EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 4, 2022, 1948-1956
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Dual B-Topological Spaces Determined by Filterbase and Some Sets in a Dual B-algebra

Katrina E. Belleza[1,*], Jimboy R. Albaracin[2]

[1] *Department of Computer, Information Science, and Mathematics, School of Arts and Sciences, University of San Carlos, Talamban, Cebu City, Philippines*
[2] *Mathematics and Statistics Programs, College of Science, University of the Philippines Cebu, Cebu City, Philippines*

Abstract. This paper presents dual B-topologies that are determined by filterbase and some sets in a dual B-algebra. Also, some properties of a filterbase in a dual B-topological space are provided. In particular, a commutative dual B-topological space and a symmetric B-topological space are topological dual B-algebras.

2020 Mathematics Subject Classifications: 46H10, 54A05, 54F65, 54H99, 55M99

Key Words and Phrases: Topological algebra, dual B-algebra, topological dual B-algebra, dual B-topological space, tdB-algebra

1. Introduction

In 1998, D.S. Lee and D.N. Ryu [5] introduced the notion of a topological BCK-algebra. Moreover, they derived a filter base generating a BCK-algebra topology. On the following year, Y.B. Jun et al. [4] gave a filterbase generating a BCI-topology and making a BCI-algebra into a topological BCI-algebra for which the filterbase is a fundamental system of neighborhoods. In 2019, K.E. Belleza and J.P. Vilela introduces and characterized the notion of a dual B-algebra [2]. Moreover on the following year, K.E. Belleza introduces the dual B-topological space and a tdB-algebra involving dual B-ideals and dual B-subalgebras.

2. Preliminaries

Definition 1. [2] A *dual B-algebra* X^D is a triple $(X^D, \circ, 1)$ where X^D is a non-empty set with a binary operation " $\circ$ " and a constant 1 satisfying the following axioms for all x, y, z in X^D:

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4594

Email addresses: kebelleza@usc.edu.ph (K. Belleza), jralbaracin@up.edu.ph (J. Albaracin)

$$\text{(DB1) } x \circ x = 1; \quad \text{(DB2) } 1 \circ x = x; \quad \text{(DB3) } x \circ (y \circ z) = ((y \circ 1) \circ x) \circ z.$$

Lemma 1. **[2]** *Let X^D be a dual B-algebra. For any x, y in X^D, $x \circ y = 1$* implies $x = y$.

Theorem 1. **[2]** *Let $X = (X, \circ, 1)$ be any algebra of type $(2, 0)$. Then X is a dual B-algebra if and only if for any $x, y, z \in X$,*

$$(i)\ x \circ x = 1; \ (ii)\ x = (x \circ 1) \circ 1; \ (iii)\ (x \circ y) \circ (x \circ z) = y \circ z.$$

Definition 2. **[2]** Let X^D be a dual B-algebra. Define a binary operation " $+$ " on X as follows: $x + y = (x \circ 1) \circ y$ for all x, y in X^D. A dual B-algebra is said to be *commutative* if $x + y = y + x$, that is, $(x \circ 1) \circ y = (y \circ 1) \circ x$ for all x, y in X^D.

Proposition 1. **[2]** *Suppose X^D is a commutative B-algebra. Then for all x, y in X^D, $x \circ (y \circ z) = y \circ (x \circ z)$.*

Let X^D be a dual B-algebra such that $x \circ y = y \circ x$ for all $x, y \in X^D$. Then we say that X^D satisfies a *symmetric* condition [2].

Lemma 2. **[2]** *Let X^D be a dual B-algebra satisfying a symmetric condition. Then for all $x, y, z \in X^D$, $(x \circ y) \circ (z \circ y) = x \circ z$.*

Definition 3. **[1]** Let X^D be a dual B-algebra and S a nonempty subset of X^D. Then S is called a *dual B-subalgebra* of X^D if S itself is a dual B-algebra with binary operation of X^D on S.

Definition 4. **[1]** Let X^D be a dual B-algebra. A subset F of X^D is called a *dual B-filter* if it satisfies the following axioms: for all x, y in X^D,

$$\text{(dF1) } 1 \in F; \quad \text{(dF2) } x \circ y \in F \text{ and } x \in F \text{ imply } y \in F.$$

Definition 5. **[3]** Let X be a set. A *topology* (or topological structure) in X is a family τ of subsets of X that satisfies the following:

(i) Each union of members of τ is also a member of τ;

(ii) Each finite intersection of members of τ is also a member of τ; and

(iii) $\varnothing$ and X are members of τ.

A couple (X, τ) consisting of a set X and a topology τ in X is called a *topological space*. We also say "τ is the topology of the space X". The members of τ are called *open sets* of (X, τ). A family $\mathcal{B} \subset \tau$ is called a *basis* for τ if each open set is the union of members of $\mathcal{B}$. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \to Y$ is called *continuous* if the inverse image of each open set in Y is open in X (that is, if f^{-1} maps τ_Y into τ_X). **[3]**

Theorem 2. **[3]** Let $\mathcal{B} \subset \tau$. The following two properties of $\mathcal{B}$ are equivalent:
(i) $\mathcal{B}$ is a basis for τ;
(ii) for each $G \in \tau$ and each $x \in G$, there is a $U \in \mathcal{B}$ with $x \in U \subset G$.

Definition 6. [**3**] Let (X, τ) be a topological space. By a *neighborhood* of an element x in X (denoted as $U(x)$) is meant any open set (that is, member of τ) containing x.

Definition 7. [**3**] Let $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ be any family of topological spaces. For each $\alpha \in \mathcal{A}$, let τ_α be the topology for Y_α. The *Cartesian product topology* in $\prod_\alpha Y_\alpha$ is that having for subbasis all sets $\langle U_\beta \rangle = \rho_\beta^{-1}(U_\beta)$, where $\rho : \prod_\alpha Y_\alpha \to Y_\alpha$, U_β ranges over all members of τ_β and β over all elements of $\mathcal{A}$.

Definition 8. [**1**] Let X^D be a dual B-algebra. A topology τ on X^D is called a *dual B-topology* and the couple (X^D, τ) is called a *dual B-topological space.*

Remark 1. Let X^D be a dual B-algebra and nonempty $A, B \in X^D$. Then

$$A \circ B = \{a \circ b \mid a \in A, b \in B\}.$$

Definition 9. [**1**] The triple $(X^D, \circ, \tau)$ is called a *topological dual B-algebra* (or *tdB*-algebra) if τ is a dual B-topology and the binary operation $\circ : X^D \times X^D \to X^D$ is continuous where the topology on $X^D \times X^D$ is the Cartesian product topology.

Theorem 3. [**1**] *Let X^D be a dual B-algebra and τ a dual B-topology. Then $(X^D, \circ, \tau)$ is a tdB-algebra if and only if for all $x, y \in X^D$ and $U(x \circ y)$, there exists $U(x)$ and $U(y)$ such that $U(x) \circ U(y) \subseteq U(x \circ y)$.*

3. Dual B-topological Space Determined by Some Sets

Suppose X^D is a dual B-algebra. For each $V \subseteq X^D$ and $x \in X^D$, let us denote the following notations:

$$\text{(i) } V[x] = \{y \in X^D \mid x \circ y \in V\}; \quad \text{(ii) } V'[x] = \{y \in X^D \mid y \circ x, x \circ y \in V\}.$$

Remark 2. Let X^D be a dual B-algebra and $V \subseteq X^D$. Then $V'[x] \subseteq V[x]$ for any $x \in X^D$.

Example 1. Consider the set $X^D = \{1, a, b, c, d, e\}$ and binary operation $\circ$ as defined in the table below.

$\circ$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	b	1	a	d	e	c
b	a	b	1	e	c	d
c	c	d	e	1	a	b
d	d	e	c	b	1	a
e	e	c	d	a	b	1

Then X^D is a dual B-algebra [1]. Let $V = \{a, d, e\}$. Then $V[b] = \{1, c, e\}$ and $V'[b] = \{c, e\}$.

Proposition 2. *Suppose X^D is a dual B-algebra and $V \subseteq X^D$ such that $1 \in V$. Then $x \in V'[x]$. In particular, $V = \{1\}$ if and only if $V[x] = \{x\} = V'[x]$ for any $x \in X$.*

Proof. Suppose X^D is a dual B-algebra and $V \subseteq X^D$ such that $1 \in V$. By (DB1), $x \circ x = 1 \in V$ for all $x \in X^D$. This implies that $x \in V'[x]$.

Suppose $V = \{1\}$. Then $V[x] = \{x\} = V'[x]$ for any $x \in X$. Let $V[x] = \{x\} = V'[x]$. Then $x \circ x \in V$. Thus, $1 \in V$. Suppose $a \in V$ such that $a \neq 1$. Then there exists $y \in X$ such that $x \circ y = a$ with $x \neq y$. Hence, $y \in V[x]$, a contradiction. Therefore, $V = \{1\}$.

Proposition 3. *Let X^D be a dual B-algebra and $U, V \subset X^D$. If $U \subseteq V$, then $U[x] \subseteq V[x]$ and $U'[x] \subseteq V'[x]$ for any $x \in X^D$.*

Proof. Suppose X^D is a dual B-algebra and $U, V \subset X^D$. Let $y \in U[x]$. Then $x \circ y \in U \subseteq V$. Hence, $y \in V[x]$ which implies that $U[x] \subseteq V[x]$. Similarly, $U'[x] \subseteq V'[x]$.

Proposition 4. *Let X^D be a dual B-algebra satisfying the symmetric condition and $V \subseteq X^D$ such that for all $p, q \in V$ and $x \in X^D$, $p \circ (x \circ q) = 1$ implies $x \in V$. Then $V'[x] \circ V'[y] \subseteq V'[x \circ y]$.*

Proof. Let $p \circ q \in V'[x] \circ V'[y]$ where $p \in V'[x]$ and $q \in V'[y]$. Then $p \circ x, x \circ p, q \circ y, y \circ q \in V$. By (DB1), Lemma 2, (DB3), symmetric condition, and (DB2), $1 = (p \circ x) \circ (p \circ x) = (p \circ x) \circ [(p \circ q) \circ (x \circ q)] = (p \circ x) \circ \big[(p \circ q) \circ [(x \circ y) \circ (q \circ y)]\big] = (p \circ x) \circ \Bigg(\big[[(x \circ y) \circ 1] \circ (p \circ q)\big] \circ (q \circ y) \Bigg) = (p \circ x) \circ \Bigg[[(x \circ y) \circ (p \circ q)] \circ (q \circ y) \Bigg]$. Since $(p \circ x), (q \circ y) \in V$ and $(p \circ x) \circ \Bigg[[(x \circ y) \circ (p \circ q)] \circ (q \circ y) \Bigg] = 1$, this implies that $(x \circ y) \circ (p \circ q) \in V$ by the hypothesis. Similarly, $(p \circ q) \circ (x \circ y) \in V$. Hence, $p \circ q \in V'[x \circ y]$. Therefore, $V'[x] \circ V'[y] \subseteq V'[x \circ y]$.

Theorem 4. *Let Ω be a family of nonempty subsets in a dual B-algebra X^D that is closed under finite intersections. Then the set $\tau = \{U \subseteq X^D \mid \forall x \in U, \exists V \in \Omega$ such that $V[x] \subseteq U\}$ is a dual B-topology on X^D.*

Proof. Let X^D be a dual B-algebra and $x \in X^D$. Note that $V[x] \subseteq X^D$ for any $V \in \Omega$. This implies that $X^D \in \tau$. Since $\varnothing$ does not contain any element, then it is vacuously true that $\varnothing \in \tau$. Suppose $U_1, U_2 \in \tau$ and $x \in U_1 \cap U_2$. Then there exist $V_1, V_2 \in \Omega$ such that $V_1[x] \subseteq U_1$ and $V_2[x] \subseteq U_2$. Since $V_1 \cap V_2 \subseteq V_1, V_2$, it follows that $(V_1 \cap V_2)[x] \subseteq V_1[x] \subseteq U_1$ and $(V_1 \cap V_2)[x] \subseteq V_2[x] \subseteq U_2$ by Proposition 3. Moreover by the hypothesis, $V_1 \cap V_2 \in \Omega$. This implies that $(V_1 \cap V_2)[x] \subseteq U_1$ and $(V_1 \cap V_2)[x] \subseteq U_2$ or $(V_1 \cap V_2)[x] \subseteq U_1 \cap U_2$. Hence, $U_1 \cap U_2 \in \tau$. Suppose $x \in \bigcup_{i \in \mathcal{A}} U_i$ where $U_i \in \tau$ for all $i \in \mathcal{A}$. Then there exists $j \in \mathcal{A}$ such that $x \in U_j$. This implies that $V_j[x] \subseteq U_j$ for some $V_j \in \Omega$. Hence, $V_j[x] \subseteq \bigcup_{i \in \mathcal{A}} U_i$. It follows that $\bigcup_{i \in \mathcal{A}} U_i \in \tau$. Therefore, τ is a dual B-topology on X^D.

Henceforth, the dual B-topology τ in the following results is the dual B-topology in Theorem 4.

Theorem 5. *Suppose X^D is a dual B-topological space and Ω is a family of subsets in a dual B-algebra X^D that is closed under finite intersections. If $\varnothing \in \Omega$, then X^D is a tdB-algebra.*

Proof. Suppose X^D is a dual B-topological space and Ω is a family of subsets in a dual B-algebra X^D that is closed under finite intersections. Let $V \subseteq X^D$. Then for all $x \in V$, there exists $\varnothing \in \Omega$ such that $\varnothing[v] = \varnothing \subseteq V$. This implies that $V \in \tau$. Hence $\mathcal{P}(X^D) \subseteq \tau$ where $\mathcal{P}(X^D)$ is the power set of X^D. Since $\tau \subseteq \mathcal{P}(X^D)$, it follows that $\tau = \mathcal{P}(X^D)$. Let $x, y \in X^D$ and $U(x \circ y) \in \tau$. Then there exist $\{x\}, \{y\} \in \tau$ such that $\{x\} \circ \{y\} = \{x \circ y\} \subseteq U(x \circ y)$. Then X^D is a tdB-algebra.

Theorem 6. *Let Ω be a family of subsets in the dual B-algebra X^D that is closed under finite intersections. Suppose that for each $U \in \Omega$, $1 \in U$ and for each $x \in U$, there exists $V \in \Omega$ such that $V[x] \subseteq U$. Then the set $\mathcal{B} = \{U[a] \mid U \in \Omega, a \in X^D\}$ is a basis for the dual B-topology.*

Proof. First, we will show that $\mathcal{B} \subseteq \tau$. Suppose $x \in U[a] \in \mathcal{B}$ for any $a \in X^D$ and $U \in \Omega$. Then $a \circ x \in U$. Moreover, there exists $V \in \Omega$ such that $V[a \circ x] \subseteq U$. Let $y \in V[x]$. Then $x \circ y \in V$. By Theorem 1(iii), $(a \circ x) \circ (a \circ y) = x \circ y \in V$. Hence, $a \circ y \in V[a \circ x] \subseteq U$. This implies that $y \in U[a]$. Moreover, $U[a] \in \tau$. That is, $V[x] \subseteq U[a]$. Consequently, $\mathcal{B} \subseteq \tau$. Now let $U \in \tau$ and $x \in U$. Then there exists $V \in \Omega$ such that $V[x] \subseteq U$. By Proposition 2 and Remark 2, $x \in V[x]$. Therefore, there exists $V[x] \in \mathcal{B}$ such that $x \in V[x] \subseteq U$. By Theorem 2, $\mathcal{B}$ is a basis for the dual B-topology τ.

Theorem 7. *Let Ω be a family of subsets in a commutative dual B-algebra X^D that is closed under finite intersections. Suppose that for each $V \in \Omega$, $1 \in V$ and for each $x \in V \in \Omega$, there exists $U \in \Omega$ such that $U[x] \subseteq V$. Then X^D is a tdB-algebra.*

Proof. Let $x, y \in X^D$ and $U \in \tau$ such that $x \circ y \in U$. Then there exists $V \in \Omega$ such that $V[x \circ y] \subseteq U$. By Remark 2 and Proposition 2 respectively, $V'[x \circ y] \subseteq V[x \circ y]$ with $x \in V'[x]$ and $y \in V'[y]$. We will show that $V'[x] \circ V'[y] \subseteq V'[x \circ y]$. Suppose $a \in V'[x]$. Then $a \circ x, x \circ a \in V$. By (DB1), Theorem 1(iii), and Proposition 1, $1 = (a \circ y) \circ (a \circ y) = (a \circ y) \circ [(x \circ a) \circ (x \circ y)] = (x \circ a) \circ [(a \circ y) \circ (x \circ y)]$ and $1 = (x \circ y) \circ (x \circ y) = (x \circ y) \circ [(a \circ x) \circ (a \circ y)] = (a \circ x) \circ [(x \circ y) \circ (a \circ y)]$. By Lemma 1, it follows that $(a \circ y) \circ (x \circ y) = x \circ a \in V$ and $(x \circ y) \circ (a \circ y) = a \circ x \in V$. This implies that $a \circ y \in V'[x \circ y]$. Hence, $V'[x] \circ y \subseteq V'[x \circ y]$. Suppose $b \in V'[y]$. Then $b \circ y, y \circ b \in V$. By Theorem 1(iii), $(x \circ b) \circ (x \circ y) = b \circ y \in V$ and $(x \circ y) \circ (x \circ b) = y \circ b \in V$. This implies that $x \circ b \in V'[x \circ y]$. Hence, $x \circ V'[y] \subseteq V'[x \circ y]$. Assume on the contrary that $V'[x] \circ V'[y] \nsubseteq V'[x \circ y]$. By Proposition 2, $V'[x] \circ y \in V'[x] \circ V'[y] \nsubseteq V'[x \circ y]$ and $x \circ V'[y] \in V'[x] \circ V'[y] \nsubseteq V'[x \circ y]$. These are contradictions. Therefore, X^D is a tdB-algebra.

Lemma 3. *Suppose Ω is an arbitrary family of dual B-filters in a dual B-algebra X^D. Then for all $a \in V \in \Omega$, $V[a] = V$.*

Proof. Suppose Ω is an arbitrary family of dual B-filters in a dual B-algebra X^D and let $a \in V \in \Omega$. Suppose $x \in V[a]$. Then $a \circ x \in V$. Since V is a dual B-filter and $a \in V$, it follows that $x \in V$ implying that $V[a] \subseteq V$. Conversely, suppose $x \in V$. Since V is a dual B-filter, V is a dual B-subalgebra of X^D. Then $a \circ x \in V$. Hence, $x \in V[a]$. Therefore, $V[a] = V$.

The next corollary follows from Lemma 3 and Theorem 7.

Corollary 1. *Let Ω be a family of dual B-filters in a commutative dual B-algebra X^D closed under finite intersections such that $1 \in V$ for all $V \in \Omega$. Then X^D is a tdB-algebra.*

4. Filterbase in a Dual B-algebra

Definition 10. Let X^D be a dual B-topological space. A *filterbase* $\mathcal{U}$ in X^D is a family $\mathcal{U} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ of subsets of X^D having two properties:
(i) $A_\alpha \neq \varnothing$ for all $\alpha \in \mathcal{A}$;
(ii) for all $\alpha, \beta \in \mathcal{A}$, there exists $\gamma \in \mathcal{A}$ such that $A_\gamma \subseteq A_\alpha \cap A_\beta$.

Remark 3. Let X^D be a dual B-topological space and $x_1 \in X^D$. The family $\{U(x_1)\}$ is a filterbase called the *neighborhood filterbase* of x_1.

Example 2. Suppose X^D is a dual B-topological space. Any family $\mathcal{W}$ of subsets of X^D containing $\varnothing$ is not a filterbase. In particular, the dual B-topology τ on X^D is not a filterbase in X^D.

Remark 4. The family of dual B-filters is not a subclass of a filterbase in a dual B-algebra X^D.

Example 3. Consider the dual B-algebra $X = \{1, a, b, c, d, e\}$ in Example 1. Let $\mathcal{F} = \{\{1, e\}, \{1, a, b\}, \{1, c\}\}$. Then $\mathcal{F}$ is a family of dual B-filters in X^D [1]. Note that $\{1, e\}, \{1, a, b\} \in \mathcal{F}$ but $\{1, e\} \cap \{1, a, b\} = \{1\} \notin \mathcal{F}$. This implies that $\mathcal{F}$ is not a filterbase.

Remark 5. A filterbase is not a subclass of a family of dual B-filters in a dual B-algebra.

Example 4. Consider the dual B-algebra $X^D = \{1, a, b, c, d, e\}$ in Example 1 and $\Omega = \{\{1, a, b\}, \{a\}\}$. Then Ω is a filterbase of X^D but $\{a\} \in \Omega$ is not a dual B-filter of X^D since $1 \notin \{a\}$.

The next results describes the dual B-topology τ determined by a filterbase Ω followed by the relationship of Ω and τ if Ω is a family of dual B-filters.

Theorem 8. *Let Ω be a filterbase in a dual B-algebra X^D. Then the family $\tau = \{O \subseteq X^D \mid \forall a \in O, \exists V \in \Omega$ such that $V'[a] \subseteq O\}$ is a dual B-topology on X^D.*

Proof. Let Ω be a filterbase in a dual B-algebra X^D. Since $V'[a] \subseteq X^D$ for all $V \in \Omega$ and $a \in X^D$, it follows that $X^D \in \tau$. Since $\varnothing$ do not have any element, then vacuously $\varnothing \in \tau$. Suppose that $O_\alpha, O_\beta \in \tau$ and $a \in O_\alpha \cap O_\beta$. Then there exist $V_\alpha, V_\beta \in \Omega$ such that $V'_\alpha[a] \subseteq O_\alpha$ and $V'_\beta[a] \subseteq O_\beta$. Since Ω is a filterbase, there exists $V \in \Omega$ such that $V \subseteq V_\alpha \cap V_\beta$. By Proposition 3, $V'[a] \subseteq (V_\alpha \cap V_\beta)'[a] \subseteq V'_\alpha[a] \subseteq O_\alpha$. Similarly, $V'[a] \subseteq O_\beta$. Hence, $V'[a] \subseteq O_\alpha \cap O_\beta$. This implies that $O_\alpha \cap O_\beta \in \tau$. Suppose $O_\alpha \in \tau$ for all $\alpha \in \mathcal{A}$ and let $a \in \bigcup_{\alpha \in \mathcal{A}} O_\alpha$. Then $a \in O_\beta \in \tau$ for some $\beta \in \mathcal{A}$. This implies that there exists $V_\beta \in \Omega$ such that $V'_\beta[a] \subseteq O_\beta$. Hence, $V'_\beta[a] \subseteq \bigcup_{\alpha \in \mathcal{A}} O_\alpha$. It follows that $\bigcup_{\alpha \in \mathcal{A}} O_\alpha \in \tau$. Therefore, τ is a dual B-topology.

Theorem 9. *Let X^D be a dual B-topological space and Ω a filterbase in X^D such that Ω is a family of dual B-filters of X^D. Then Ω is a proper subclass of τ.*

Proof. Suppose X^D is a dual B-topological space and Ω a filterbase in X^D such that Ω is a family of dual B-filters of X^D. Note that $\varnothing \notin \Omega$ by Definition 10(i) but $\varnothing \in \tau$. This implies that $\Omega \neq \tau$. Let $O \in \Omega$ and $x \in O$. It remains to show that $O'[x] \subseteq O$. Suppose $a \in O'[x]$. Then $a \circ x, x \circ a \in O$. Since O is a dual B-filter and $x \in O$, it follows that $a \in O$. Hence, $O'[x] \subseteq O$. This implies that $O \in \tau$. Therefore, Ω is a proper subclass of τ.

Theorem 10. *Suppose X^D is a dual B-topological space and let Ω be a filterbase in X^D such that for all $V \in \Omega$ and for all $p, q \in V$, (i) $p \circ 1 \in V$; and (ii) $(p \circ x) \circ q = 1$ implies $x \in V$. Then Ω is the neighborhood filterbase of $1 \in X^D$. That is, Ω is a family of neighborhoods of* 1 ($\forall V \in \Omega, 1 \in V$ and $V \in \tau$).

Proof. Suppose X^D is a dual B-topological space and let Ω be a filterbase in X^D and $p \in V$. By (i), $p \circ 1 \in V$. By (DB1) and (ii), $(p \circ 1) \circ (p \circ 1) = 1$ implying that $1 \in V$.
Claim: $V'[p] \subseteq V$.
Let $x \in V'[p]$. Then $x \circ p, p \circ x \in V$. This implies that $p \circ x = v$ for some $v \in V$. By (DB1) and (ii), $1 = v \circ v = (p \circ x) \circ v$ implying that $x \in V$. This proves the claim.
By the claim, $V \in \tau$. Therefore, Ω is the neighborhood filterbase of $1 \in X^D$.

Lemma 4. *Suppose X^D is a dual B-topological space and let Ω be a filterbase in X^D such that for all $V \in \Omega$ and for all $p, q \in V$, (i) $p \circ 1 \in V$; and (ii) $(p \circ x) \circ q = 1$ implies $x \in V$. Then $V'[a]$ is open in X^D for all $a \in X^D$.*

Proof. Suppose X^D is a dual B-topological space and let Ω be a filterbase in X^D. Suppose $x \in V'[a]$ for any $a \in X^D$. Then $a \circ x, x \circ a \in V$. Note that by Theorem 10, $V \in \tau$. By Theorem 8, there exist $U_\alpha, U_\beta \in \Omega$ such that $U'_\alpha[a \circ x], U'_\beta[x \circ a] \subseteq V$. Since Ω is a filterbase in X^D, there exist $W \in \Omega$ such that $W \subseteq (U_\alpha \cap U_\beta)$. This implies that $W \subseteq U_\alpha$ and $W \subseteq U_\beta$. By Proposition 3, it follows that $W'[a \circ x] \subseteq U'_\alpha[a \circ x] \subseteq V$ and $W'[x \circ a] \subseteq U'_\beta[x \circ a] \subseteq V$.
Claim: $W'[x] \subseteq V'[a]$.
Suppose $y \in W'[x]$. Then $x \circ y, y \circ x \in W$. By (DB1) and Theorem 1 (iii), $1 = (x \circ y) \circ$

$(x \circ y) = [(a \circ x) \circ (a \circ y)] \circ (x \circ y)$. Similarly, $1 = (y \circ x) \circ (y \circ x) = [(a \circ y) \circ (a \circ x)] \circ (y \circ x)$. Hence by (DB2), $\big(1 \circ [(a \circ x) \circ (a \circ y)]\big) \circ (x \circ y) = 1$ and $\big(1 \circ [(a \circ y) \circ (a \circ x)]\big) \circ (y \circ x) = 1$. By Theorem 10 and hypothesis (ii), $(a \circ x) \circ (a \circ y) \in W$ and $(a \circ y) \circ (a \circ x) \in W$. This implies that $a \circ y \in W'[a \circ x] \subseteq U'_\alpha[a \circ x] \subseteq V$. Similarly, $y \circ a \in W'[x \circ a] \subseteq U'_\beta[x \circ a] \subseteq V$. It follows that $y \in V'[a]$. This proves the claim. Therefore, $V'[a] \in \tau$. That is, $V'[a]$ is open in X^D for all $a \in X^D$.

The next theorem identifies a dual B-topological space determined by a filterbase to be a tdB-algebra provided some conditions.

Theorem 11. *Suppose X^D is a dual B-topological space satisfying the symmetric condition and Ω a filterbase in X^D such that for all $V \in \Omega$ and for all $p, q \in V$, (i) $p \circ 1 \in V$; and (ii) $(p \circ x) \circ q = 1$ implies $x \in V$. Then X^D is a tdB-algebra.*

Proof. Suppose X^D is a dual B-topological space satisfying the symmetric condition and Ω a filterbase in X^D. Let $x \circ y \in O \in \tau$ for any $x, y \in X^D$. By Theorem 8, there exists $V \in \Omega$ such that $V'[x \circ y] \subseteq O$. Note that by Lemma 4, Theorem 10, and Proposition 2, $V'[x], V'[y] \in \tau$ with $x \in V'[x]$ and $y \in V'[y]$. By Proposition 4, $V'[x] \circ V'[y] \subseteq O$. Therefore by Theorem 3, X^D is a tdB-algebra.

The last corollary follows from Theorem 11 and Definition 4 of a dual B-filter.

Corollary 2. *Suppose X^D is a dual B-topological space satisfying the symmetric condition and Ω a filterbase in X^D such that for all $V \in \Omega$, V is a dual B-filter. Then X^D is a tdB-algebra.*

5. Conclusion

Given a dual B-algebra X^D and a family Ω of nonempty subsets of X^D that is closed under finite intersection, we can construct a dual B-topology on X^D given by $\tau = \{U \subseteq X^D \mid \forall x \in U, \exists V \in \Omega$ such that $V[x] \subseteq U\}$. If the empty set is a member of Ω, then X^D is a tdB-algbera. Furthermore, if Ω is a filterbase of X^D, then $\tau = \{O \subseteq X^D \mid \forall a \in O, \exists V \in \Omega$ such that $V'[a] \subseteq O\}$ is also a dual B-topology on X^D. If the condition is imposed to Ω such that for all $V \in \Omega$ and for all $p, q \in V$, (i) $p \circ 1 \in V$; and (ii) $(p \circ x) \circ q = 1$ implies $x \in V$, then X^D is a tdB-algebra. Generally, in this paper we constructed two dual B-topologies on X^D and proved with some conditions that X^D with these topologies is a tdB-algebra.

Acknowledgement

This research is financially supported through an approved research load by the Research, Development, Extension and Publications Office (RDEPO) of the University of San Carlos during the 2nd term of A.Y. 2020-2021. The author would like to extend their sincerest gratitude for this support.

References

[1] Belleza, K., and Albaracin, J., *On Dual B-filters and Dual B-subalgebras in a Topological Dual B-algebra*, Journal of Mathematics and Computer Science, **28** No.1 (2023), 1-10.

[2] Belleza, K. and Vilela, J., *The Dual B-Algebra*, European Journal of Pure and Applied Mathematics, **12** No.4 (2019), 1497-1507.

[3] Dugunji, J., *Topology*, Allyn and Bacon Inc., Atlantic Avenue, Boston (1966).

[4] Jun, Y.B. et al., *On Topological BCI-Algebras*, Information Sciences, **116** (1999), 253-261.

[5] Lee, D.S. and Ryu, D.N., *Notes on Topological BCK-Algebras*, Scientiae Mathematicae Japonicae, **1** No. 2 (1998), 231-235.