EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 4, 2022, 1808-1821
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Epi-completely regular topological spaces

Ibtesam Alshammari[1,*]

[1] *Department of Mathematics, Faculty of Science, University of Hafr Al Batin, Saudi Arabia*

Abstract. The purpose of this work is to introduce and study a new topological property called epi-complete-regularity. A space $(X, \mathcal{T})$ is called an epi-completely-regular space if there exists a topology $\mathcal{T}'$ on X which is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is Tychonoff. This new property is investigated and some examples are presented in this work to illustrate its relationships with other kinds of normality and complete-regularity.

2020 Mathematics Subject Classifications: 54A10, 54B10, 54C10, 54D10, 54D20, 54D15, 54D70

Key Words and Phrases: Epi-normal, epi-regular, epi-almost normal, epi-quasi normal, epi-partially normal, completely regular and epi-mildly normal

1. Introduction

The notion of epi-normality was introduced by Arhangel'skii during his visiting to Department of Mathematics in King Abdulaziz University, Saudi Arabia on 2012. The notion of epi-normality has been studied by Kalantan and Alzahrani in 2016 [15]. Then, Alzahrani studied the notion of epi-regularity in 2018 [5]. Kalantan and Alshammari studied the notion of epi-mild normality in 2018 [18]. At the beginning of 2020, Alshammari studied the notion of epi-almost normality [3]. Thabit studied the notion of epi-partial normality in 2021 [32]. At the end of 2021, Thabit and others studied the notion of epi-quasi normality [31]. The space X means a topological space in whole paper. We need to recall that: a subset A of a space X is said to be a *closed domain* subset if it is the closure of its own interior [20]. The complement of a closed domain subset is called open domain. A subset A of a space X is called π-*closed* if it is a finite intersection of closed domain subsets [33]. The complement of a π-closed subset is called π-open. Two subsets A and B of a space X are said to be *separated* if there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$ [11, 12, 23]. If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T}'$ is called a topology *coarser* than $\mathcal{T}$, and $\mathcal{T}$ is called *finer* [12]. A T_4-space is a T_1 normal space, a T_3-space is a T_1 regular space and a Tychonoff space is a

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4598

Email addresses: iealshamri@hotmail.com, iealshamri@uhb.edu.sa (I. Alshammari)

T_1 completely regular space. A space X is said to be *π-normal* [14], if any pair of disjoint closed subsets A and B of X, one of which is π-closed, can be separated. A space X is said to be *almost-normal* [14, 28], if any pair of disjoint closed subsets A and B of X, one of which is closed domain, can be separated. A space X is said to be *mildly normal* [29], if any pair of disjoint closed domain subsets A and B of X can be separated. A space X is said to be *partially normal* [4], if any pair of disjoint closed subsets A and B of X, one of which is closed domain and the other is π-closed, can be separated. A space $(X, \mathcal{T})$ is said to be *epi-normal* [15] (resp. *epi-mildly normal* [18], *epi-almost normal* [3], *epi-regular* [5], *epi-quasi normal* [31], *epi-partially normal* [32]), if there exists a topology $\mathcal{T}'$ on X coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is a T_4 (resp. Hausdorff mildly-normal, Hausdorff almost-normal, T_3, Hausdorff-quasi-normal, Hausdorff partially-normal) space. A space X is said to be *Hausdorff* or *a T_2-space*, if for each distinct two points $x, y \in X$ there exist two open subsets U and V of X such that $x \in U$, $y \in V$ are $U \cap V = \emptyset$ [12]. A space X is said to be *completely Hausdorff* or *Urysohn* [12, 30], if for each distinct two points $x, y \in X$ there exist two open subsets U and V of X such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. A space X is said to be *almost completely-regular* if for each $x \in X$ and each closed domain subset F of X such that $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$ [28]. A space X is said to be *almost-regular* if for each $x \in X$ and each closed domain subset F of X such that $x \notin F$, there exist two disjoint open subsets U and V such that $x \in U$ and $F \subseteq V$ [27]. A space X is said to be *sub-metrizable* [13], if there exists a metric d on X such that the topology $\mathcal{T}_d$ on X generated by d is coarser than $\mathcal{T}$. The topology on X generated by the family of all open domain subsets of X, denoted by $\mathcal{T}_s$, is coarser than $\mathcal{T}$, and $(X, \mathcal{T}_s)$ is called the *semi-regularization* of X. A space $(X, \mathcal{T})$ is called *semi-regular* if $\mathcal{T} = \mathcal{T}_s$ [22]. A space X is called *H-closed* [12], if X Hausdorff almost-compact [19, 24]. A space X is called *C-normal* [8] (resp. *C-regular* [6], *C-Tychonoff* [7]) if there exist a normal (resp. regular, Tychonoff) space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. A space X is called *L-normal* [16] (resp. *CC-normal* [17]) if there exist a normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each Lindelöf (resp. countably compact) subspace $A \subseteq X$. A space X is called *L-regular* [6] (resp. *L-Tychonoff* [7]) if there exist a regular (resp. Tychonoff) space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$. The basic definitions and any undefined terms in this article can be found in [31] and [32].

In this paper, I introduce and study a new topological property called epi-complete regularity. I show that this new property is different from epi-normality, epi-regularity, epi-mild normality, epi-quasi normality, epi-partial normality and epi-almost normality. Some properties, counterexample and relationships of this property are investigated. This paper contains three main sections starting from section 2. In section 2, the definition of epi-complete regularity is introduced and some examples are presented. Some properties of epi-complete regularity are studied and given in section 3.

2. Preliminaries

First, I present the main definition of this study:

Definition 1. A space $(X, \mathcal{T})$ is called an *epi-completely-regular* space if there exists a topology $\mathcal{T}'$ on X which is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is Tychonoff.

From Definition 1, note that: every epi-completely regular space is Hausdorff and any Tychonoff space is epi-completely-regular, but the converses are not true in general, for example: the irregular lattice topology, Example 6 is a Hausdorff space which is not epi-completely regular. The Smirnov's deleted sequence topology, Example 10, and the half disc topology, Example 5, are epi-completely regular spaces which are not Tychonoff. Now, I present the next results:

Theorem 1. *Every epi-completely-regular space is Urysohn.*

Proof. Let $(X, \mathcal{T})$ be an epi-completely-regular space. Then, there exists a topology $\mathcal{T}'$ on X that is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is T_1-completely-regular. Thus, $(X, \mathcal{T}')$ is Tychonoff. Hence, $(X, \mathcal{T}')$ is Uryshon (completely Hausdorff). Since $\mathcal{T}' \subseteq \mathcal{T}$, we conclude: $(X, \mathcal{T})$ is Urysohn.

Observe that: any Urysohn space is not necessary to be epi-completely regular. For example, the Tychonoff corkscrew topology, Example 9, and the irregular lattice topology, Example 6, are Urysohn spaces which are not epi-completely-regular. Thus, the converse of Theorem 1 is not true in general.

Theorem 2. *Every epi-completely-regular space is epi-regular.*

Proof. Let $(X, \mathcal{T})$ be an epi-completely-regular space. Then, there exists a topology $\mathcal{T}'$ on X coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is T_1-completely-regular. Since every completely-regular space is regular [12], we get: $(X, \mathcal{T}')$ is a T_1-regular space. Hence, $(X, \mathcal{T}')$ is T_3-space. Therefore, $(X, \mathcal{T})$ is epi-regular.

Note that: the converse of Theorem 2 is not necessarily true in general. For example, the Tychonoff corkscrew topology, Example 9, is an epi-regular space which is not epi-completely-regular. Also, complete regularity and epi-complete regularity are different from each other, for example, the half disc topology, Example 5, is an epi-completely-regular space, which is not completely-regular and any uncountable indiscrete space is a completely-regular space which is not epi-completely-regular.

Theorem 3. *Every epi-almost-normal space is epi-completely-regular.*

Proof. Let $(X, \mathcal{T})$ be an epi-almost-normal space. Then, there exists a topology $\mathcal{T}'$ on X which is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is a Hausdorff almost-normal space. Since every almost-normal T_1-space is almost-regular [27], we have: $(X, \mathcal{T}')$ is Hausdorff almost-normal almost-regular. Since every almost-normal almost-regular space is almost-completely regular [28], we get: $(X, \mathcal{T}')$ is Hausdorff almost-completely regular. Let the semi regularization of $(X, \mathcal{T}')$ be $(X, \mathcal{T}'_s)$. Then, $(X, \mathcal{T}'_s)$ is a Hausdorff completely-regular space because the semi regularization of a Hausdorff almost completely regular space is Hausdorff completely regular [22]. Since $\mathcal{T}'_s \subseteq \mathcal{T}' \subseteq \mathcal{T}$, we conclude: $\mathcal{T}'_s$ is a topology on X that is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}'_s)$ is Hausdorff completely-regular and hence Tychonoff. Therefore, $(X, \mathcal{T})$ is epi-completely-regular.

Since every epi-completely-regular space is epi-regular (Theorem 2), every sub-metrizable space is epi-normal and every epi-normal space is epi-almost-normal [3, 15], we obtain:

Corollary 1.

(1) Every sub-metrizable space is epi-completely-regular.

(2) Every epi-normal space is epi-completely-regular.

Thus, we conclude the following implications:

$$\text{epi-normal} \Longrightarrow \text{epi-almost-normal} \Longrightarrow \text{epi-completely-regular} \Longrightarrow \text{epi-regular}$$

The next example is an epi-completely regular space which is not epi-normal.

Example 1. Consider the Example 10 in [26], let $G = D^{\omega_1}$, where $D = \{0, 1\}$ with the discrete topology. Let H be a subspace of G consisting of all points of G with at most countably many non zero coordinates. Put $X = G \times H$. Raushan Buzyakova proved that X cannot be mapped onto a normal space Y by a bijective continuous function [9]. It can be observed that: H is a T_2-Fréchet space and hence it is a k-space. G is also a T_2-compact space. Hence, $X = H \times G$ is a k-space [26]. Since X is Tychonoff, we get X is epi-completely regular. The space X is not C-normal [26]. Since every C-Tychonoff Fréchet Lindelöf space is C-normal, we conclude: X is not Lindelöf. Since X is not C-normal, we obtain X is neither CC-normal, sub-metrizable nor epi-normal. The space X is not a locally compact space as well. Thus, the space X is an epi-completely regular space which is neither C-normal, CC-normal, epi-normal, sub-metrizable nor locally compact.

Observe that: any C-Tychonoff (resp. C-normal) space is not necessary to be epi-completely regular. Here is a counterexample:

Example 2. The countable complement topology $(\mathbb{R}, \mathcal{CC})$ is both C-Tychonoff and C-regular space [6, 7], which is neither epi-completely-regular, epi-regular nor epi-mildly normal because it is not Hausdorff.

The following example is a normal space, which is not epi-completely regular.

Example 3. The left ray topology $(\mathbb{R}, \mathcal{L})$, the right ray topology $(\mathbb{R}, \mathcal{R})$ [30] are normal spaces, which are not epi-completely regular because they are not Hausdorff.

Note that: complete regularity (resp. L-regularity) does not imply to epi-complete-regularity in general as shown by the next example.

Example 4. The double pointed reals topology [30, Example 62], is both a regular and completely regular space [30], which is not epi-completely regular because it is not Hausdorff.

The next example is an epi-completely-regular space which is neither Tychonoff nor completely-regular.

Example 5. *The half disc topology* [30, Example 78] is not Tychonoff. The semi regularization of X is the closed upper half plane with the Euclidean topology $\mathcal{U}$ on $\mathbb{R}$ that is a topology coarser than $\mathcal{T}$ and $(X,\mathcal{U})$ is a T_4-space. Thus, X is epi-normal. Hence, X is epi-completely-regular. Since $(X,\mathcal{T})$ is an almost-completely regular space if and only if $(X,\mathcal{T}_s)$ is completely-regular [22], we get: the half disc topology is almost-completely regular. Therefore, the half disc topology is an epi-completely-regular space, which is neither completely-regular, Tychonoff nor almost-normal.

A Urysohn epi-mildly normal Lindelöf space is not necessary to be epi-completely-regular, for example:

Example 6. *The irregular lattice topology* [30, Example 79], is a Urysohn Lindelöf space, which is neither normal, completely regular nor semi-regular [30]. It is also a mildly-normal space, which is not partially-normal [4]. Hence, it is neither quasi-normal, almost-normal nor semi-normal. Since every almost-regular Lindelöf space is quasi-normal [21], and X is a Lindelöf non quasi-normal space, it is not almost-regular. Since $(X,\mathcal{T})$ is a Hausdorff mildly-normal space, it is epi-mildly normal. Hence, the irregular lattice topology is a Urysohn epi-mildly-normal space, which is neither epi-almost-normal, epi-regular nor epi-completely-regular.

An almost-completely regular space is not necessarily epi-completely-regular. For example:

Example 7. *The telophase topology* [30, Example 73], is a T_1-compact, paracompact space, which is neither Hausdorff, normal nor semi-regular [30]. Clearly that: X is an almost-regular space. Since it is an almost-regular paracompact space, it is almost-normal. Since every almost-normal T_1 space is almost-completely regular, we have: the telophase topology is T_1-almost-completely regular. Since the telophase topology is not Hausdorff, it is neither epi-completely-regular, epi-mildly normal nor epi-regular. Therefore, the telophase topology is an almost-completely regular space, which is neither epi-completely-regular, epi-mildly-normal nor epi-regular.

An epi-completely-regular space need not be almost-normal nor quasi-normal. Here is an example:

Example 8. *The Thomas' plank topology* [30, Example 93], Let $X = \bigcup_{i=0}^{\infty} L_i$, where $L_0 = (0,1)\times\{0\}$ and $L_i = [0,1)\times\{\frac{1}{i}\}$ for each $i \geq 1$. For each $i \geq 1$, each point $(x,\frac{1}{i}) \in L_i,\ x \neq 0$, we have $\{(x,\frac{1}{i})\}$ is an open subset of X. For each $i \geq 1$, the basic open subset of the points $(0, \frac{1}{i}) \in L_i$ is a subset W_i of L_i such that $L_i - W_i$ is finite. The basic open subset of any point $(x, 0) \in L_0$ is of the form $U_i(x, 0) = \{(x, 0)\} \cup \{(x, \frac{1}{n}) : n > i\}$. It can be observed that: each basic open subsets of X is clopen (closed-and-open). Hence, $(X, \mathcal{T})$ is a zero-dimensional, Hausdorff, regular, completely-regular, semi-regular, Urysohn, locally-compact and Tychonoff space, and it is neither normal nor paracompact [30]. Hence, the Thomas' plank topology is an almost-regular and almost-completely regular space. Since it is Hausdorff, we have: the Thomas' plank topology is epi-completely-regular and epi-regular space. Since X is Hausdorff locally-compact, we obtain: X is a k-space. Thus, X is C-normal. It can be observed that: each L_i, $i \geq 1$ is open because L_0 is closed [30]. Also, $A = \{(0, \frac{1}{n}) : n \geq 1\}$ is a closed subset of X [30]. Since $A \cap L_0 = \emptyset$, we get: A and L_0 are disjoint closed subsets of X, which cannot be separated [30]. Let $U = \bigcup_{n \in \mathbb{N}} L_{2n}$ and $V = \bigcup_{n \in \mathbb{N}} L_{2n+1}$. Then, U and V are disjoint open subsets of X. Thus, $\overline{U} = U \cup L_0$ and $\overline{V} = V \cup L_0$. Hence, $\overline{U}$ and $\overline{V}$ are closed-domains in X such that $\overline{U} \cap \overline{V} = L_0$. Therefore, L_0 is a π-closed subset of X. Since A and L_0 cannot be separated, we obtain: X is not π-normal.

Claim 1: Any singleton $\{(x, 0)\}$ is π-closed and any singleton $\{(0, \frac{1}{i})\}$, $i \geq 1$ is also π-closed in X.

Proof of the Claim 1: Let $U_x = \{(x, \frac{1}{2n}) : n \in \mathbb{N}\}$ and $V_x = \{(x, \frac{1}{2n+1}) : n \in \mathbb{N}\}$. Then, U_x and V_x are disjoint open subsets of X such that $\overline{U_x} = U_x \cup \{(x, 0)\}$ and $\overline{V_x} = V_x \cup \{(x, 0)\}$. Therefore, $\overline{U_x}$ and $\overline{V_x}$ are closed domain subsets of X and $\overline{U_x} \cap \overline{V_x} = \{(x, 0)\}$. Thus, $\{(x, 0)\}$ is π-closed in X for each $x \in (0, 1)$. Now, fix a sequence $\langle (x^i_k, \frac{1}{i}) \rangle$ of distinct points of L_i. Consider the two subsequences $U_i = \{(x^i_{2k}, \frac{1}{i}) : k \in \mathbb{N}\}$ and $V_i = \{(x^i_{2k+1}, \frac{1}{i}) : k \in \mathbb{N}\}$. Then, U_i and V_i are disjoint open subsets of X, $U_i, V_i \subset L_i$ for each $i \geq 1$, $\overline{U_i} = U_i \cup \{(0, \frac{1}{i})\}$ and $\overline{V_i} = V_i \cup \{(0, \frac{1}{i})\}$. Since $\overline{U_i}$ and $\overline{V_i}$ are closed-domains of X, we get: $\{(0, \frac{1}{i})\}$ is π-closed for each $i \geq 1$. Now, let $G = \bigcup_{i \geq 1} U_i$ and $H = \bigcup_{i \geq 1} V_i$. Then, G and H are disjoint open subsets of X such that $\overline{G} = G \cup A \cup \{(x_{2k}, 0) : k \in \mathbb{N}\}$ and $\overline{H} = H \cup A \cup \{(x_{2k+1}, 0) : k \in \mathbb{N}\}$, where $A = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$. Then, $\overline{G}$ and $\overline{H}$ are closed-domains in X such that $\overline{G} \cap \overline{H} = A$. Hence, A is π-closed. Since $A \cap L_0 = \emptyset$ and they cannot be separated [30], we obtain that: X is not quasi-normal. It is easy to show that X cannot be semi-normal.

Claim 2: The Thomas' plank topology is not almost-normal.

Proof of the Claim 2: It can be observed that, $A_1 = \{(0, \frac{1}{2n}) : n \in \mathbb{N}\}$ is a closed subset of X and $U = \bigcup_{n \in \mathbb{N}} L_{2n}$ is an open-domain subset of X such that $A_1 \subseteq U$. Then, for each open subset W of X such that $A_1 \subset W$, we have: $A_1 \subseteq W \subseteq \overline{W} \nsubseteq U$ because there are some points $(x, 0) \in \overline{W}$, and $(x, 0) \notin U$ for each $(x, 0) \in L_0$. Hence, X is not almost-normal. Note that: $A = \{(0, \frac{1}{n}); n \in \mathbb{N}\}$ and L_0 are disjoint π-closed subsets that cannot be separated. If $U = \bigcup_{n \in \mathbb{N}} L_n$ is π-open subset of X such that $A \subseteq U$. For each open set W of X, we have: $A \subseteq W \subseteq \overline{W} \nsubseteq U$ and $A \subseteq W \subseteq \text{int}(\overline{W}) \nsubseteq U$. Thus, X is neither quasi-normal nor semi-normal. Therefore, the Thomas' plank topology is an epi-completely-regular space, which is neither almost-normal, semi-normal nor quasi-normal.

Note that: an epi-regularity does not imply to epi-complete-regularity as shown by the next example:

Example 9. *The Tychonoff corkscrew topology:* [30, Example 90], Let $X = S \cup \{a^+, a^-\}$, where $(S, \mathcal{T})$ is homeomorphic to the deleted Tychonoff plank topology [30]. The basic open subset U of a^+ contains all points of X which lies above a certain level k. That means: $U = \{x \in X : L(x) > k+1\}$. The basic open subset V of a^- contains all points of X which lies below a certain level k. That means: $V = \{x \in X : L(x) < k+1\}$. The space $(X, \mathcal{T})$ is a Hausdorff, regular and semi-regular space, which is neither Tychonoff, Urysohn, locally-compact, Lindelöf, first-countable, normal nor completely-regular [30]. Since $(X, \mathcal{T})$ is a Hausdorff regular space, it is epi-regular. Since every regular almost-normal space is completely-regular [28], and X is regular non completely-regular, we obtain: $(X, \mathcal{T})$ is not almost-normal.
Claim 1: Any Hausdorff topology $\mathcal{T}'$ on X, which is coarser than $\mathcal{T}$, cannot be completely-regular.
Proof of the Claim 1: Let $\mathcal{T}'$ be any Hausdorff topology on X which is coarser than $\mathcal{T}$. I show $(X, \mathcal{T}')$ is not a completely-regular space. Let A be any closed subset of $(X, \mathcal{T}')$ and $a^+ \notin A$. Then, A is a closed subset of $(X, \mathcal{T})$ and $a^+ \notin A$. Thus, $X \setminus A$ is an open subset of $(X, \mathcal{T})$ containing a^+. But a^+ cannot be separated by a continuous function from a closed subset A of X consisting the complement of the basis neighborhood of a^+ [30]. Thus, $(X, \mathcal{T}')$ is not a completely-regular space. Therefore, any Hausdorff topology $\mathcal{T}'$ on X, which is coarser than $\mathcal{T}$ cannot be completely-regular. Hence, $(X, \mathcal{T})$ is not epi-completely-regular. Hence, X is not epi-almost-normal. Since every T_1-semi-regular almost-completely regular space is epi-completely-regular (Corollary 7), and X is T_1-semi-regular non epi-completely-regular, we obtain that: X is not almost-completely regular. Therefore, the Tychonoff corkscrew topology is an epi-regular space, which is neither epi-completely-regular, almost-completely regular nor epi-almost-normal.

Note that: the Mrówka space $\Psi(\mathcal{A})$ [15, Example 2.10], is a Tychonoff, first-countable and locally compact space, which is neither normal, countably-compact nor epi-normal. Hence, it is an epi-completely-regular space, which is not epi-normal. The space presented in [15, Example 3.1], is a sub-metrizable, epi-normal, Tychonoff and C-normal space, which is not mildly-normal. The space presented in [6, Example 2.8], is an epi-completely-regular space, which is neither C-normal nor epi-normal. Now, since every Hausdorff locally compact space is Tychonoff [12], we get:

Corollary 2. Every Hausdorff locally-compact space is epi-completely-regular.

The converse of Corollary 2 cannot be true in general. Here is a counterexample:

Example 10. *The Smirnov's deleted sequence topology* [30, Example 64], is a Urysohn space, which is neither semi-regular, completely-regular, locally-compact nor almost-normal. Since any closed domain subset of X is just the closed domain in the Euclidean topology and $\mathcal{U} \subseteq \mathcal{T}$ [30], we obtain: X is both almost-regular and almost-completely regular. The Smirnov's deleted sequence topology is not almost-normal because the closed domain subset $B = [-1, 0]$ is disjoint from the closed subset $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, and they cannot be separated. Since $\mathcal{U} \subseteq \mathcal{T}$, $\mathcal{U}$ is the Euclidian topology on $\mathbb{R}$, which is coarser than $\mathcal{T}$, and $(\mathbb{R},\mathcal{U})$ is a T_4-space, we obtain: X is epi-normal (in fact it is sub-metrizable [5]). Since the Smirnov's deleted sequence topology is a Lindelöf non regular space, it is not L-regular [6]. Therefore, the Smirnov's deleted sequence is an epi-completely-regular space, which is neither completely-regular, almost-normal, L-regular nor locally-compact. The Niemytzki plane topology, the sorgenfrey line square and the Michael line are Tychonoff and hence epi-completely-regular spaces [30], which are not locally-compact.

Example 11. *The deleted Tychonoff plank* [30, Example 87], is a Tychonoff locally-compact space. Hence, it is an epi-completely-regular space. The deleted Tychonoff plank is neither almost-normal nor sub-metrizable [6, 8]. Therefore, the deleted Tychonoff plank topology is an epi-completely-regular space, which is not sub-metrizable.

Example 12. *The odd-even topology* [30, Example 6], is a completely regular and normal space, which is not epi-completely regular being not Hausdorff.

Every Hausdorff semi-regular almost-compact (resp. H-closed) space is not necessary to be epi-completely regular. Here is a counterexample:

Example 13. *The minimal Hausdorff topology* [30, Example 100], is a Hausdorff, semi-regular, second-countable and almost-compact space, which is neither Urysohn, regular, normal nor compact [30]. Since X is a semi-regular non regular space, we have: X is not almost-regular. Since X is a T_1 non almost-regular space, it is not almost-normal. Hence, X is a quasi-normal space, which is not semi-normal [31]. Since X is not Urysohn, it is neither epi-almost-normal, epi-regular, epi-completely-regular nor epi-normal. Therefore, the minimal Hausdorff topology is a semi-regular, Hausdorff and epi-quasi-normal almost-compact H-closed space [31], which is neither almost-regular, epi-regular, epi-completely-regular nor Urysohn.

Observe that: a normal compact space need not be epi-completely regular. For example: the excluded point topology [30, Example 15], and the either-or-topology [30, Example 17], are normal compact spaces, which are neither epi-completely-regular, epi-regular nor epi-normal.

3. Some properties of epi-complete regularity

In this section, I present the following results:

Theorem 4. *Epi-complete regularity is a topological property.*

Proof. Let $(X,\mathcal{T}) \cong (Y,\mathcal{S})$ and $(X,\mathcal{T})$ be an epi-completely-regular space. There are a homeomorphism $f : X \to Y$ and a topology $\mathcal{T}'$ on X that is coarser than $\mathcal{T}$ such that $(X,\mathcal{T}')$ is Tychonoff. Define $\mathcal{S}'$ on Y by $\mathcal{S}' = \{f(U) : U \in \mathcal{T}'\}$. Then, $\mathcal{S}'$ is a topology on Y, which is coarser than $\mathcal{S}$, and $(Y,\mathcal{S}')$ is Tychonoff. Thus, $(Y,\mathcal{S})$ is epi-completely-regular.

Theorem 5. *Epi-complete regularity is an additive property.*

Proof. Let X_s be an epi-completely-regular space for each $s \in S$. Then, there exists a topology $\mathcal{T}_s{}'$ on X_s, which is coarser than $\mathcal{T}_s$, such that $(X_s, \mathcal{T}_s{}')$ is a T_1-completely-regular space. Since both T_1 and complete-regularity are additive properties, we obtain: $(X, \underset{s\in S}{\oplus} \mathcal{T}_s{}')$ is T_1-completely regular (Tychonoff). Since $\underset{s\in S}{\oplus} \mathcal{T}_s{}'$ is a topology coarser than $\underset{s\in S}{\oplus} \mathcal{T}_s$, we get: $(X, \underset{s\in S}{\oplus} \mathcal{T}_s)$ is epi-completely-regular.

Theorem 6. *Epi-complete regularity is a hereditary property.*

Proof. Let $(X, \mathcal{T})$ be an epi-completely-regular space, and $(M, \mathcal{T}_M)$ be a subspace of X. Then, there exists a topology $\mathcal{T}'$ on X that is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is T_1-completely-regular. To show $(M, \mathcal{T}_M)$ is epi-completely-regular, define $\mathcal{T}_M{}'$ on M by: $\mathcal{T}_M{}' = \{U \cap M : U \in \mathcal{T}'\}$. Then, $\mathcal{T}_M{}' \subseteq \mathcal{T}_M$. Hence, $\mathcal{T}_M{}'$ is a topology on M which is coarser than $\mathcal{T}_M$. Since $(X, \mathcal{T}')$ is a T_1-completely-regular space and $(M, \mathcal{T}_M{}')$ is a subspace of X, we obtain: $(M, \mathcal{T}_M{}')$ is a T_1-completely-regular subspace. Therefore, $(M, \mathcal{T}_M)$ is epi-completely-regular.

Theorem 7. *A product space $X = \underset{\alpha\in\Lambda}{\Pi} X_\alpha$, $X_\alpha \neq \emptyset$ for each $\alpha \in \Lambda$, is an epi-completely-regular space if and only if each factor X_α is epi-completely-regular for each $\alpha \in \Lambda$.*

Proof. Let $(\underset{\alpha\in\Lambda}{\Pi} X_\alpha, \mathcal{T})$ be an epi-completely-regular space, $X_\alpha \neq \emptyset$ for each $\alpha \in \Lambda$. There exists a topology $\mathcal{T}'$ which is coarser than $\mathcal{T}$ such that $(\underset{\alpha\in\Lambda}{\Pi} X_\alpha, \mathcal{T}')$ is T_1-completely-regular. Thus, we have each factor $(X_\alpha, \mathcal{T}_\alpha')$ is a T_1-completely-regular space [12], where $\mathcal{T}_\alpha{}'$ is a topology coarser than $\mathcal{T}_\alpha$ for each $\alpha \in \Lambda$. Thus, $(X_\alpha, \mathcal{T}_\alpha)$ is an epi-completely regular space for each $\alpha \in \Lambda$. Conversely, suppose that $(X_\alpha, \mathcal{T}_\alpha)$ is an epi-completely-regular space for each $\alpha \in \Lambda$. Then, for each $\alpha \in \Lambda$, there exists a topology $\mathcal{T}_\alpha{}'$ that is coarser than $\mathcal{T}_\alpha$ such that $(X_\alpha, \mathcal{T}_\alpha{}')$ is a T_1-completely-regular space. Thus, the product space $(\underset{\alpha\in\Lambda}{\Pi} X_\alpha, \mathcal{T}')$ is T_1-completely-regular, where $\mathcal{T}'$ is coarser than $\mathcal{T}$. Therefore, $(\underset{\alpha\in\Lambda}{\Pi} X_\alpha, \mathcal{T})$ is epi-completely-regular.

Corollary 3. *Epi-complete regularity is a multiplicative property.*

Theorem 8. *Every epi-completely regular nearly-compact (resp. nearly-paracompact) space is epi-normal.*

Proof. Let $(X, \mathcal{T})$ be an epi-completely-regular nearly-compact (resp. nearly-paracompact) space. Then, there exists a topology $\mathcal{T}'$ on X which is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is a Tychonoff compact (resp. paracompact) space. Thus, $(X, \mathcal{T}')$ is a T_1-normal space. Hence, $(X, \mathcal{T}')$ is a T_4-space. Therefore, $(X, \mathcal{T})$ is epi-normal.

Now, we recall the definition of the Alexandroff duplicate space. For any space X, let $X' = X \times \{1\}$. Clearly that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For an element $x \in X$, the element $(x, 1) \in X'$ and for a subset $B \subseteq X$, let $B \times \{1\} = \{(x, 1) : x \in B\} \subseteq X'$. For each $(x, 1) \in X'$, let $\mathcal{B}((x, 1)) = \{\{(x, 1)\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U \times \{1\} \setminus \{(x, 1)\}) :$

U is open in X with $x \in U\}$. Let $\mathcal{T}$ denote the unique topology on $A(X)$ which has $\{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}((x,1)) : (x,1) \in X'\}$ as its neighborhood system. The space $A(X)$ with this topology is called the Alexandroff duplicate of X [2].

Theorem 9. *The Alexandroff duplicate $A(X)$ of an epi-completely-regular space X is epi-completely-regular.*

Proof. Let $(X, \mathcal{T})$ be an epi-completely-regular space. Then, there exists a topology $\mathcal{T}'$ on X that is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is T_1-completely-regular. Since T_1 and complete-regularity are preserved by the Alexandroff duplicate space [2], we obtain: $A(X, \mathcal{T}')$ is also a T_1-completely-regular space, which is coarser than $A(X, \mathcal{T})$ by the topology of the Alexandroff duplicate. Hence, $A(X)$ is epi-completely-regular.

Since every subspace of a cube is completely-regular [12], we get:

Corollary 4. Every T_1-subspace of a cube is epi-completely-regular.

Since every C_2-paracompact Fréchet space is epi-normal, and any Mrôwka space $\Psi(\mathcal{A})$ is Tychonoff [18], we obtain:

Corollary 5.

(1) Every C_2-paracompact first-countable space is epi-completely-regular.

(2) Any Mrôwka space $\Psi(\mathcal{A})$ is epi-completely-regular.

Note that: a space $(X, \mathcal{T})$ is an almost-completely regular space if and only if the semi-regularization $(X, \mathcal{T}_s)$ of $(X, \mathcal{T})$ is completely-regular [22]. Also, complete-regularity is not a semi-regularization property, but almost-complete regularity is [22]. For example, the half disc topology $(X, \mathcal{T})$ is not completely-regular [30], and its semi-regularization $(X, \mathcal{T}_s)$ is the usual topology on the closed upper half plane, which is completely-regular.

Theorem 10. *If $(X, \mathcal{T})$ is an almost-completely regular space such that the semi-regularization $(X, \mathcal{T}_s)$ of $(X, \mathcal{T})$ is T_1, then $(X, \mathcal{T})$ is epi-completely-regular.*

Proof. Let $(X, \mathcal{T})$ be an almost-completely regular space and the semi-regularization $(X, \mathcal{T}_s)$ of $(X, \mathcal{T})$ be T_1. Since the semi-regularization of an almost-completely regular space is completely-regular [22], we get: $(X, \mathcal{T}_s)$ is T_1-completely-regular. Thus, $(X, \mathcal{T}_s)$ is Tychonoff. Since $\mathcal{T}_s$ is a topology on X which is coarser than $\mathcal{T}$, we obtain: $(X, \mathcal{T})$ is epi-completely-regular.

Since every extremally-disconnected space is T_1-π-normal [14], we get: every extremally-disconnected space is T_1-almost-completely regular. Since every extremally-disconnected semi-regular space is Tychonoff [3], we conclude:

Corollary 6.

(a) Every Hausdorff extremally-disconnected space is epi-completely-regular.

(b) Every extremally-disconnected semi-regular space is epi-completely-regular.

In fact, an epi-completely-regular space is not necessary to be extremally-disconnected. For example: the rational sequence topology [30, Example 65], is a semi-regular epi-completely-regular space being Tychonoff, which is not extremally disconnected. The next result is obvious:

Theorem 11. *If the semi-regularization space* $(X, \mathcal{T}_s)$ *of a space* $(X, \mathcal{T})$ *is an epi-completely-regular space, then* $(X, \mathcal{T})$ *is epi-completely-regular.*

Theorem 12. *Every Hausdorff almost-completely regular space is epi-completely-regular.*

Proof. Let $(X, \mathcal{T})$ be a Hausdorff almost-completely regular space. Let $(X, \mathcal{T}_s)$ be the semi-regularization of $(X, \mathcal{T})$. Then, $(X, \mathcal{T}_s)$ is a Hausdorff completely regular space because the semi-regularization of a Hausdorff almost-completely regular space is Hausdorff completely regular [22]. Thus, $(X, \mathcal{T}_s)$ is Tychonoff. Since $\mathcal{T}_s \subseteq \mathcal{T}$, we conclude: $(X, \mathcal{T})$ is epi-completely regular.

The next results are obvious:

Corollary 7.

(1) Every T_1-semi-regular (resp. semi-normal) almost-completely regular space is epi-completely-regular.

(2) Any nearly-paracompact Hausdorff space is epi-normal.

(3) Every almost-regular Hausdorff Lindelöf space is epi-quasi-normal.

Since every epi-completely-regular space is epi-regular, and every epi-regular space is C-regular, we get: every epi-completely-regular space is C-regular, but the converse is not true in general. For example: *the odd-even topology*, Example 12, is a normal and completely-regular space [30], which is not T_1. Thus, the odd-even topology is a C-regular space, which is not epi-completely regular.

Theorem 13. *Every semi-regular almost-normal* T_1*-space is Tychonoff.*

Proof. Let X be a semi-regular T_1-almost-normal space. Then, X is almost-regular. Since every semi-regular almost-regular is regular, we obtain: X is a T_1-regular almost-normal space. Hence, X is a T_1-completely-regular space because every regular almost-normal space is completely-regular [10]. Therefore, X is Tychonoff.

From Theorem 13, we conclude the next corollary:

Corollary 8. Every semi-regular almost-normal T_1-space is epi-completely regular.

The next theorem has been presented in [25, Theorem 9 - 1.17, page 306]:

Theorem 14. [25], *A space* $(X, \mathcal{T})$ *is a* T_1*-space if and only if* $\mathcal{T}$ *contains the finite complement topology on* X*. i.e.* $\mathcal{CF} \subseteq \mathcal{T}$ *and* $(X, \mathcal{CF})$ *is the finite complement topology on* X*.*

From Theorem 14, we conclude:

Corollary 9. If $(X, \mathcal{T})$ is a T_1-space, then there exists a topology $\mathcal{T}'$ coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is T_1 almost completely regular (resp. almost regular). We can say that $(X, \mathcal{T})$ is epi-almost completely regular (resp. epi-almost regular).

Recall that: any closed extension space $(X^p, \mathcal{T}^*)$ of a given space $(X, \mathcal{T})$ is always connected, π-normal, almost normal, separable and it cannot be T_1 [1]. Thus, we conclude:

Corollary 10. Every closed extension space $(X^p, \mathcal{T}^*)$ of a given space $(X, \mathcal{T})$ cannot be epi-completely regular.

The next problem is still open in this work:

Problem:

- Is there an example of an epi-completely-regular space, which is not epi-mildly-normal?.

4. Conclusion

A new version of complete regularity called epi-complete regularity has been studied in this work. I have shown that epi-complete regularity is different from both epi-regularity and epi-normality. I have proved that epi-complete regularity is a topological, productive, hereditary and additive property. Some properties, counterexamples and relationships with some other forms of topological properties have been presented and proved.

References

[1] Dina Abuzaid, Suad Al-Qarhi, and Lutfi Kalantan. Closed extension topological spaces. *European Journal of Pure and Applied Mathematics (UJPAM).*, 15(2):672–680, 2022.

[2] Khulod Almontashery and Lutfi Kalantan. Results about the alexandroff duplicate space. *Appl. Gen. Topol.*, 17(2):117–122, 2016.

[3] Ibtesam Alshammari. Epi-almost normality. *Journal of Mathematical Analysis,*, 11:52–57, 2020.

[4] Ibtesam Alshammari, Lutfi Kalantan, and Sadeq Ali Thabit. Partial normality. *Journal of Mathematical Analysis*, 10:1–8, 2019.

[5] S. Alzahrani. Epiregular topological spaces. *Afr. Mat.*, 29:803–808, 2018.

[6] Samirah Alzahrani. c-regular topological spaces. *Journal of Mathematical Analysis JMA*, 9:141–149, 2018.

[7] Samirah Alzahrani. c-tychonoff and l-tychonoff topological spaces. *European Journal of Pure and Applied Mathematics*, 11(3):882–892, 2018.

[8] Samirah Alzahrani and Lutfi Kalantan. c-normal topological property. *Filomat*, 31:2:407–411, 2017.

[9] R. Z. Buzyakova. An exampe of a product of two normal groups that can not be condensed onto a normal space. *Moscow Univ. Math. Bull.*, 52(3):page 42, 1961.

[10] A. K. Das. Simultaneous generalizations of regularity and normality. *European Journal of Pure and Applied Mathematics*, 4(1):34–41, 2011.

[11] J. Dugundji. *Topology.* Allyn and Bacon, Inc., 470 Atlantic Avenue, Boston, 1966.

[12] R. Engelking. *General Topology*, volume 6. Berlin: Heldermann (Sigma series in pure mathematics), Poland, 1989.

[13] G. Gruenhage. Generalized metric spaces. *In: Handbook of Set-theoretic topology, K. Kunen and J. Vaughan, eds., North-Holland, Amsterdam*, pages 423–501, 1984.

[14] L. Kalantan. π-normal topological spaces. *Filomat*, 22-1:173–181, 2008.

[15] L. Kalantan and S. Alzahrani. Epinormality. *J. nonlinear Sci. Appl.*, (9):5398–5402, 2016.

[16] L. Kalantan and M. Saeed. l-normality. *Topology Proceedings*, 50:141–149, 2017.

[17] Lutfi Kalantan and Manal Alhomieyed. cc-normal topological spaces. *Turk. J. Math.*, 41:749–755, 2017.

[18] Lutfi Kalantan and Ibtesam Alshammari. Epi-mild normality. *open Mat.J.*, 16:1170–1175, 2018.

[19] J. K. Kohli and A. K. Das. A class of spaces containing all generalized absolutely closed (almost compact) spaces. *Applied General Topology*, 7(2):233–244, 2006.

[20] C. Kuratowski. *Topology I*, volume 4th ed. in France. Hafner, New York, 1958.

[21] S. Lal and M. S. Rahman. A note of quasi-normal spaces. *Indian Journal of Mathematics*, 32(1):87–94, 1990.

[22] M. Mršević, I. L. Reilly, and M.K. Vamanamurthy. On semi regularization topologies. *J. Austral. Math. Soc., (Series A)*, 38:40–54, 1985.

[23] C. Patty. *foundation of topology.* PWS-KENT Publishing Company, Boston, 1993.

[24] J. R. Porter and J. D. Thomas. On h-closed and minimal hausdorff spaces. *Trans. Amer. Math. Soc.*, 138:159–170, 1996.

[25] M. D. Raisinghania and R. S. Aggarwal. *Topology for Post-Graduate students in Indian Universities*. S. Chand and Company LTD, Ram Nagar, New Delhi-110055, India, 1973.

[26] Maha Mohammed Saeed. Countable normality. *Journal of Mathematical Analysis*, 9:116–123, 2018.

[27] M. K. Singal and S. Arya. On almost regular spaces. *Glasnik Matematicki*, 4(24):89–99, 1969.

[28] M. K. Singal and S. P. Arya. On almost normal and almost completely regular spaces. *Glasnik Matematicki*, 5(5):141–152, 1970.

[29] M. K. Singal and A. R. Singal. Mildly normal spaces. *Kyungpook Mathematical Journal*, 13-1:27–31, 1973.

[30] A. L. Steen and J. A. Seebach. *Counterexamples in Topology*. Dover Publications, INC., New York, 1995.

[31] Sadeq Ali Thabit, Ibtesam Alshammari, and Wafa Alqurashi. Epi-quasi normality. *Open Mathematics (De Gruyter Open Access)*, 19:1755–1770, 2021.

[32] Sadeq Ali Saad Thabit. Epi-partial normality. *Journal of Physics: Conference Series, IOP Publishing Ltd (J. Phys.: Conf. Ser)*, 1900(012013):1–11, 2021.

[33] V. Zaitsev. On certain classes of topological spaces and their bicompactifications. *Doklady Akademii Nauk SSSR*, 178:778–779, 1968.