EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 4, 2022, 1917-1936
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

The possible solutions for the two KdV-type equations using a semi-analytical Kamal-iteration method

Amal Jasim Mohammed[1,*], Sohaib Talal Al-Ramadhani[1], Rabeea Mohammed Hani Darghoth[2]

[1] *College of Education For Pure Sciences, Mathematics Department, University of Mosul, Iraq*
[2] *College of Basic Education, Mathematics Department, University of Mosul, Iraq*

Abstract. In this paper, we used a semi-analytical method that combines the Kamal transform and a modified iteration method to solve the nonlinear homogeneous and nonhomogeneous modified Kerteweg-de Vries and the fifth-order Kerteweg-de Vries equations. The modified iteration method has been used to with a condition that guarantees fast convergence of the approximate solution. The analysis and convergence of the combined method have been discussed. Furthermore, the numerical simulations are presented to illustrate the effectiveness of the proposed semi-analytical method.

2020 Mathematics Subject Classifications: 35Q53, 35Q35,41A58, 74H15, 74J35

Key Words and Phrases: KdV, mKdV, FKdV equations, Modified Iteration method, Kamal transform

1. Introduction

The Kerteweg-de Vries equation (KdV eq.) is a nonlinear integrable system that has various applications in fluid mechanics, hydromagnetics plasma, electrical transmission lines, and others, see for example [8]. Thus, numerical and semi-analytical methods are designed to solve the nonlinear partial differential equations (PDEs), where; several mechanism methods were employed to study and find out a suitable solution for the PDEs. Then, some of these methods are analytical and others are semi-analytical (numerical with analytic technique) and numerical, for instance, homotopy analysis, differential transformation methods, and many others. These methods are used in different branches of sciences such as; physics, chemistry, and engineering which is why the numerical and semi-analytical methods have become more common in use than pure analytical methods [1, 7, 10, 16]. For example, the integral transforms; by Laplace, Fourier, Hilbert, and Sumida, have

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4599

Email addresses: a.j.moha7@uomosul.edu.iq. (A.J. Mohammed),
s.alramadhani@uomosul.edu.iq. (S.T. Al-Ramadhani), r.darghoth@uomosul.edu.iq. (R.M.H. Darghoth)

been smoothly used to solve linear homogeneous or nonhomogeneous ordinary and partial differential equations (PDEs) [4, 25]. For instance, heat, wave, telegraph equations, advection problems. Kamal transform (KT) is used to solve various types of differential and integral equations, see [2, 5, 15, 17, 26]. However, all these transforms cannot be used directly if they are employed for the nonlinear homogeneous or nonhomogeneous ordinary and PDEs, because there are no transforms for the nonlinear part. For that reason, some researchers have used the integral transform with the help of decomposition, variational methods and others to solve this problem [6, 23, 24].

The problem in this project is based on the nonlinear partial KdV eq. [3, 9, 11, 28] of the form

$$(\partial_t - 6\ w(x,t)\ \partial_x + \partial_{xx})w(x,t) = 0, \qquad -\infty < x < \infty \quad t > 0, \tag{1}$$

where, the initial condition $w(x,0) = F(x), F(x)$ is a known function. A differentiable solution of eq. (1) is $w = w(x,t)$ which is travelling wave solution tends to zero as $|x| \to \infty$. This kind of equation is classified with the group of integrable equations such as; the nonlinear Schrödinger, Sine-Gordon, Kadomtsev-Petviashvili and Bogoyavlensky-Konopelchenko equations [12, 13, 19, 27]. In addition, the family of the nonlinear PDEs of the KdV eq. comes from the theory of dispersive wave motions. The solutions of these equations have been used to describe several physical phenomena like; "widespread class, elastic scattering property" and others (see [26]).

Here, we are interested in employing the modified iteration method (**MI**) [21], which was used for the homogeneous and nonhomogeneous nonlinear ordinary differential equations (ODEs). In this paper, we used a combination of the **KT** [5, 17] and **MI** [21] method which we denote for as **K-MI**. Using this method, we have explored the acceptable solution for two types of the family of KdV eq.: the modified KdV eq. (mKdV eq.) and the fifth order of KdV eq. (FKdV eq.).

The outline of this paper is arranged as follows. The mathematical model has been presented in Section 2, where, we displayed the **KT** and some of its related theorems. Next, we have shown the combined method **K-MI** and applied it to the general description of nonlinear PDEs. The limit of iteration of this method has been discussed. In section 3 we have applied the **K-MI** method to several examples to show the success of the method in reaching accuracy of the results. Section 4 is related to comparing the classical iterations method to the modified one, both combined with the Kamal transform. Finally, our discussion and conclusion are given in section 5 and 6 respectively.

2. Mathematical Method

2.1. Kamal transform

This transformation is one of the integral transforms that is employed to find out the solution to differential equations [22]. We have presented some of its properties. We have denoted the integral transform of the Kamal by $\mathcal{K}$ and it is specified as:

$$\mathcal{K}\left[S(t)\right] = \int_0^\infty S(t)\ e^{-\frac{t}{v}} t = G(v), \tag{2}$$

where, $l1 \leq \upsilon \leq l2$. This transform defined for functions on the set $\mathcal{A}$

$$\mathcal{A} = \left\{ S(t) : \exists\ M, l1,\ l2 > 0, |S(t)| < M\ e^{\frac{|t|}{lj}}, \text{if } t \in (-1)^l \times [0, \infty) \right\} \tag{3}$$

Theorem 1. Assume $G(\upsilon)$ is the **KT** of $S(t)$ then [17]:

$$\mathcal{K}\left[S^{(m)}(t)\right] = \upsilon^{(-m)} G(\upsilon) - \sum_{k=0}^{m-1} \upsilon^{k-m+1} S^{(k)}(0), \quad \text{for} \quad m \geq 0, \tag{4}$$

where $(S^{(m)})$ is the m^{th} derivative. For $m = 1$, $\quad \mathcal{K}[S'(t)] = \upsilon^{(-1)} G(\upsilon) - S(0)$.

Theorem 2. Let a be a constant and $a \geq 0$ then [17]:

$$\mathcal{K}[a] = a\ \upsilon. \tag{5}$$

Theorem 3. For an integer number $m \geq 0$, the $\mathcal{K}[t^m]$ is [17]:

$$\mathcal{K}[t^m] = m!\ \upsilon^{m+1} \tag{6}$$

For more information about the properties of this transformation see [5, 22] and many others. In the next section, we use the above theorems to calculate the appropriate solution for the nonlinear PDEs with the help of the **MI** method.

2.2. Kamal-Modified Iteration method for the KdV equation types

Consider the nonlinear PDEs of the form

$$\dot{\mathcal{L}}[w(x,t)] + M[w(x,t)] + G(x,t) = 0, \tag{7}$$

where, $\dot{\mathcal{L}}$ is the time dependent operator, M is containing only the dependants of the variable x and $w(x, 0) = F(x)$ is the initial condition (IC) of eq. (7). Here, $F(x)$ and $G(x, t)$ are known functions.
First, we apply **KT** to eq. (7)

$$\mathcal{K}[\mathcal{L}[w(x,t)]] + \mathcal{K}[M[w(x,t)]] + \mathcal{K}[G(x,t)] = 0, \tag{8}$$

next, we need to use theorems (1 and 2) and the IC in eq. (8) as

$$W(x,\upsilon) = \upsilon\ w(x,0) - \upsilon\ \mathcal{K}[M[w(x,t)] + G(x,t)]. \tag{9}$$

To find $w(x,t)$, we need to take the inverse of Kamal transform ($\mathcal{K}^{-1}$) of eq. (9), which yields

$$w(x,t) = w(x,0) - \mathcal{K}^{-1}[\upsilon \mathcal{K}[M[w(x,t)] + G(x,t)]]. \tag{10}$$

Second, we use the **MI** method on eq. (10) to deduce the solution of eq. (7).

Let $w(x, 0) = w_0(x, t)$, be the first approximate solution then, eq. (10) becomes

$$w_{n+1}(x,t) = w_0(x,0) - \mathcal{K}^{-1}[\upsilon \mathcal{K}[M[w_n(x,t)] + G(x,t)]]. \tag{11}$$

Remark 1. During the calculation of classical iteration method, some terms are created, we need to cancel the un necessary terms which make noise to our calculation (see section (4). For that reason, we have defined the condition of the **MI** method as

$$M[w_n(x,t)] = Q_n(x,t) + O(t^{n+1}), \qquad n = 0, 1, 2, \dots \tag{12}$$

where, $O(t^{n+1})$ are higher order terms to be neglected. Therefore, eq. (11) becomes:

$$w_{n+1}(x,t) = w_0(x,0) - \mathcal{K}^{-1}[\upsilon\mathcal{K}[Q_n(x,t) + G(x,t)]]. \tag{13}$$

Suppose that $G(x,t)$ is smooth enough, so it has Taylor's expansion in the variable t (about $t = 0$)

$$G(x,t) = G_n(x,t) + O(t^{n+1}), \tag{14}$$

where,

$$\begin{aligned} G_n =& \sum_{k=0}^{n+1} S_k t^k = S_0 + S_1 t + S_2 t^2 + \cdots + S_{n+1} t^{n+1}, \\ S_k =& \frac{1}{k!}\left[\frac{k}{t^k} G(x,t)\right], \end{aligned} \tag{15}$$

so, eq. (13) becomes

$$w_{n+1}(x,t) = w_0(x,0) - \mathcal{K}^{-1}[\upsilon\mathcal{K}[Q_n + G_n]]. \tag{16}$$

On the other hand, eq. (16) can be written as

$$w_{n+1}(x,t) = w_0(x,0) - \mathcal{K}^{-1}[\upsilon\mathcal{K}[Q_{n-1} + G_{n-1}]] - \mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_n - Q_{n-1}) + (G_n - G_{n-1})]], \tag{17}$$

set $n = n - 1$ in (16), then we have used it in eq. (17), yields,

$$w_{n+1}(x,t) = w_n(x,0) - \mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_n - Q_{n-1}) + (G_n - G_{n-1})]], \tag{18}$$

here, $Q_{-1} = 0$, and $G_{-1} = 0$.
Now, let $T[w_n(x,t)]$ is define as

$$T[w_n(x,t)] = -\mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_n - Q_{n-1}) + (G_n - G_{n-1})]], \tag{19}$$

if we assume $\eta_{n+1} = T[\eta_0 + \eta_1 + \eta_2 + \cdots + \eta_n]$, such that

$$\begin{aligned} \eta_0 =& w_0, \\ \eta_1 =& T[\eta_0], \\ \eta_2 =& T[\eta_0 + \eta_1] = T[w_1], \\ & \vdots \\ \eta_{n+1} =& T[\eta_0 + \eta_1 + \eta_2 + \cdots + \eta_n], \end{aligned} \tag{20}$$

then, eq. (18) will be

$$w_{n+1}(x,t) = w_n(x,0) + T[w_n(x,t)], \tag{21}$$

where,

$$\begin{aligned} w_1 =& w_0 + T[w_0] = \eta_0 + \eta_1, \\ w_2 =& w_1 + T[w_1] = \eta_0 + \eta_1 + T[\eta_0 + \eta_1], \\ =& \eta_0 + \eta_1 + \eta_2, \\ & \vdots \\ w_n =& \eta_0 + \eta_1 + \eta_2 + \cdots + \eta_n. \end{aligned} \tag{22}$$

Therefore, we have obtained an approximate solution $w_{n+1}(x,t)$ as a series solution. Hence, $w(x,t)$ is the sum of the iterative solutions, i.e:

$$w(x,t) = \lim_{n\to\infty} w_n(x,t) = \sum_{k=0}^{\infty} \eta_k. \tag{23}$$

3. Numerical Examples

In this section we clearly show the effectiveness of the **K-MI** method to solve the certain PDEs mKdV eq. and FKdV eq. The technique of semi-analytical (numerical with analytic) **K-MI** method is used on the homogeneous and nonhomogeneous mKdV eq. and FKdV eq. This method started by the initial condition $w(x,0)$ as a zero iteration solution $w_0(x,t)$, then, we applied the **KT** and **MI** method on the PDE.

Example 1. Consider the third order homogeneous mKdV eq. [26]

$$w_t(x,t) + 6\ w^2\ w_x(x,t) + w_{3x}(x,t) = 0, \tag{24}$$

using the IC

$$w(x,0) = \frac{2\ \lambda\ \mathrm{e}^{\lambda x}}{1+\mathrm{e}^{2\lambda x}}.$$

Compering eq. (24) to eq. (7), It is clear that

$$\dot{w} = \frac{\partial w(x,t)}{\partial t}, \quad M[w(x,t)] = 6\ w^2\ w_x(x,t) + w_{3x}(x,t), \text{ and } \quad G(x,t) = 0, \tag{25}$$

using the **K-MI** method and following the procedure in section (2.2), we obtain

$$M[w_n(x,t)] = 6\ w_n^2\ w_{\{n,x\}}(x,t) + w_{\{n,3x\}}(x,t) = Q_n(x,t) + O(t^{n+1}), \tag{26}$$

the first iteration $w_1(x,t)$ of the homogeneous mKdV eq. can be found from substituting $w_0(x,t)$ in eq. (22). When $n=0$,

$$\eta_0 = w_0 = \frac{2\ \lambda\ \mathrm{e}^{\lambda x}}{1+\mathrm{e}^{2\lambda x}}, \tag{27}$$

to find Q_0, substitute eq. (27) in eq. (26) yields

$$\begin{aligned} Q_0 =& M[w_0(x,t)] = 6\ w_0^2\ w_{\{0,x\}}(x,t) + w_{\{0,3x\}}(x,t) + O(t^1), \\ =& -2\,\frac{\lambda^4 \mathrm{e}^{\lambda x}\left(-1+\mathrm{e}^{2\lambda x}\right)}{\left(1+\mathrm{e}^{2\lambda x}\right)^2}. \end{aligned} \tag{28}$$

Here, Q_0 in (28) is time independent, so, substitute the value of Q_0 in eq. (19) to find η_1

$$\begin{aligned} \eta_1 =& T[w_0] = -\mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_0 - Q_{-1})]], \quad \text{where } Q_{-1} = 0, \\ =& -\mathcal{K}^{-1}[\upsilon\mathcal{K}[-2\,\frac{\lambda^4 \mathrm{e}^{\lambda x}\left(-1+\mathrm{e}^{2\lambda x}\right)}{\left(1+\mathrm{e}^{2\lambda x}\right)^2}]], \end{aligned}$$

since $-2\,\frac{\lambda^4 \mathrm{e}^{\lambda x}\left(-1+\mathrm{e}^{2\lambda x}\right)}{\left(1+\mathrm{e}^{2\lambda x}\right)^2}$ is time independent, which means it is constant with respect to t, then $\mathcal{K}$ transform of a constant is υ (Theorem 2)

$$\begin{aligned} =& \mathcal{K}^{-1}[\upsilon(2\,\frac{\lambda^4 \mathrm{e}^{\lambda x}\left(-1+\mathrm{e}^{2\lambda x}\right)}{\left(1+\mathrm{e}^{2\lambda x}\right)^2}\ \upsilon)], \quad \text{Theorem 2}, \\ =& \mathcal{K}^{-1}[\upsilon^2(2\,\frac{\lambda^4 \mathrm{e}^{\lambda x}\left(-1+\mathrm{e}^{2\lambda x}\right)}{\left(1+\mathrm{e}^{2\lambda x}\right)^2})], \quad \text{Theorem 3}, \end{aligned}$$

and $\mathcal{K}^{-1}[\upsilon^2]$ is t (Theorem 3), so we have

$$\eta_1 = -2\,\frac{\lambda^4 \mathrm{e}^{\lambda x}\left(1-\mathrm{e}^{2\lambda x}\right)}{\left(1+\mathrm{e}^{2\lambda x}\right)^2}\ t. \tag{29}$$

Then, the first iteration w_1 is

$$\begin{aligned} w_1(x,t) =& \eta_0 + \eta_1 \\ =& \frac{2\ \lambda\ \mathrm{e}^{\lambda x}}{1+\mathrm{e}^{2\lambda x}} - 2\,\frac{\lambda^4 \mathrm{e}^{\lambda x}\left(1-\mathrm{e}^{2\lambda x}\right)}{\left(1+\mathrm{e}^{2\lambda x}\right)^2}\ t. \end{aligned} \tag{30}$$

When $n = 1$, we have substituted w_1 in eq. (26) to find Q_1. In this case, we neglected the coefficients of t with indices grater than or equal 2 ($O(t^2)$). Here, Q_1 is t independent,

$$\begin{aligned} Q_1 =& 6\ w_1^2\ w_{\{1,x\}}(x,t) + w_{\{1,3x\}}(x,t) + O(t^2), \\ =& -\frac{A\lambda^3\left(-2\,\mathrm{e}^{10\,\lambda x} - 17\,\mathrm{e}^{8\,\lambda x} - 28\,\mathrm{e}^{6\,\lambda x} + \mathrm{e}^{12\,\lambda x} - 17\,\mathrm{e}^{4\,\lambda x} - 2\,\mathrm{e}^{2\,\lambda x} + 1\right)t}{B^7} - \\ & \frac{A\left(4\,\mathrm{e}^{10\,\lambda x} + 5\,\mathrm{e}^{8\,\lambda x} + \mathrm{e}^{12\,\lambda x} - 5\,\mathrm{e}^{4\,\lambda x} - 4\,\mathrm{e}^{2\,\lambda x} - 1\right)}{B^7} + \frac{A\left(\mathrm{e}^{2\lambda x} - 1\right)}{B^2}, \end{aligned} \tag{31}$$

where, $A = 2\lambda^4\ e^{\lambda x}$ and $B = (1 + e^{2\lambda x})$, and using equation (19), we have

$$\eta_2 = T[w_1] = -\mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_1 - Q_0)]],$$

$$= \frac{\lambda^7 \left(e^{5\,\lambda x} - 6\, e^{3\,\lambda x} + e^{\lambda x} \right)}{B^3}\ t^2. \tag{32}$$

So, the second iteration w_2 is

$$\begin{aligned} w_2(x,t) =& \eta_0 + \eta_1 + \eta_2 \\ =& \frac{2\ \lambda\ e^{\lambda x}}{B} - \frac{A \left(1 - e^{2\lambda x}\right)}{B^2}\ t + \frac{A\ \lambda^3 \left(e^{4\,\lambda x} - 6\, e^{2\,\lambda x} + 1\right)}{B^3}\ \frac{t^2}{2}. \end{aligned} \tag{33}$$

When $n \to \infty$, the $\mathrm{n^{th}}$ iteration (approximate solution) will be

$$\begin{aligned} w(x,t) &= \lim_{n\to\infty} \sum_{k=0}^{n} \eta_k \\ w(x,t) =& \frac{2\ \lambda\ e^{\lambda x}}{B} - \frac{A \left(1 - e^{2\lambda x}\right)}{B^2}\ t + \frac{A\ \lambda^3 \left(e^{4\,\lambda x} - 6\, e^{2\,\lambda x} + 1\right)}{B^3}\ \frac{t^2}{2} - \\ & \frac{A\ \lambda^5 \left(e^{6\,\lambda x} - 23\, e^{4\,\lambda x} + 23\, e^{2\,\lambda x} - 1\right)}{B^4} \frac{t^3}{3} + \ldots \end{aligned} \tag{34}$$

the series solution (34) of the mKdV eq. is the Taylor's expansion in the variable t of the function

$$w(x,t) = \sqrt{c}\ \mathrm{sech}(\sqrt{c}(x - c\ t)), \quad c > 0, \tag{35}$$

which is exactly the same as the one obtained in [26]. Figure (1) shows the solution deduced from the convergent Taylor's series in eq. (35).

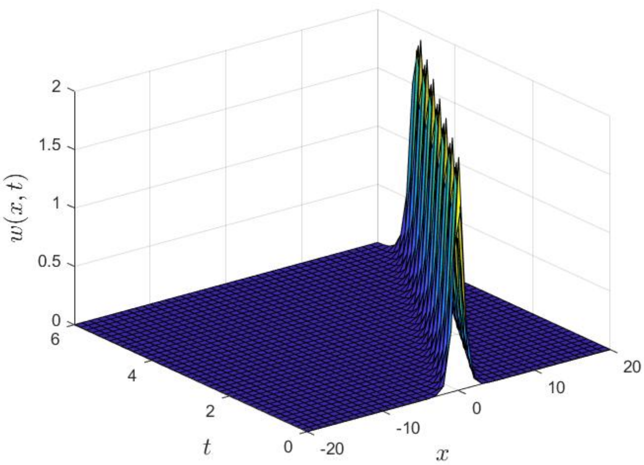

Figure 1: The exact solution of eq. (24) induced from **K-MI** method eq. (35) when $c = 4$. This is soliton type of solution.

Example 2. Consider the third order nonhomogeneous mKdV eq. [20]

$$w_t(x,t) - w^2\ w_x(x,t) + w_{3x}(x,t) = x\ - 3\,t^2x - \left(x\ t - t^3x\right)^2\left(t\ - t^3\right), \tag{36}$$

using the IC

$$w(x,0) = 0.$$

It is clear that, when we compare eq. (36) to eq. (7), we have

$$\dot{w} = \frac{\partial w(x,t)}{\partial t}, \quad M[w(x,t)] = w_{3x}(x,t) - w^2\ w_x(x,t), \tag{37}$$

$$\text{and} \quad G(x,t) = -3\,t^2x + x - \left(-t^3x + tx\right)^2\left(-t^3 + t\right). \tag{38}$$

Similarly as in example (1), we have used **K-MI** method and following the procedure in section (2.2), we have

$$M[w_n(x,t)] = w_{\{n,3x\}}(x,t) - w^2_{\{n,x\}}(x,t) = Q_n(x,t) + O(t^{n+1}), \tag{39}$$

$$G_n(x,t) = -t^9x^2 + 3\,t^7x^2 - 3\,t^5x^2 + t^3x^2 + 3\,t^2x - x + O(t^{n+1}), \tag{40}$$

here, the function G_n is a polynomial of degree (9) in the variable t.

When $n = 0$, then $\eta_0 = w_0 = 0$, using (39) and (40) to find

$$Q_0(x,t) = 0, \quad G_0(x,t) = -x + \underbrace{-t^9x^2 + 3\,t^7x^2 - 3\,t^5x^2 + t^3x^2 + 3\,t^2x}_{neglect}, \tag{41}$$

Hence, substituting Q_0 and G_0 in eq (19), we have

$$\begin{aligned}\eta_1 =& T[\eta_0 = w_0] \\ =& -\mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_0 - Q_{-1}) + (G_0 - G_{-1})]], \quad G_{-1} = 0 \\ =& \ xt,\end{aligned} \tag{42}$$

therefore, the first iteration w_1 is

$$w_1(x,t) = \eta_0 + \eta_1 = xt. \tag{43}$$

When $n = 1$, then, $Q_1(x,t) = 0$, and $G_1(x,t) = -x + \underbrace{xt^3}_{neglect}$, so,

$$\eta_2 = T[w_1] = -\mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_1 - Q_0) + (G_1 - G_0)]] = 0, \tag{44}$$

therefore, the second iteration w_2 is

$$w_2(x,t) = \eta_0 + \eta_1 + \eta_2 = x\ t. \tag{45}$$

When $n = 2$

$$Q_2(x,t) = \underbrace{-x^2t^3}_{neglect}, \quad G_2(x,t) = -x + 3\ x\ t^2 + \underbrace{-t^9x^2 + 3\,t^7x^2 - 3\,t^5x^2 + t^3x^2}_{neglect}. \tag{46}$$

So, $Q_2 = 0$ and $G_2 = 3\ x\ t^2 - x$

$$\eta_3 = T[w_2] = -\mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_2 - Q_1) + (G_2 - G_1)]] = -x\ t^3, \tag{47}$$

therefore, the third iteration w_3 is

$$w_3(x,t) = \eta_0 + \eta_1 + \eta_2 + \eta_3 = x\ t - x\ t^3. \tag{48}$$

Finally, when $n \to \infty$, the series solution (48) of the nonhomogeneous mKdV equation (36) has the sum:

$$\begin{aligned} w(x,t) &= \lim_{n\to\infty} \sum_{k=0}^{n} \eta_k \\ &= 0 + xt + 0 - x\ t^3 - \frac{1}{4}\ x\ t^4(x-1) + \frac{1}{4}\ x\ t^4(x-1) + \cdots = x\ t - x\ t^3. \end{aligned} \tag{49}$$

which is the exact solution. The $w(x,t)$ in eq. (49) is a polynomial of degree (3) function. Equation (36) was solved in [20] and Figure (2) display the exact solution which concluded from the Taylor's series in eq. (49).

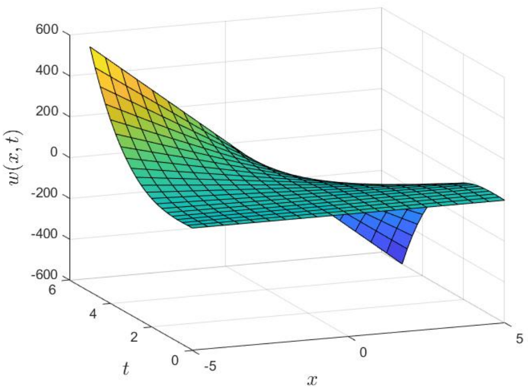

Figure 2: The exact solution of the nonhomogeneous mKdV equation (36) is a polynomial type.

In the next example, we apply the **K-MI** method on the nonlinear homogeneous and nonhomogeneous FKdV eq.

Example 3. Consider the nonlinear homogeneous FKdV eq.

$$w_t(x,t) + w^2\ w_{xx}(x,t) + w_x w_{3x}(x,t) - 20w^2\ w_{3x}(x,t) + w_{5x}(x,t) = 0, \tag{50}$$

with the IC

$$w(x,0) = \frac{1}{x}, \quad x \neq 0.$$

That is clear from equation (50), we have

$$\dot{w} = \frac{\partial w(x,t)}{\partial t},\ M[w(x,t)] = w^2\ w_{xx}(x,t) + w_x w_{3x}(x,t) - 20w^2\ w_{3x}(x,t) + w_{5x}(x,t),\ \text{and } G(x,t) = 0, \tag{51}$$

following the same structures as above examples, the Q_n is

$$\begin{aligned} M[w_n(x,t)] =& w_n^2\ w_{\{n,xx\}}(x,t) + w_{\{n,x\}} w_{\{n,3x\}}(x,t) - 20w_n^2 w_{\{n,3x\}}(x,t) + \\ & w_{\{n,5x\}}(x,t) = Q_n(x,t) + O(t^{n+1}), \end{aligned} \tag{52}$$

when $n = 0$,

$$\eta_0 = w_0 = \frac{1}{x}, \qquad Q_0(x,t) = \frac{-1}{x^2}, \tag{53}$$

then,

$$\eta_1 = T[w_0] - \mathcal{K}^{-1}[\upsilon \mathcal{K}[(Q_0 - Q_{-1})]] = \frac{t}{x^2} \tag{54}$$

therefore, the first iteration w_1 is

$$w_1(x,t) = \eta_0 + \eta_1 = \frac{1}{x} + \frac{t}{x^2}. \tag{55}$$

When $n = 1$,

$$Q_1(x,t) = \underbrace{\frac{(6\,x + 480)\,t^3}{x^9} + \frac{\left(2\,x^2 + 1080\,x\right)t^2}{x^9}}_{neglect} - 2\,\frac{t}{x^3} - \frac{1}{x^2}, \tag{56}$$

so, $Q_1 = -2\,\frac{t}{x^3} - \frac{1}{x^2}$, then

$$\eta_2 = T[w_1] = -\mathcal{K}^{-1}[\upsilon \mathcal{K}[(Q_1 - Q_0)]] = \frac{t^2}{x^3}, \tag{57}$$

therefore, the second iteration w_2 is

$$w_2(x,t) = \eta_0 + \eta_1 + \eta_2 = \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3}. \tag{58}$$

When $n = 2$

$$Q_2(x,t) = -\frac{1}{x^2} - 2\,\frac{t}{x^3} - 3\,\frac{t^2}{x^4}, \tag{59}$$

so,

$$\eta_3 = T[w_2] = -\mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_2 - Q_1)]] = \frac{t^3}{x^4}. \tag{60}$$

Then, the third iteration w_3 is

$$w_3(x,t) = \eta_0 + \eta_1 + \eta_2 + \eta_3 = \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + \frac{t^3}{x^4}. \tag{61}$$

Finally, when $n \to \infty$, the series solution (61) of the homogeneous FKdV eq. (50)

$$\begin{aligned} w(x,t) &= \lim_{n\to\infty} \sum_{k=0}^{n} \eta_k \\ &= \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + \frac{t^3}{x^4} + \dots, \end{aligned} \tag{62}$$

when we take $\frac{1}{x}$ as a common factor from eq. (62), we have

$$w(x,t) = \frac{1}{x}\left(1 + \frac{t}{x} + \frac{t}{x^2} + \dots\right), \tag{63}$$

the second part of eq.(63) is the Taylor's expansion of,

$$w(x,t) = \frac{1}{x(1-\frac{t}{x})},$$

then, the Taylor's expansion of the exact solution is

$$w(x,t) = \frac{1}{x-t}. \tag{64}$$

Equation (50) was solved in [14, 20], we have reached the same result with the **K-IM** method. According to Taylor's series, Figure (3) display the exact solution of the homogeneous FKdV eq. (50). The function in eq. (64) has singularities at $(t = x)$.

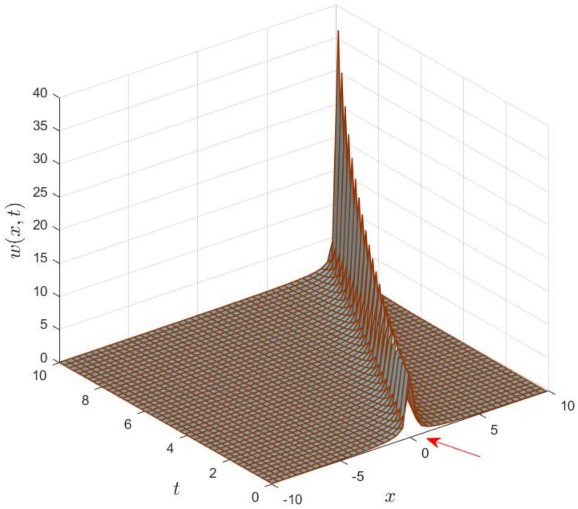

Figure 3: The singular solution of the homogeneous FKdV eq. (50). The solution is growing up rapidly on time. In addition, the solution when $|x| \to \infty$ is decay to zero.

Example 4. Consider the nonlinear nonhomogeneous FKdV eq.

$$w_t(x,t) + w\ w_x(x,t) + w\ w_{3x}(x,t) + w_{5x}(x,t) = 2\ \cos(x+t), \tag{65}$$

with the IC

$$w(x,0) = \sin(x).$$

Compering (65) to eq. (7), this in turn gives

$$\dot{w} = \frac{\partial w(x,t)}{\partial t},\ M[w(x,t)] = w\ w_x(x,t) + w\ w_{3x}(x,t) + w_{5x}(x,t),\ \text{and}\ G(x,t) = 2\cos(x+t). \tag{66}$$

Similarly, as above examples, we have

$$M[w_n(x,t)] = w_n\ w_{\{n,x\}}(x,t) + w_n w_{\{n,3x\}}(x,t) + w_{\{n,5x\}}(x,t) = Q_n(x,t) + O(t^{n+1}), \tag{67}$$

and

$$G(x,t) = 2\cos(x+t), \tag{68}$$

when $n = 0$,

$$\eta_0 = w_0 = \sin(x), \qquad Q_0(x,t) = \cos(x). \tag{69}$$

Using the Taylor's expansion in the function $G_0(x,t)$, for $\mathrm{e}^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \dots$, then,

$$G(x,t) = \cos(x) - \sin(x)\,t - \frac{\cos(x)}{2}t^2 + \frac{\sin(x)}{6}t^3 + \frac{\cos(x)}{24}t^4 - \frac{\sin(x)}{120}t^5 + \dots,$$

$$G_0(x,t) = \cos(x). \tag{70}$$

Substituting Q_0 and G_0 in eq (19), we have

$$\eta_1 = T[w_0] = -\mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_0 - Q_{-1}) + (G_0 - G_{-1})]] = \cos(x)t, \tag{71}$$

therefore, the first iteration w_1 is,

$$w_1(x,t) = \eta_0 + \eta_1 = \sin(x) + \cos(x)\ t. \tag{72}$$

When $n = 1$,

$$Q_1(x,t) = \cos(x) - \sin(x)\ t, \tag{73}$$

and $G_1 = -2\cos(x) + 2\sin(x)\,t$ after cancelling the neglected terms. So,

$$\eta_2 = T[w_1] = -\mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_1 - Q_0) + (G_1 - G_0)]] = -\frac{1}{2}\sin(x)\,t^2, \tag{74}$$

therefore, the second iteration w_2 is,

$$w_2(x,t) = \eta_0 + \eta_1 + \eta_2 = \sin(x) + \cos(x)\,t - \frac{1}{2}\sin(x)\,t^2. \tag{75}$$

When $n = 2$

$$Q_2(x,t) = \cos(x) - \sin(x)\,t - 1/2\cos(x)\,t^2, \tag{76}$$

and

$$G_2(x,t) = -2\cos(x) + 2\sin(x)\,t + \cos(x)\,t^2, \tag{77}$$

so,

$$\eta_3 = T[w_2] = -\mathcal{K}^{-1}[\upsilon\mathcal{K}[(Q_2 - Q_1) + (G_2 - G_1)]] = -\frac{1}{6}\cos(x)\,t^3, \tag{78}$$

then, the third iteration w_3 is,

$$w_3(x,t) = \sin(x) + \cos(x)\,t - \frac{1}{2}\sin(x)\,t^2 - \frac{1}{6}\cos(x)\,t^3. \tag{79}$$

Finally, when $n \to \infty$, the series solution (79) of the homogeneous FKdV equation (65)

$$\begin{aligned}
w(x,t) &= \lim_{n\to\infty}\sum_{k=0}^{n}\eta_k \\
&= \eta_0 + \eta_1 + \eta_2 = \sin(x) + \cos(x)\,t - \frac{1}{2}\sin(x)\,t^2 - \frac{1}{6}\cos(x)\,t^3 + \dots, \\
&= (\sin(x) - \frac{1}{2}\sin(x)\,t^2 + \frac{1}{24}\sin(x)\,t^4 + \dots) + \\
&\quad (\cos(x)\,t - \frac{1}{6}\cos(x)\,t^3 + \frac{1}{120}\cos(x)\,t^5 + \dots)
\end{aligned} \tag{80}$$

the first part of eq. (80) represents the Taylor's expansion of $\sin(x)\ \cos(t)$ where $\sin(x)$ is a common factor, and the second part represents the Taylor's expansion of $\cos(x)\ \sin(t)$, then the Taylor's expansion of the exact solution

$$\begin{aligned} w(x,t) &= \sin(x)\ \cos(t) + \cos(x)\ \sin(t) \\ &= \sin(x+t), \end{aligned} \tag{81}$$

which is the periodic exact solution. Figure (4) shows the exact solution deduce from the convergent of Taylor's series in eq. (81).

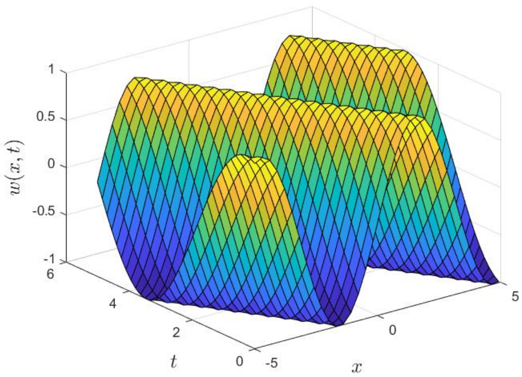

Figure 4: The periodic solution of the nonhomogeneous FKdV eq. (65).

4. The effects of noise terms on Kamal transform with classical iteration method

Here, the **K-CI** method is applied on the homogenous FKdV eq. and the result does not converge well. In previous examples, the homogenous FKdV eq. (50) was solved by the **K-MI** method. So, we present the following example to prove that the **K-CI** method is not a good choice to solve the KdV eq. types.

Example 5. Consider the homogeneous FKdV eq.

$$w_t(x,t) + w^2\ w_{xx}(x,t) + w_x w_{3x}(x,t) - 20w^2\ w_{3x}(x,t) + w_{5x}(x,t) = 0, \tag{82}$$

with given IC

$$w(x,0) = \frac{1}{x},$$

taking **KT** and it's inverse on eq. (82), lead to

$$w(x,t) = \frac{1}{x} - \mathcal{K}^{-1}[\upsilon\mathcal{K}[w^2\ w_{xx}(x,t) + w_x w_{3x}(x,t) - 20w^2\ w_{3x}(x,t) + w_{5x}(x,t)]], \tag{83}$$

next, applying the classical iteration method on eq. (83)

$$w_{n+1}(x,t) = \frac{1}{x} - \mathcal{K}^{-1}[\upsilon\mathcal{K}[w_n^2\ w_{\{n,xx\}}(x,t) + w_{\{n,x\}}w_{\{n,3x\}}(x,t) - 20w_n^2\ w_{\{n,3x\}}(x,t) + w_{\{n,5x\}}(x,t)]], \tag{84}$$

where, $w_0(x,t) = \frac{1}{x}$. Then when $n = 0$, in eq. (84) the first iteration will be

$$w_1(x,t) = \frac{1}{x} - \mathcal{K}^{-1}[\upsilon\mathcal{K}[w_0^2\ w_{\{0,xx\}}(x,t) + w_{0,x\}}w_{\{0,3x\}}(x,t) - 20w_0^2\ w_{\{0,3x\}}(x,t) + w_{\{0,5x\}}(x,t)]],$$

$$w_1(x,t) = \frac{1}{x} + \frac{t}{x^2}, \tag{85}$$

and when $n = 1$, yields

$$w_2(x,t) = x^{-1} + \frac{t}{x^2} + \frac{t^2}{x^3} - \frac{3}{2}\frac{t^4}{x^8} - \frac{2}{3}\frac{t^3}{x^7} - 120\frac{t^4}{x^9} - 360\frac{t^3}{x^8}. \tag{86}$$

Figure (5) display the sketch of $w_2(x,t)$ by eq. (86) when $n = 1$, we can see clearly it

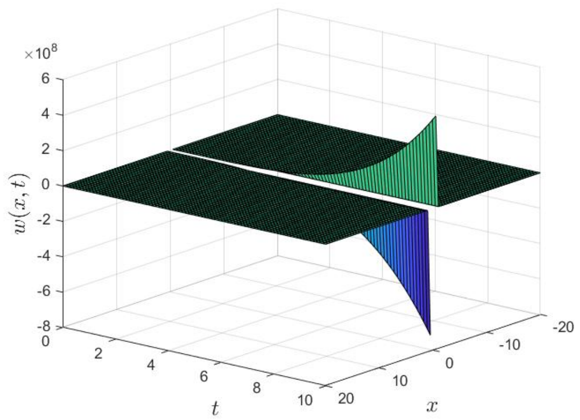

Figure 5: the plot of $w_2(x,t)$ given by (86) when $n = 1$.

is far from our expectation, so we need to find the next iteration to see whether the next iteration will be better. When $n = 2$, the third iteration is

$$w_3(x,t) = 1621620\,t^3x^{26} + 1621620\,t^2x^{27} + 1621620\,tx^{28} + 1621620\,x^{29} - 2779920\,t^7x^{19} - 8108100\,t^6x^{20} - 8432424\,t^5x^{21} - 3513510\,t^4x^{22} - 277992000\,t^7x^{18} - 778377600\,t^6x^{19} - - 1868106240\,t^5x^{20} - 2821618800\,t^4x^{21} + 25945920\,t^9x^{14} + 65540475\,t^8x^{15} + 3397680\,t^7x^{16} - - 20900880\,t^6x^{17} - 11459448\,t^5x^{18} - 2702700\,t^4x^{19} + 7005398400\,t^9x^{13} + 23724300600\,t^8x^{14} +$$

$$33016183200\,t^7x^{15} + 11578366800\,t^6x^{16} - 38776177440\,t^5x^{17} - 13945932000\,t^4x^{18} + 479999520000\,t^9x^{12} + 2270916648000\,t^8x^{13} + 4433416416000\,t^7x^{14} + 5876750880000\,t^6x^{15} - 1727998272000\,t^5x^{16} - 11734042320000\,t^4x^{17} - 51744420\,t^{11}x^9 - 100135035\,t^{10}x^{10} + 116105990\,t^9x^{11} + 129639510\,t^8x^{12} + 28622880\,t^7x^{13} - 19185238800\,t^{11}x^8 - \dots, \tag{87}$$

Figure (6) shows the sketch of $w_3(x,t)$ when $n = 2$. In addition, as $n \to \infty$, the w_{n+1} has

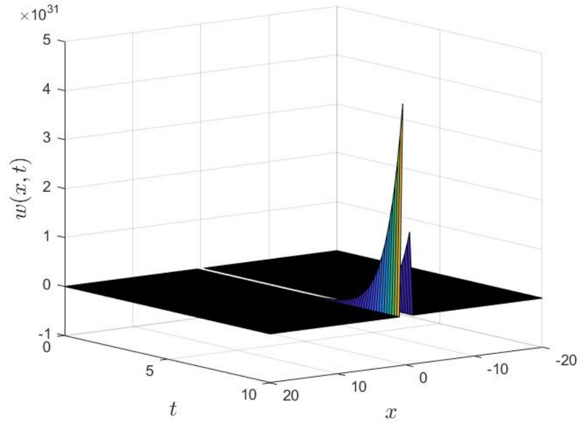

Figure 6: The Figure shows the un stability of the approximate solution due to the higher order noise terms in the iteration in eq. (87) when $n = 2$.

unnecessary (noise) terms which makes the approximate solution looks divergent from the exact solution. This case will happen when we try to use the **CI** method instead of the **MI** method for the above examples (1,2) and (4).

5. Discussion

Kamal transform has been used recently in a combination with another method to solve nonlinear differential equations [18, 22]. In this work, we use **KT** in combination with the **IT** method to solve the KdV eq. Our analysis shows that using the **KT** and the classical iteration method can lead to divergent series of approximate solutions due to the high order terms induced during the iterations causing high noise and error in the calculation as shown in section (4). However, modifying the iteration method by removing the noise terms and combining them with the **KT** leads to better approximation results since the series of the solutions converges fast to the accurate solution as shown in the examples in section (3). The semi-analytical **K-MI** method is used to solve the homogeneous and nonhomogeneous mKdV eq. and FKdV eq. This method begins with the initial condition $w(x,0)$ as the first approximate solution $w_0(x,t)$, then, we apply the **KT** and **MI** method to construct the other iterations. Each iteration is an approximate solution of the PDE. The approximate solution is a polynomial in the variable t of a degree equal to the order of the iteration, hence, it is not bounded as $|t| \to \infty$. However, in many cases, one can deduce an infinite series by taking the limit of the iteration. When the deduced series is comparable to the Taylor expansion of a known function, then the exact solution is inferred from our semi-analytic method, as we have shown in our examples. In example (1), the exact solution of the homogeneous KdV equation is shown in Figure (1). The solution is called soliton which is a traveling wave that maintains its amplitude for a long distance while vanishing as $|x| \to \infty$. In example (2), the exact solution of the nonhomogeneous mKdV equation is an odd polynomial that is not bounded as shown in Figure (2). In example (3) the solution of the homogeneous FKdV equation has a discontinuity at $x = t$ while decaying to zero as $|x| \to \infty$, hence, the equation in example (3) admits a singular solution. The solution of the nonhomogenous FKdV eq. in example (4) is a periodic sine wave as shown in Figure (4). Figure (5), illustrates the approximate solution of equation (87) obtained after 2 iterations ($n = 0, 1$) using the **CI** method. Both approximate solutions w_1 and w_2 are polynomials in the variable t, therefore, they are unbounded along time progress. However, the error of the approximate solution of the classical iterations method is higher than the error of the modified iterations method. The error induced by the polynomial terms of degree higher than the iteration order $n = 3$ which cause high noise in the calculation, see Figure(6). Removing the noise terms in the modified iterations method leads to better convergence towards the exact solution, see Figure(3).

6. Conclusion

In this work, the semi-analytical Kamal transform combined to the modified iteration method (**K-MI** method) has been used to extract approximate solutions for various examples of the non-homogeneous KdV equation. In addition, by considering the limit series of the approximate solutions as Taylors' expansions of known functions, we inferred the exact solutions of the considered equations. The exact solutions of the considered equations varied from polynomial, periodic and even discontinuous singular solution. Moreover, we showed the outweigh of the modified iterations method over the classical iterations method since the latter method can suffer from slow convergence of the approximations to the exact solution. Due to the generated noise terms, those are eliminated in the **MI** method.

Acknowledgements

This work is supported by the "College of Education for Pure Sciences and the College of Basic Education in the University of Mosul, Iraq".

References

[1] S Abbasbandy. Homotopy analysis method for the kawahara equation. *Nonlinear Analysis: Real World Applications*, 11(1):307–312, 2010.

[2] K Abdelilah and Sedeeg Hassan. The use of kamal transform for solving partial differential equations. *Advances in theoretical and applied mathematics*, 12(1):7–13, 2017.

[3] Mark J Ablowitz, Xu-Dan Luo, and Ziad H Musslimani. Discrete nonlocal nonlinear schrödinger systems: Integrability, inverse scattering and solitons. *Nonlinearity*, 33(7):3653, 2020.

[4] Adebayo O Adewumi, Saheed O Akindeinde, Adebayo A Aderogba, and Babatunde S Ogundare. Laplace transform collocation method for solving hyperbolic telegraph equation. *International Journal of Engineering Mathematics*, 2017:1–9, 2017.

[5] Sudhanshu Aggarwal, Nidhi Sharma, and Raman Chauhan. Application of kamal transform for solving linear volterra integral equations of first kind. *International Journal of Research in Advent Technology*, 6(8):2081–2088, 2018.

[6] Artan F Alidema. Applications of double fuzzy sumudu adomain decompositon method for two-dimensional volterra fuzzy integral equations. *European Journal of Pure and Applied Mathematics*, 15(3):1363–1375, 2022.

[7] Seval Catal. Response of forced euler-bernoulli beams using differential transform method. *Structural engineering and mechanics: An international journal*, 42(1):95–119, 2012.

[8] DG Crighton. Applications of kdv. In *KdV'95*, pages 39–67. Springer, 1995.

[9] Philip G Drazin and Robin Stanley Johnson. *Solitons: an introduction*, volume 2. Cambridge university press, 1989.

[10] MA El-Tawi and HN Hassan. A new application of using homotopy analysis method for solving stochastic quadratic nonlinear diffusion equation. *Int. J. of Appl Math and Mech*, 9(16):35–55, 2013.

[11] Clifford S Gardner, John M Greene, Martin D Kruskal, and Robert M Miura. Method for solving the korteweg-devries equation. *Physical review letters*, 19(19):1095, 1967.

[12] Georgi G Grahovski, Amal J Mohammed, and Hadi Susanto. Nonlocal reductions of the ablowitz–ladik equation. *Theoretical and Mathematical Physics*, 197(1):1412–1429, 2018.

[13] Georgi G Grahovski, Junaid I Mustafa, and Hadi Susanto. Nonlocal reductions of the multicomponent nonlinear schrödinger equation on symmetric spaces. *Theoretical and Mathematical Physics*, 197(1):1430–1450, 2018.

[14] Sujit Handibag and BD Karande. Existence the solutions of some fifth-order kdv equation by laplace decomposition method. 2013.

[15] Olufemi Elijah Ige, Razak Adekola Oderinu, and Tarig Mohyeldin Elzaki. Adomian polynomial and elzaki transform method of solving fifth order korteweg-de vries equation. *Caspian Journal of Mathematical Sciences (CJMS) peer*, 8(2):103–119, 2019.

[16] Subrat Kumar Jena and S Chakraverty. Differential quadrature and differential transformation methods in buckling analysis of nanobeams. *Curved and Layered Structures*, 6(1):68–76, 2019.

[17] Abdelilah Kamal and H Sedeeg. The new integral transform kamal transform. *Advances in Theoretical and Applied Mathematics*, 11(4):451–458, 2016.

[18] Rachana Khandelwal, Padama Kumawat, and Yogesh Khandelwal. Kamal decomposition method and its application in solving coupled system of nonlinear pde's. *Malaya J. Mat.*, 6(3):619–625, 2018.

[19] Amal Jasim Mohammed and Ahmed Farooq Qasim. A new procedure with iteration methods to solve a nonlinear two dimensional bogoyavlensky-konopelchenko equation. *Journal of Interdisciplinary Mathematics*, 25(2):537–552, 2022.

[20] Shams Eldeen Ahmed Abdelmajed Mohmmed et al. *Solution of linear and nonlinear partial differential equations by mixing Adomian decomposition method and Sumudu transform.* PhD thesis, Sudan University of Science and Technology, 2016.

[21] Maheshwar Pathak and Pratibha Joshi. Modified iteration method for numerical solution of nonlinear differential equations arising in science and engineering. *Asian-European Journal of Mathematics*, 14(09):2150151, 2021.

[22] A Emimal Kanaga Pushpam and C Dhinesh Kumar. Kamal decomposition method for solving nonlinear delay differential equations. *Bull. Pure Appl. Sci.-Math. Stat. 38e*, 231, 2019.

[23] Dimple Rani and Vinod Mishra. Modification of laplace adomian decomposition method for solving nonlinear volterra integral and integro-differential equations based on newton raphson formula. *European Journal of Pure and Applied Mathematics*, 11(1):202–214, 2018.

[24] Metomou Richard and Weidong Zhao. Padé-sumudu-adomian decomposition method for nonlinear schrödinger equation. *Journal of Applied Mathematics*, 2021, 2021.

[25] Subashini Vilu, Rokiah Rozita Ahmad, and Ummul Khair Salma Din. Variational iteration method and sumudu transform for solving delay differential equation. *International Journal of Differential Equations*, 2019, 2019.

[26] Abdul-Majid Wazwaz. *Partial differential equations and solitary waves theory.* Springer Science & Business Media, 2010.

[27] Zhengping Yang and Wei-Ping Zhong. Analytical solutions to sine-gordon equation with variable coefficient. *Romanian Reports in Physics*, 66(2):262–273, 2014.

[28] Norman J Zabusky and Martin D Kruskal. Interaction of" solitons" in a collisionless plasma and the recurrence of initial states. *Physical review letters*, 15(6):240, 1965.