all $n > n_0$. Choose $N_0 = max\{n_0, n_1\}$. For $n, m > N_0$, $\sigma_i(d_n, d_m) \leq \sigma_i(d_n, c_n) + \sigma_i(c_n, c_m) + \sigma_i(c_m, d_m) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Therefore, $\sigma_i(d_n, d_m) < \varepsilon$ for all $n, m > N_0$. Hence $\{d_n\} \in \mathcal{C}(\sigma_i)$.
Case 2: If $\sigma_j(c, d) < \sigma_i(c, d)$, then the proof is completed by replacing i by j as in the proof of Case-1.
(b). It is obvious.

Theorem 52. *Let (X, Ω) be a GMS, Ω satisfy the $\mathcal{S}$-property and the pair of sequences $\{c_n\}$ and $\{d_n\}$ be uniformly asymptotic with respect to $\sigma_j \in \Omega$. Then the following hold.*
(a) If $\{d_n\} \in \mathfrak{C}(\sigma_i)$, then $\{d_n\} \in \mathcal{C}(\sigma_i)$.
(b) If $\{d_n\} \in \mathfrak{P}(\sigma_i)$, then $\{d_n\} \in \mathcal{C}(\sigma_i)$.

In [14], Preecha Yupapin et. al have obtained new set of properties of nowhere dense sets. Inspired by this, rest of this section, we check some of the collections are whether satisfies the stack property or not, in a generalized metric space.

Let (X, μ) be a GTS. A family $\mathcal{E}$ of subsets of X is called a *stack* [13] if $C \in \mathcal{E}$ whenever $D \in \mathcal{E}$ and $D \subset C$.

Theorem 53. *Let (X, Ω) be a GMS and $\eta = \{\{c_n\}_{n\in\mathbb{N}} \mid \{c_n\}_{n\in\mathbb{N}}$ is a cofinally-Cauchy sequence in $X\}$. Then η is a stack.*

Proof. Let $\{d_n\}_{n\in\mathbb{N}}$ be a subsequence of $\{c_n\}_{n\in\mathbb{N}}$ in X where $\{d_n\}_{n\in\mathbb{N}} \in \eta$. Then there is a metric $\sigma \in \Omega$ such that $\{d_n\}_{n\in\mathbb{N}} \in \mathfrak{C}(\sigma)$ and so for every $\varepsilon > 0$, there exists an infinite subset N_ε of $\mathbb{N}$ such that for every $n, j \in N_\varepsilon$ we have $\sigma(d_n, d_j) < \varepsilon$. By hypothesis, $\sigma(c_n, c_j) < \varepsilon$ for every $n, j \in N_\varepsilon$. Thus, for every $\varepsilon > 0$, there exists an infinite subset N_ε of $\mathbb{N}$ such that for every $n, j \in N_\varepsilon$ we have $\sigma(c_n, c_j) < \varepsilon$. Therefore, $\{c_n\}_{n\in\mathbb{N}} \in \mathfrak{C}(\sigma)$. Hence $\{c_n\}_{n\in\mathbb{N}} \in \eta$. Thus, η is a stack in (X, Ω).

Theorem 54. *Let (X, Ω) be a generalized metric space and $\eta = \{\{c_n\}_{n\in\mathbb{N}} \mid \{c_n\}_{n\in\mathbb{N}}$ is a pseudo-Cauchy sequence in $X\}$. Then η is a stack.*

Proof. Let $\{l_n\}_{n\in\mathbb{N}}$ be a subsequence of $\{k_n\}_{n\in\mathbb{N}}$ in X where $\{l_n\}_{n\in\mathbb{N}} \in \eta$. Then there is a metric $\sigma \in \Omega$ such that $\{l_n\}_{n\in\mathbb{N}} \in \mathfrak{P}(\sigma)$ and so for every $\varepsilon > 0$ and for every $n \in \mathbb{N}$, there exist $s, t \in \mathbb{N}, s \neq t$ such that $s, t > n$ and $\sigma(l_s, l_t) < \varepsilon$. By hypothesis, $\sigma(k_s, k_t) < \varepsilon$. Thus, for every $\varepsilon > 0$ and for every $n \in \mathbb{N}$, there exist $s, t \in \mathbb{N}, s \neq t$ such that $s, t > n$ and $\sigma(k_s, k_t) < \varepsilon$. Therefore, $\{k_n\}_{n\in\mathbb{N}} \in \mathfrak{P}(\sigma)$. Hence $\{k_n\}_{n\in\mathbb{N}} \in \eta$. Thus, η is a stack in (X, Ω).

Theorem 55. *Let (X, Ω) be a generalized metric space. If $\eta = \{\Omega_0 \mid \Omega_0$ is a kernel in $X\}$, then η is a stack.*

Proof. Let $\Omega_1 \subset \Omega_0$ where $\Omega_1 \in \eta$. Then Ω_1 is a kernel. Let $K \in \tilde{\mu}_\Omega$. Then there exists $\sigma \in \Omega_1$ such that $i_\sigma K \neq \emptyset$. Since $\Omega_1 \subset \Omega_0$ we have there exists a metric $\sigma \in \Omega_0$ such that $i_\sigma K \neq \emptyset$. Hence Ω_0 is a kernel.

5. Conclusions

The various relations between the three types of sequences in a generalized metric space has evaluated. Further, the necessity of uniformly asymptotic condition has been examined to prove some new relationship between the functions defined in Definition 41. Hence the meaning of the collection of sequences has been analyzed.

References

[1] M. Aggarwal and S. Kundu. More on variants of complete metric spaces. *Acta math. Hungar.*, 151(2):391–408, 2017.

[2] G. Beer. Between compactness and completeness. *Topology Appl.*, 155:503 – 514, 2008.

[3] G. Beer. Between the cofinally complete spaces and the uc spaces. *Houston J. Math.,*, 38:999 – 1015, 2012.

[4] M. M. Bonsangue, F. Van Breugel, and J.J.M.M. Rutten. Generalized metric spaces; completion, topology, and power domains via the yoneda embedding. *Theoretical Computer Science*, 193, 1998.

[5] T. Dosenovi, S. Radenovi, and S. Sedghi. generalized metric spaces: Survey , twms. *J. Pure Appl. Math.*, 9(1):3–17, 2018.

[6] Ekici Erdal. Generalized hyperconnectedness. *Acta Mathematica Hungarica*, 133:140–147, 2011.

[7] Korczak-Kubiak Ewa, Loranty Anna, and Pawlak Ryszard J. Baire generalized topological spaces, generalized metric spaces and infinite games. *Acta Mathematica Hungarica*, 140(3):203–231, 2013.

[8] Zdeněk Frolík and Praha. Baire spaces and some generalizations of complete metric spaces. *Czechoslov. Math. J.,*, 11(2):237–248, 1961.

[9] M. Isabel Garrido and Ana S. Mero noi. New types of completeness in metric spaces. *Ann. Acad. Sci. Fenn.*, 39:733–758, 2014.

[10] Lee J.-G., Şenel G., Baek J.-I., Han S.H., and Hur K. Neighborhood structures and continuities via cubic sets. *Axioms*, 11, 2022.

[11] Tanvi Jain and S. Kundu. Atsuji completions: Equivalent characterisations. *Topol. Appl.*, 154:28–38, 2007.

[12] M. Jleli and B. Samet. On a new generalized of metric spaces. *arXiv:*, pages 1 – 20, 2018.

[13] W. K. Min. On weak neighborhood systems and spaces. *Acta Mathematica Hungarica*, 121(3):283–292, 2008.

[14] Preecha Yupapin, Vadakasi Subramanian, and Yasser Farhat. On nowhere dense sets. *European Journal of Pure and Applied Mathematics*, 15(2):403–414, 2022.

[15] Şenel Guzide and Cagman Naim. Soft closed sets on soft bitopological space. *Journal of New Results in Science*, 3(5):57–66, 2014.

[16] Şenel Guzide and Cagman Naim. Soft topological subspaces. *Annals of Fuzzy Mathematics and Informatics*, 10(4):525–535, 2015.

[17] Şenel Güzide. A new approach to hausdorff space theory via the soft sets. *Mathematical Problems in Engineering*, 9:1–6, 2016.

[18] Şenel Güzide, Lee Jeong Gon, and Kul Hur. Distance and similarity measures for octahedron sets and their application to mcgdm problems. *Mathematics*, 8, 2020.

[19] Şenel Güzide, Lee Jeong Gon, and Kul Hur. Advanced soft relation and soft mapping. *International Journal of Computational Intelligence Systems*, 14(1):461–470, 2021.