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Transversal Hop Domination in Graphs

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Abstract. Let G be a graph. A set $S \subseteq V(G)$ is a hop dominating set of G if for every $v \in V(G)\setminus S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The minimum cardinality $\gamma_h(G)$ of a hop dominating set is the hop domination number of G. Any hop dominating set of G of cardinality $\gamma_h(G)$ is a γ_h-set of G. A hop dominating set S of G which intersects every γ_h-set of G is a transversal hop dominating set. The minimum cardinality $\widehat{\gamma}_h(G)$ of a transversal hop dominating set in G is the transversal hop domination number of G. In this paper, we initiate the study of transversal hop domination. First, we characterize graphs G whose values for $\widehat{\gamma}_h(G)$ are either n or $n-1$, and we determine the specific values of $\widehat{\gamma}_h(G)$ for some specific graphs. Next, we show that for every positive integers a and b with $a \geq 2$ and $b \geq 3a$, there exists a connected graph G on b vertices such that $\widehat{\gamma}_h(G) = a$. We also show that for every positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G for which $\gamma_h(G) = a$ and $\widehat{\gamma}_h(G) = b$. Finally, we investigate the transversal hop dominating sets in the join and corona of two graphs, and determine their corresponding transversal hop domination numbers.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Hop dominating set, transversal hop dominating set, transversal hop domination number

1. Introduction

The concept of domination in graphs was first introduced by Ore [16] in 1958 and C. Berge [2] in 1962. Thereafter, domination as well as its numerous variations have become among the most extensively studied research areas in graph theory.

Given a family $\mathscr{C}$ of sets, a transversal of $\mathscr{C}$ is a set containing at least one element from each member of $\mathscr{C}$. Transversals in graphs have received high attention since the last 30 years. In 1991, T. Andreae et al. [21] studied the clique-transversal sets of line graphs.

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In 1996, the vertex transversals that dominate is introduced in [5]. The independent transversal domination is being investigated in [6, 20, 22]. Recently, A. Alwardi et al.[18, 19] investigated the transversal domination in graphs. In this paper, we introduce and initiate the study of transversal hop domination.

All graphs considered here are finite, simple and undirected. For basic graph terminologies, we refer the readers to [3]. For a graph $G = (V(G), E(G))$, $V(G)$ and $E(G)$ are its *vertex set* and *edge set*, respectively. For $S \subseteq V(G)$, $|S|$ refers to the cardinality of S. In particular, $|V(G)|$ is the *order* of G.

Given two graphs G and H with disjoint vertex sets, the *union* of G and H is the graph $G\cup H$ whose vertex set is $V(G\cup H) = V(G)\cup V(H)$ and edge set $E(G\cup H) = E(G)\cup E(H)$. The *join* of G and H is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. In $G \circ H$, we denote by H^v that copy of H which is being joined to the vertex v of G. We also denote by $H^v + v$ that subgraph $\langle\{v\} \cup V(H^v)\rangle$ of $G \circ H$ induced by $\{v\} \cup V(H^v)$.

For a vertex v of G, the *open neighborhood* of v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, while the *closed neighborhood* of v in G is the set $N_G[v] = N_G(v) \cup \{v\}$. Any vertex $u \in N_G(v)$ is called a *neighbor* of v. The *degree* of a vertex v of G, denoted by $deg_G(v)$, is the number $|N_G(v)|$ of neighbors of v. The *distance* between two vertices $u, v \in V(G)$ is the number of edges in a shortest path that joins vertex u to vertex v, and is denoted by $d_G(u, v)$. Such shortest u-v path is called *u-v geodesic*. We define $diam(G) = \max\{d_G(u, v) : u, v \in V(G)\}$. Any geodesic of length equal to $diam(G)$ is called a *diametral path*.

For $S \subseteq V(G)$, $N_G(S) = \cup_{v\in S}N_G(v)$ and $N_G[S] = N_G(S) \cup S$. If $N_G[S] = V(G)$ (resp. $N_G(S) = V(G)$), then S is a *dominating set* (resp. *total dominating set*) of G. For total dominating sets, G necessarily has no isolated vertex. The smallest cardinality of a dominating (resp. total dominating) set S of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$) is called the *domination number* (resp. *total domination number*). A dominating (resp. total dominating) set S of G with $|S| = \gamma(G)$ (resp. $|S| = \gamma_t(G)$) is called a γ-set (resp. γ_t-set) of G. The reader is referred to the following references, namely [4, 7–12, 14], for the history and a bit of the succeeding developments of the theory of domination in graphs.

For two vertices u and v of G, v is a *hop neighbor* of vertex u if $d_G(u, v) = 2$. The set $N_G(u, 2) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G refers to $N_G[u, 2] = N_G(u, 2)\cup\{u\}$. For $S \subseteq V(G)$, the *open hop neighborhood* and *closed hop neighborhood* of S refer to the sets $N_G(S, 2) = \cup_{u\in S}N_G(u, 2)$ and $N_G[S, 2] = N_G(S, 2) \cup S$, respectively. In case $N_G[S, 2] = V(G)$, then S is a *hop dominating set* (or *HD-set*) of G. Provided G has no isolated vertex, S is a *total hop dominating set* (or *tHD-set*) of G if $N_G(S, 2) = V(G)$. The minimum cardinality of a HD-set (resp. tHD-set) of G, denoted by $\gamma_h(G)$ (resp. $\gamma_{th}(G)$), is called the *hop domination number* (resp. *total hop domination number*) of G. Any HD-set (resp. tHD-set) with cardinality $\gamma_h(G)$ (resp. $\gamma_{th}(G)$) is called a *γ_h-set* (resp. *γ_{th}-set*). References [15] and [17] are excellent references for hop domination and total hop domination, respectively.

A set $S \subseteq V(G)$ is a $(1,2)^*$-*dominating set of* G (resp. $(1,2)^*$-*total dominating set*) if it is both a dominating (resp. a total dominating) set and a hop dominating set of G. The smallest cardinality of a $(1,2)^*$ -dominating (resp. $(1,2)^*$-total dominating) set of G, denoted by $\gamma^*_{1,2}(G)$ (resp.$\gamma^{*t}_{1,2}(G)$) is called the $(1,2)^*$ -*domination number* (resp. $(1,2)^*$-*total domination number*) of G. A $(1,2)^*$ -dominating (resp. $(1,2)^*$ -total dominating) set S with $|S| = \gamma^*_{1,2}(G)$ (resp. $|S| = \gamma^{*t}_{1,2}(G)$) is called a $\gamma^*_{1,2}$-set (resp. $\gamma^{*t}_{1,2}$-set) of G. The concept of $(1,2)^*$-domination (a variation of $(1,2)$-domination) is introduced in [1].

A subset S of $V(G)$ is a *point-wise non-dominating set* (or *PND-set*) of G if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $v \notin N_G(u)$. The smallest cardinality of a PND-set of G, denoted $pnd(G)$, is called the *point-wise non-domination number* of G. Any point-wise non-dominating set S of G with $|S| = pnd(G)$ is called a *pnd*-set of G. The concept of point-wise non-domination was introduced in [1].

A hop dominating set S of G which intersects every γ_h-set of G is called a *transversal hop dominating set* or (*THD-set*). In other words, a THD-set is a hop dominating set of G which is represented by every γ_h-set of G. The minimum cardinality of a THD-set of G is called the *transversal hop domination number of* G and is denoted by $\widehat{\gamma}_h(G)$. Any THD-set S of G with $|S| = \widehat{\gamma}_h(G)$ is called a $\widehat{\gamma}_h$-set.

2. Preliminary Results

Clearly, $\gamma_h(G) \leq \widehat{\gamma}_h(G) \leq n$ for all connected graphs G of order n. In particular, $\widehat{\gamma}_h(G) = 1$ if and only if $G = K_1$.

Proposition 1. *Let G be a connected graph of order n. Then*

(*i*) *$\widehat{\gamma}_h(G) = n$ if and only if G is a complete graph.*

(*ii*) *For $n \geq 3$, $\widehat{\gamma}_h(G) = n-1$ if and only if G is one of the graphs P_4, C_4 and $\overline{K_2}+K_{n-2}$.*

Proof. It is clear that if $G = K_n$, then $\gamma_h(G) = \widehat{\gamma}_h(G) = n$. Conversely, suppose $\widehat{\gamma}_h(G) = n$. The conclusion is clear if $n = 1, 2$. Assume $n \geq 3$. Suppose G is not complete. Since G is connected, there exist $u, v \in V(G)$ such that $d_G(u, v) = 2$. Let $S = V(G) \setminus \{v\}$, and let T be a $\widehat{\gamma}_h$-set of G. Clearly, S is an HD-set of G. If $v \notin T$, then $T \subseteq S$. Suppose that $v \in T$. Since $\{v\}$ is not a hop dominating set of G, there exists $w \in T$ with $w \neq v$. Thus, $w \in S \cap T$. Since T is arbitrary, S is a THD-set of G. Consequently, $\widehat{\gamma}_h(G) \leq |S| = n - 1$, a contradiction. Thus, G is a complete graph.

If G is one of the graphs P_4, C_4 and $\overline{K_2} + K_{n-2}$ where $n \geq 3$, then $\widehat{\gamma}_h(G) = n - 1$. Conversely, assume that $\widehat{\gamma}_h(G) = n - 1$. Suppose $diam(G) \geq 4$. Then G has order $n \geq 5$. Let $u, v \in V(G)$ such that $d_G(u, v) = 4$. Then $\{u, v\}$ is not a γ_h-set of G. It follows that $S = V(G) \setminus \{u, v\}$ is a THD-set of G. Thus, $\widehat{\gamma}_h(G) \leq |S| = n - 2$, a contradiction. Hence, $diam(G) \leq 3$. Consider the following cases:

Case 1: Suppose that $diam(G) = 3$. Then G has order $n \geq 4$. Let $P = [u, w, x, v]$ be a diametral path in G. Let $y \in V(G) \setminus V(P)$ that is adjacent to any of the vertices in P. If $wy \in E(G)$, then $S = V(G) \setminus \{u, y\}$ is a HD-set of G. Since $\{u, y\}$ is not a HD-set of G, S is a THD-set of G. Thus, $\widehat{\gamma}_h(G) \leq |S| = n-2$, a contradiction. Similar contradiction is attained if $xy \in E(G)$. Suppose that $uy \in E(G)$. Necessarily, $2 \leq d_G(y,v) \leq 3$. Since $\{w,y\}$ is not a HD-set of G, $S = V(G) \setminus \{w,y\}$ is a THD-set of G. Thus, $\widehat{\gamma}_h(G) \leq |S| = n-2$, a contradiction. A similar contradiction is attained if $yv \in E(G)$. Therefore, $G = P = P_4$.

Case 2: Suppose that $diam(G) = 2$ and $G \neq C_4$. Then G has order $n \geq 3$. Let $u, v \in V(G)$ such that $d_G(u,v) = 2$. First, we claim that $ux, vx \in E(G)$ for all $x \in V(G) \setminus \{u,v\}$. This is clear if $n = 3$ i.e., $G = P_3$. Assume $n \geq 4$. Let $[u,w,v]$ be a geodesic in G, and let $y \in V(G) \setminus \{u,w,v\}$. Suppose that $uy \notin E(G)$. Then $d_G(u,y) = 2$, and say $[u,z,y]$ is a u-y geodesic in G. The desired contradiction is attained as we consider the following subcases:

Subcase 2.1: Suppose that $z = w$. Observe that $S = V(G) \setminus \{v,y\}$ is a HD-set of G. Since $T = \{v,y\}$ does not hop-dominate w, T is not a HD-set of G. Thus, S is a THD-set of G. Consequently, $\widehat{\gamma}_h(G) \leq |S| = n-2$, a contradiction.

Subcase 2.2: Suppose that $w \neq z$ and $wy \in E(G)$. Then $S = V(G) \setminus \{v,y\}$ is a HD-set of G. Since $T = \{v,y\}$ is not a HD-set of G, S is a THD-set of G. This means that $\widehat{\gamma}_h(G) \leq |S| = n-2$, a contradiction.

Subcase 2.3: Suppose that $w \neq z$ and $d_G(y,w) = 2$. Let $[y,x,w]$ be a y-w geodesic in G. If $x = z$, then $S = V(G) \setminus \{u,y\}$ is a THD-set of G as $\{u,y\}$ does not hop-dominate z. If $x \neq z$, then $S = V(G) \setminus \{w,y\}$ is a THD-set of G as $\{w,y\}$ does not hop-dominate x. Accordingly, $\widehat{\gamma}_h(G) \leq |S| = n-2$, a contradiction.

Therefore, $ux \in E(G)$ for all $x \in V(G) \setminus \{u,v\}$. Similarly, $vx \in E(G)$ for all $x \in V(G) \setminus \{u,v\}$.

Next, we claim that $H = \langle V(G) \setminus \{u,v\} \rangle$ is complete. Suppose not, and let $x, y \in V(H)$ with $d_H(x,y) = 2$. Let $[x,z,y]$ be a geodesic in H. In particular, $[u,x,v]$ is a geodesic in G by the above claim. Observe also that $S = V(G) \setminus \{u,y\}$ is a HD-set of G. By the first claim, $uz \in E(G)$. Thus, $\{u,y\}$ does not hop-dominate z. This means that S is a THD-set of G. Consequently, $\widehat{\gamma}_h(G) \leq |S| = n-2$, a contradiction. Therefore, H is complete. Accordingly, $G = \langle \{u,v\} \rangle + H = \overline{K_2} + K_{n-2}$. □

Theorem 1. *If G is a disconnected graph with components $G_1, G_2, \ldots, G_m$ then*

$$\widehat{\gamma}_h(G) = \min_{1 \leq k \leq m} \{\widehat{\gamma}_h(G_k) + \sum_{j=1, j \neq k}^{m} \gamma_h(G_j)\}.$$

In particular, $\widehat{\gamma}_h(\overline{K_n}) = n$.

Proof. For each $k \in \{1,2,\ldots,m\}$, let D_k and S_k be a $\widehat{\gamma}_h$-set and a γ_h-set, respectively, of G_k. Then $D_k \cup \left(\cup_{j=1,j\neq k}^{m} S_j\right)$ is a THD-set of G for all $k \in \{1,2,\ldots,m\}$. Thus, $\widehat{\gamma}_h(G) \leq \min_{1 \leq k \leq m}\{\widehat{\gamma}_h(G_k) + \sum_{j=1,j\neq k}^{m} \gamma_h(G_j)\}$.

To get the other inequality, let S be any THD-set of G. Then $T_k = S \cap V(G_k)$ is a HD-set for all $k \in \{1, 2, \ldots, m\}$. We claim that T_k is a THD-set of G_k for at least one $k \in \{1, 2, \ldots, m\}$. Suppose not, and let, for each $k \in \{1, 2, \ldots, m\}$, S_k be a γ_h-set of G_k for which $T_k \cap S_k = \varnothing$. Since $\cup_{k=1}^{m} S_k$ is a γ_h-set of G, $S \cap (\cup_{k=1}^{m}) S_k \neq \varnothing$. Since $S = \cup_{k=1}^{m} T_k$, this is impossible and our claim holds. Hence, $|S| = \sum_{k=1}^{m} |T_k| \geq \min_{1 \leq k \leq m} \{\widehat{\gamma}_h(G_k) + \sum_{j=1, j\neq k}^{m} \gamma_h(G_j)\}$. □

Theorem 2. *Let G be any nontrivial graph. Then $4 \leq \widehat{\gamma}_h(G) + \widehat{\gamma}_h(\overline{G}) \leq 2n$, and these bounds are sharp.*

Proof. Let G be any nontrivial graph. Then, we have $2 \leq \widehat{\gamma}_h(G) \leq n$. Similarly, $2 \leq \widehat{\gamma}_h(\overline{G}) \leq n$. Therefore, $4 \leq \widehat{\gamma}_h(G) + \widehat{\gamma}_h(\overline{G}) \leq 2n$.

To show sharpness of the bounds, consider $G = K_2$ for the lower bound and $G = K_n$ for the upper bound. □

3. On some specific graphs

Proposition 2. *Let $G = K_{m_1, m_2, \ldots, m_k}$ be a complete multipartite graph such that $1 \leq m_1 \leq m_2 \leq \ldots \leq m_k$, and $k \geq 2$. Then*

$$\widehat{\gamma}_h(G) = m_1 + k - 1.$$

In particular, for $m, n \geq 1$, $\widehat{\gamma}_h(K_{m,n}) = 1 + \min\{m, n\}$.

Proof. Let $U_1, U_2, \ldots U_k$ be the partite sets of G with $|U_i| = m_i$ for each $i \in \{1, 2, \ldots, k\}$. Then $\gamma_h(G) = k$, and $S \subseteq V(G)$ is a γ_h-set of G if and only if $|S \cap U_i| = 1$ for each $i \in \{1, 2, \ldots, k\}$. Thus, $T \subseteq V(G)$ is a THD-set of G if and only if $T = U_i \cup \left(\cup_{j=1; j\neq i}^{k} S_j\right)$ for some $i \in \{1, 2, \ldots, k\}$ and $\varnothing \neq S_j \subseteq U_j$ for all $j \neq i$. Consequently, $\widehat{\gamma}_h(G) = m_1 + k - 1$. □

Proposition 3. *For a path P_n on n vertices,*

$$\widehat{\gamma}_h(P_n) = \begin{cases} 2, & \text{if } n = 3, 5 \\ 3, & \text{if } n = 4 \\ 2r, & \text{if } n = 6r \\ 2r + 1, & \text{if } n = 6r + 1 \\ 2r + 2, & \text{if } n = 6r + s,\ s = 2, 3, 4, 5 \end{cases}$$

Proof. Let $P_n = [v_1, v_2, \ldots, v_n]$. The case where $n = 3, 4, 5$ can easily be verified. Let $n \geq 6$ and let r and s be integers for which $n = 6r + s$ with $0 \leq s \leq 5$. Let $S = \{v_3, v_4, v_9, v_{10}, \ldots, v_{6r-3}, v_{6r-2}\}$. If $s = 0$, then S is the unique $\widehat{\gamma}_h$-set of P_n. In this case, $\widehat{\gamma}_h(P_n) = \gamma_h(P_n) = 2r$. If $s = 1$, then every γ_h-set contains the vertex v_4. Thus, $\widehat{\gamma}_h(P_n) = \gamma_h(P_n) = 2r + 1$. Suppose that $s \in \{2, 3, 4, 5\}$. Then $T = S \cup \{v_{n-2}, v_{n-1}\}$ is a $\widehat{\gamma}_h$-set of P_n. Thus, $\widehat{\gamma}_h(P_n) = |T| = 2r + 2$. □

Proposition 4. *For a cycle C_n of length n,*

$$\widehat{\gamma}_h(C_n) = \begin{cases} 3, & \text{if } n = 4,5 \\ 2r+2, & \text{if } n = 6r,\ 6r+1 \\ 2r+3, & \text{if } n = 6r+s,\ s = 2,3,4,5 \end{cases}$$

Proof. Let $C_n = [v_1, v_2, \dots, v_n, v_1]$. The case where $n = 4,5$ can easily be verified. Let $n \geq 6$ and write $n = 6r + s$ where $0 \leq s \leq 5$. Let $S = \{v_3, v_4, \dots, v_{6r-3}, v_{6r-2}\}$. Then $S \cup \{v_{n-4}, v_n\}$, $S \cup \{v_{n-2}, v_n\}$, $S \cup \{v_{n-2}, v_{n-1}, v_n\}$, $S \cup \{v_{n-3}, v_{n-1}, v_n\}$, $S \cup \{v_{n-4}, v_{n-3}, v_{n-2}\}$ and $S \cup \{v_{n-5}, v_{n-3}, v_{n-2}\}$ are $\widehat{\gamma}_h$-sets of C_n provided $s = 0$, $s = 1$, $s = 2$, $s = 3$, $s = 4$ and $s = 5$, respectively. Since $|S| = 2r$, the result follows. □

Proposition 5. *Let $G = K_m(a_1, a_2, \cdots, a_m)$ be a multi-star graph. Then $\widehat{\gamma}_h(G) = m$ for $m \geq 2$.*

Proof. Let $V(K_m) = \{v_1, v_2, \dots, v_m\}$ and for each $i = 1, 2, 3 \dots, m$, let $v_i x^i_j$ ($j = 1, 2, \dots, a_i$) be the pendant edges joined with v_i (Figure 1 shows the particular G with $m = 4$). Then the γ_h-sets of G are of the form $\{v_i, x^i_j\}$. Hence G has a unique $\widehat{\gamma}_h$-set,

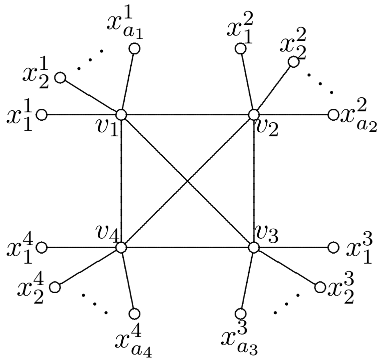

Figure 1: Multi-star $K_4(a_1, a_2, a_3, a_4)$

namely, the $V(K_m)$. Thus, $\widehat{\gamma}_h(G) = m$. □

Proposition 6. *For the Petersen graph P, $\widehat{\gamma}_h(P) = 6$.*

Proof. Let $V(P) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5\}$ where x_1, x_2, x_3, x_4, x_5 are the vertices of the outer cycle and y_1, y_2, y_3, y_4, y_5 are the corresponding vertices of the inner cycle. Then the γ_h-sets of P are of the form $\{x_i, y_i\}$ where $x_i y_i \in E(P)$, $\{x_i, x_j\}$ where $x_i x_j \in E(P)$, $i \neq j$ and $\{y_i, y_j\}$ where $y_i y_j \in E(P)$, $i \neq j$ for all $i, j = 1, 2, \dots, 5$. Thus, $S = \{x_i, x_j, x_k, y_i, y_j, y_k\}$ is a $\widehat{\gamma}_h$-set of P where x_i and x_j are adjacent in P and the vertex $x_k \notin N_P(x_i) \cup N_P(x_j)$ and y_k is adjacent to the vertex x_k in P and $y_i, y_j \notin N_P(x_i) \cup N_P(x_j)$. Therefore, $\widehat{\gamma}_h(P) = |S| = 6$. □

Proposition 7. *Let G be a firefly graph with $t \geq 1$ pendant paths, $s \geq 1$ triangles and $n-2s-2t-1 \geq 1$ pendant edges. Then, $\widehat{\gamma}_h(G) = 2$.*

Proof. Let G be a firefly graph as shown in Figure 2, where a is the vertex common to the triangles $[a, a_{2k-1}, a_{2k}, a]$ $(k = 1, 2, \ldots, s)$, pendant edges $[a, w_k]$ $(k = 1, 2, \ldots, n-2s-2t-1)$ and pendant paths $[a, u_k, v_k]$ $(k = 1, 2, \ldots, t)$ in G. If $t = 1$, then the γ_h-sets of G are the sets $\{a, u_1\}$, $\{u_1, v_1\}$ and the sets of the form $\{a, w_i\}$ for $i \geq 1$. Hence, $\{a, u_1\}$ is the unique $\widehat{\gamma}_h$-set of G. Therefore, $\widehat{\gamma}_h(G) = 2$.

Assume $t > 1$. Then the γ_h-sets of G are of the form $\{a, w_i\}$ and $\{a, u_i\}$ for $i \geq 1$. Since the vertex a is in every γ_h-set of G. Therefore, $\widehat{\gamma}_h(G) = 2$. □

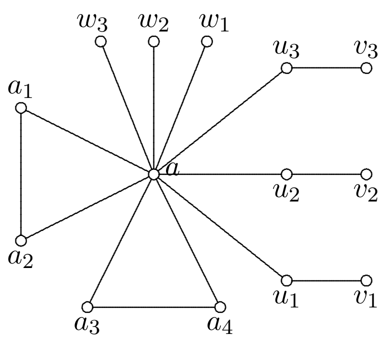

Figure 2: firefly graph $F_{2,3,3}$

4. Realization Problems

Theorem 3. *Let a and b be two positive integers with $a \geq 2$ and $b \geq 3a$. Then there exists a connected graph G on b vertices such that $\widehat{\gamma}_h(G) = a$.*

Proof. Write $b = 3a + r$ for some integer $r \geq 0$. Consider the corona $K_a \circ K_2$, and let $x \in V(K_a)$. Obtain G from $K_a \circ K_2$ by joining to $K_2^x + x$, the complete graph K_r (see G in Figure 3 where $a = 4$ and $r = 3$). Then G is a connected graph on b vertices. If

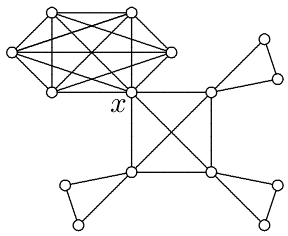

Figure 3: Graph G where $a = 4$, $r = 3$ and $|V(G)| = 3a + r = 15$

$a = 2$, then $\gamma_h(G) = a$ and $V(K_a)$ is the unique γ_h-set of G. In this case, $V(K_a)$ is also the unique $\widehat{\gamma}_h$-set of G so that $\widehat{\gamma}_h(G) = a$. If $a \geq 3$, then except for the case of $a = 3$

which also includes $V(K_a)$ as a γ_h-set, the γ_h-sets of G are all sets of the form $\{u, y, v\}$ for distinct vertices $u, v \in V(K_a)$ and $y \in V(H^u) \cup V(H^v)$ and $\{p, q, z\}$ where $z \in V(K_a)$ and $p, q \in V(K_2^z)$. Thus, $V(K_a)$ is a $\widehat{\gamma}_h$-set of G. Therefore, $\widehat{\gamma}_h(G) = a$. □

Theorem 4. *For any positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G for which $\gamma_h(G) = a$ and $\widehat{\gamma}_h(G) = b$.*

Proof. If $a = b$, then we consider the complete graph K_a. By Proposition 1, $\gamma_h(K_a) = a = \widehat{\gamma}_h(K_b)$. Suppose that $a < b$. Then $b = a + n$ for some positive integer n. Consider the graph G as shown in Figure 4. G is obtained from the rectangular grid graph $L(a, 3)$ by adding the edges $w_i v_i$ and $y_i z_i$ and the paths $[w_i, x_i^k, y_i]$ for each $i \in \{1, 2, \ldots, a\}$ and $k \in \{1, 2, \ldots, n\}$. For each $i \in \{1, 2, 3, \ldots, a\}$, let $U_i = \{x_i, x_i^1, x_i^2 \ldots, x_i^n\}$. Then $\gamma_h(G) = a$ and S is a γ_h-set of G if and only if $|S \cap U_i| = 1$ for all $i = 1, 2, 3, \ldots, a$ and $S \setminus \cup_{i=1}^a U_i = \varnothing$. In particular, the set $S^* = \{x_1, x_2, x_3, \ldots, x_{a-1}, x_a\}$ is a γ_h-set of G. Put $D = S^* \cup U_1$. For each γ_h-set S of G, $S \cap D \neq \varnothing$. Clearly, D is a $\widehat{\gamma}_h$-set of G. Therefore, $\widehat{\gamma}_h(G) = |D| = a + n = b$. □

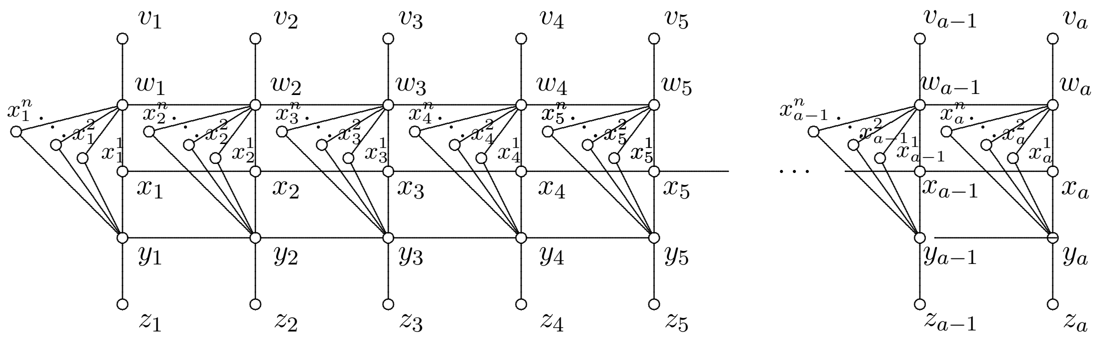

Figure 4: Graph G with $\gamma_h(G) = a$ and $\widehat{\gamma}_h(G) = b$

Corollary 1. *For each positive integer n, there exists a connected graph G such that $\widehat{\gamma}_h(G) - \gamma_h(G) = n$. That is, the difference $\widehat{\gamma}_h - \gamma_h$ can be made arbitrarily large.*

5. In the join of graphs

To attain a precise characterization of transversal hop dominating sets in the join of graphs, we give the following definition. A set $S \subseteq V(G)$ is a *transversal point-wise non-dominating set* (or *$TRPND$-set*) of G if S is a PND-set of G that intersects every *pnd*-set of G. The minimum cardinality of a $TRPND$-set of G, denoted $trpnd(G)$, is the *transversal point-wise non-domination number* of G. Any transversal point-wise non-dominating set of G of cardinality $trpnd(G)$ is referred to as a *trpnd*-set. For complete graphs, complete bipartite graphs, paths and cycles, we have

$trpnd(K_n) = n$ for all $n \geq 1$;

$trpnd(K_{m,n}) = 1 + min\{m, n\}$ for all m, n not both equal to 1;

$$trpnd(P_n) = \begin{cases} n-1, & \text{if } n = 3, 4 \\ n-2, & \text{if } n \geq 5; \end{cases} \text{ and}$$

$$trpnd(C_n) = \begin{cases} n-1, & \text{if } n = 4, \\ n-3, & \text{if } n = 6, \\ n-2, & \text{if } n = 5 \text{ and } n \geq 7. \end{cases}$$

The following theorem is due to Canoy et al. [13] from which we draw the succeeding lemma.

Theorem 5. [13] Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are PND-sets of G and H, respectively.

Lemma 1. *Let G and H be any two graphs. Then S is a γ_h-set of $G+H$ if and only if $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ are pnd-sets of G and H, respectively.*

Theorem 6. *Let G and H be any two graphs. Then $S \subseteq V(G+H)$ is a THD-set of $G+H$ if and only if $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ for which one of the following holds:*

(i) S_G is a $TRPND$-set of G and S_H is a PND-set of H.

(ii) S_H is a $TRPND$-set of H and S_G is a PND-set of G.

Proof. Suppose that S is a THD-set of $G+H$. Put $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Since $S = S_G \cup S_H$ is a hop dominating set of $G+H$, $S_G \neq \varnothing$ and $S_H \neq \varnothing$. Moreover, by Theorem 5, S_G and S_H are PND-sets of G and H, respectively. Suppose that S_G and S_H are, respectively, not $TRPND$-sets of G and H. There exist *pnd*-sets A and B of G and H, respectively, for which $A \cap S_G = \varnothing$ and $B \cap S_H = \varnothing$. By Lemma 1, $A \cup B$ is a *pnd*-set of $G+H$. Being a $TRPND$-set, $S \cap (A \cup B) \neq \varnothing$, which is impossible. Thus, the conclusion follows.

Conversely, suppose $S = S_G \cup S_H$ where S_G and S_H are as described in (i). Let $T = T_G \cup T_H$ be a γ_h-set of $G+H$. Write $T_G = T \cap V(G)$ and $T_H = T \cap V(H)$. By Lemma 1, T_G is a *pnd*-set of G. Thus, $S_G \cap T_G \neq \varnothing$. This means that $S \cap T \neq \varnothing$. Therefore, S is a THD-set of $G+H$. Similarly, if (ii) holds for S_G and S_H, then S is a THD-set of $G+H$. □

Corollary 2. *Let G and H be any two graphs. Then*

$$\widehat{\gamma}_h(G+H) = min\{trpnd(G) + pnd(H), trpnd(H) + pnd(G)\}.$$

6. In the corona of graphs

(*i*) If H has no isolated vertex, then $\widehat{\gamma}_h(K_2 \circ H) = 2$.

(*ii*) If H has $k \geq 1$ isolated vertices, then $\widehat{\gamma}_h(K_2 \circ H) = k + 2$.

Proposition 8. *Let G be a connected graph of order $n \geq 3$, and let H be any graph. Then*

$$\widehat{\gamma}_h(G \circ H) \leq n,$$

and equality is attained if $G = K_n$.

Proof. It is easy to verify that $V(G)$ is a hop dominating set of $G \circ H$. Let S be a γ_h-set of $G \circ H$. Suppose that $S \cap V(G) = \varnothing$. Then $|S \cap V(H^v)| \geq 1$, say $u^v \in S \cap V(H^v)$, for each $v \in V(G)$. Let $[z, v, w]$ be a path in G. Define $S^* = (S \setminus \{u^z, u^w\}) \cup \{v\}$. To show that S^* is a hop dominating set of $G \circ H$, it is enough to consider only the vertices in $V(G) \cap (N_G(z) \cup N_G(w))$. Let $a \in V(G) \cap N_G(z)$ with $a \neq v$. If $av \in E(G)$, then $d_{G \circ H}(a, u^v) = 2$. On the other hand, if $av \notin E(G)$, then $d_{G \circ H}(a, v) = 2$. Thus, S^* hop dominates $V(G) \cap N_G(z)$. Similarly, S^* hop dominates $V(G) \cap N_G(w)$. This means that S^* is a hop dominating set of $G \circ H$ with $|S^*| < |S|$, a contradiction. Therefore, $S \cap V(G) \neq \varnothing$ for all γ_h-sets S of $G \circ H$ so that $V(G)$ is a THD-set of $G \circ H$. Consequently, $\widehat{\gamma}_h(G \circ H) \leq n$.

Now, consider $G = K_n$. If H has an isolated vertex, then S is a γ_h-set of $G \circ H$ if and only if $S = \{u, v\}$ where $v \in V(G)$ and u is an isolated vertex of H^v. In this case, $V(G)$ is a $\widehat{\gamma}_h$-set of $G \circ H$. Suppose that H has no isolated vertices. Then S is a γ_h-set of $G \circ H$ if and only if one of the following holds for S:

(*i*) $S = V(G)$ (whenever $n = 3$);

(*ii*) $S = V(H^v + v)$ (whenever $H = K_2$);

(*iii*) $S = \{v, u, w\}$ where $v \in V(G)$ and u and w belong to distinct components of H^v (whenever H has at least 2 components);

(*iv*) $S = \{v, u, w\}$ where $v \in V(G)$ and both u and w belong the same component of H^v such that for each $z \in V(H^v) \setminus \{u, w\}$ we have $uz \notin E(H^v)$ or $wz \notin E(H^v)$;

(*v*) $S = \{v, u, w\}$ where $u, v \in V(G)$ and $w \in V(H^v) \cup V(H^u)$.

Therefore, $V(G)$ is a $\widehat{\gamma}_h$-set so that $\widehat{\gamma}_h(G \circ H) = n$. □

The inequality in Proposition 8 can be strict. If $G = K_{1,4}$ and $H = K_2$, then $\widehat{\gamma}_h(G \circ H) = 2 < n$.

Proposition 9. [1] *Let G be a graph. Then $1 \leq pnd(G) \leq |V(G)|$. Moreover,*

(*i*) *$pnd(G) = |V(G)|$ if and only if G is a complete graph;*

(*ii*) *$pnd(G) = 1$ if and only if G has an isolated vertex; and* (*iii*) *$pnd(G) = 2$ if and only if G has no isolated vertex and there exist distinct vertices a and b of G such that $N_G(a) \cap N_G(b) = \varnothing$.*

Theorem 7. [13] *Let G and H be any two graphs. A set $C \subseteq V(G \circ H)$ is a hop dominating set of $G \circ H$ if and only if*

$$C = A \cup (\cup_{v \in V(G) \cap N_G(A)} S_v) \cup (\cup_{w \in V(G) \setminus N_G(A)} E_w),$$

where

(*i*) *$A \subseteq V(G)$ such that for each $w \in V(G) \setminus A$, there exists $x \in A$ with $d_G(w, x) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $V(H^y) \cap C \neq \varnothing$,*

(*ii*) *$S_v \subseteq V(H^v)$ for each $v \in V(G) \cap N_G(A)$, and*

(*iii*) *$E_w \subseteq V(H^w)$ is a point-wise non-dominating set of H^w for each $w \in V(G) \setminus N_G(A)$.*

Lemma 2. *Let G be a connected K_3-free graph and H be a nontrivial connected graph, and let $S \subseteq V(G \circ H)$. Then S is a γ_h-set of $G \circ H$ if and only if $S \subseteq V(G)$ and is a $\gamma^{*t}_{1,2}$-set of G.*

Proof. Let $S \subseteq V(G \circ H)$ be a γ_h-set of $G \circ H$. First, we claim that $S \subseteq V(G)$. Assume, to the contrary, that $S_v = S \cap V(H^v) \neq \varnothing$ for some $v \in V(G)$. Suppose that $S \cap N_G(v) = \varnothing$. By Theorem 7, S_v is a PND-set of H^v. By Proposition 9, $|S_v| \geq 2$. Choose $w \in N_G(v)$, and define $S^* = (S \setminus S_v) \cup \{w\}$. Let $x \in V(G \circ H) \setminus S^*$. If $x \in V(H^y)$ for $y \neq v$ and $z \in S$ is such that $d_{G \circ H}(x, z) = 2$, then $z \in S^*$. If $x \in V(H^v)$, then $d_{G \circ H}(w, x) = 2$. Suppose that $x \in V(G)$. Then $x \in V(G) \setminus S$ and $x \neq w$. There exists $z \in S$ such that $d_{G \circ H}(x, z) = 2$. If $z \in S_v$, then $d_{G \circ H}(x, w) = 2$ since G is K_3-free. If $z \notin S_v$, then $z \in S \setminus S_v$ so that $z \in S^*$. This shows that S^* is a hop dominating set of $G \circ H$. Since $|S^*| < |S|$, this is a contradiction. Suppose, $S \cap N_G(v) \neq \varnothing$. Let $w \in S \cap N_G(v)$. Define $T = S \setminus S_v$. Following similar argument, T is a hop dominating set of $G \circ H$. Since $|T| < |S|$, this is a contradiction. Therefore, $S \subseteq V(G)$.

Now, let $x \in V(G)$. Pick $u \in V(H^x)$. Then $u \notin S$ and there exists $v \in S$ for which $d_{G \circ H}(u, v) = 2$. Necessarily, $d_G(x, v) = 1$. Therefore, S is a $(1, 2)^*$-total dominating set of G. Hence, $|S| \geq \gamma^{*t}_{1,2}(G)$. Suppose that $A \subseteq V(G)$ is a $\gamma^{*t}_{1,2}$-set of G. Then A is a hop dominating set of $G \circ H$ so that $|S| \leq |A| = \gamma^{*t}_{1,2}(G)$. Therefore, S is a $\gamma^{*t}_{1,2}$-set of G.

Conversely, suppose that $S \subseteq V(G)$ is a $\gamma^{*t}_{1,2}$-set of G. Then S is a hop dominating set of G. Let $v \in V(G)$ and $x \in V(H^v)$. Since S is a total dominating set of G, $uv \in E(G)$ for some $u \in S$. Then $d_{G \circ H}(x, u) = 2$. Since x and v are arbitrary, S hop dominates $V(H^v)$ for all $v \in V(G)$. Thus S is a hop dominating set of $G \circ H$. Let S^* be a γ_h-set of G. By the above result, $S^* \subseteq V(G)$ and is a $\gamma^{*t}_{1,2}$-set of G. Therefore, $|S| = |S^*|$ and S is a γ_h-set of $G \circ H$. □

Corollary 3. *For all connected K_3-free graphs G and nontrivial connected graphs H,*

$$\gamma_h(G \circ H) = \gamma^{*t}_{1,2}(G).$$

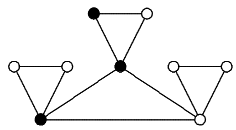

Figure 5: The corona $K_3 \circ K_2$

It is worth noting that the necessity part of Lemma 2 need not be true if G contains a K_3. Consider, for example, the corona $K_3 \circ K_2$ as shown in Figure 5. Observe that the blackened vertices constitute a γ_h-set of $K_3 \circ K_2$.

For what follows, we give the following definition. A subset $S \subseteq V(G)$ is said to be a *transversal* $(1,2)^*$*-total dominating set* of G if S is a $(1,2)^*$-total dominating set of G which intersects every $\gamma^{*t}_{1,2}$-set of G. The minimum cardinality of a transversal $(1,2)^*$-total dominating set, denoted $\widehat{\gamma}^{*t}_{1,2}(G)$, is the transversal $(1,2)^*$-total domination number of G. Any transversal $(1,2)^*$-total dominating set of G of cardinality $\widehat{\gamma}^{*t}_{1,2}(G)$ is referred to as a $\widehat{\gamma}^{*t}_{1,2}$-set.

Theorem 8. *If G is a connected K_3-free graph of order $n \geq 2$, then*

$$\widehat{\gamma}_h(G \circ H) = \widehat{\gamma}^{*t}_{1,2}(G)$$

for all nontrivial connected graphs H.

Proof. This is clear if $n = 2$. Suppose that $n \geq 3$. First, let $S \subseteq V(G)$ be a $\widehat{\gamma}^{*}_{1,2}$-set of G. Following the sufficiency proof of Lemma 2, S is a hop dominating set of $G \circ H$. By Lemma 2, S is a THD-set of $G \circ H$. Consequently, $\widehat{\gamma}_h(G \circ H) \leq |S| = \widehat{\gamma}^{*t}_{1,2}(G)$.

Now, let $S \subseteq V(G \circ H)$ be a $\widehat{\gamma}_h$-set of $G \circ H$. Let $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for each $v \in V(G)$. By Theorem 7 and Proposition 9, S_v is a PND-set of H^v, hence $|S_v| \geq 2$, for all $v \in V(G) \setminus N_G(A)$. For each $v \in V(G) \setminus N_G(A)$, choose $u_v \in N_G(v)$. Define

$$C = A \cup \{v, u_v : v \in V(G) \setminus N_G(A)\}.$$

Clearly, $|C| \leq |S|$.

Claim 1: C is a hop dominating set of G.

Let $x \in V(G) \setminus C$. Since S is a hop dominating set of $G \circ H$ and $x \notin S$, there exists $y \in S$ for which $d_{G \circ H}(x, y) = 2$. If $y \in A$, then $y \in C$. Suppose that $y \notin A$ and $A \cap N_G(x, 2) = \varnothing$. Then there exists $v \in V(G)$ such that $xv \in E(G)$ and $y \in V(H^v)$. Since G is K_3-free, $v \in V(G) \setminus N_G(A)$. Thus, there exists $u_v \in N_G(v)$ such that $v, u_v \in C$. Since G is K_3-free, $d_G(x, u_v) = 2$. This shows that C is a hop dominating set of G.

Claim 2: C is a total dominating set of G.

Let $v \in V(G)$. If $v \in N_G(A)$, then there exists $u \in C$ such that $uv \in E(G)$. Suppose that $v \notin N_G(A)$. Then there exists $u_v \in N_G(v)$ such that $v, u_v \in C$. In this case, u_v is the desired vertex in C for which $vu_v \in E(G)$. Accordingly, C is a total dominating set of G.

Claim 1 and Claim 2 all show that C is a $(1,2)^*$-total dominating set of G. Let $T \subseteq V(G)$ be a $\gamma^{*t}_{1,2}$-set of G. By Corollary 3, T is a γ_h-set of $G \circ H$. Thus, $S \cap T \neq \varnothing$. This means that $A \cap T \neq \varnothing$. Therefore, $C \cap T \neq \varnothing$ and C is a transversal $(1,2)^*$-total dominating set of G. Consequently, $\widehat{\gamma}^{*t}_{1,2}(G) \leq |C| \leq |S| = \widehat{\gamma}_h(G \circ H)$. □

Example 1. *Let H be a nontrivial connected graph.*

1. *For a path P_n on n vertices,*

$$\widehat{\gamma}_h(P_n \circ H) = \widehat{\gamma}^{*t}_{1,2}(P_n) = \begin{cases} 2, & \text{if } n = 2, 3; \\ 2r, & \text{if } n = 4r; \\ 2r+1, & \text{if } n = 4r+1; \\ 2r+2, & \text{if } n = 4r+s, \ s = 2, 3. \end{cases}$$

2. *For a cycle C_n of order $n \geq 4$,*

$$\widehat{\gamma}_h(C_n \circ H) = \widehat{\gamma}^{*t}_{1,2}(C_n) = \begin{cases} 2r+1, & \text{if } n = 4r, \ 4r+1 \\ 2r+2, & \text{if } n = 4r+2, \ 4r+3 \end{cases}$$

3. *For the complete bipartite graph $K_{m,n}$ with $m, n \geq 2$,*

$$\widehat{\gamma}_h(K_{m,n} \circ H) = \widehat{\gamma}^{*t}_{1,2}(K_{m,n}) = 1 + min\{m, n\}.$$

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References

[1] S. Arriola and Jr. S. Canoy. $(1,2)^*$-domination in graphs. *Advances and Applications in Discrete Mathematics*, 18(2):179–190, 2017.

[2] C. Berge. Theory of graphs and its Applications. *Methuen, London*, 1962.

[3] F. Buckley and F. Harary. *Distance in Graphs.* Addison-Wesley, Redwood City, CA, 1990.

[4] E. Cockayne and S. Hedetniemi. Towards a theory of domination in graphs. *Networks*, 7(3):247–261, 1977.

[5] N. Alon M.R. Fellows and D.R. Hare. Vertex transversals that dominate. *Journal of Graph Theory*, 21(1):21–31, 1996.

[6] I.S. Hamid. Independent transversal domination in graphs. *Discussiones Mathematicae Graph Theory*, 32:5–7, 2012.

[7] JW.J. Desormeaux T.W. Haynes and M.A. Henning. An extremal problem for total domination stable graphs upon edge removal. *Discrete Appl. Math*, 159:1048–1052, 2011.

[8] T.W. Haynes S.T. Hedetniemi and P.J. Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker, Inc., New York, 1990.

[9] M. Henning and A. Yeo. *Total domination in graphs*. Springer, 2013.

[10] F.P. Jamil and H.N. Maglanque. Cost effective domination in the join, corona and composition of graphs. *European Journal of Pure and Applied Mathematics*, 12(3):978–998, 2019.

[11] F.P. Jamil and R.P. Malalay. On disjunctive domination in graphs. *Quaestiones Mathematicae*, 43(2):149–168, 2020.

[12] R.G. Eballe R. Aldema E.M. Paluga R.F. Rulete F.P. Jamil. Global defensive alliances in the join, corona and composition of graphs. *Ars Combinatoria*, 107:225–245, 2012.

[13] J.G.Canoy R. Mollejon and S. Canoy Jr. Hop dominating sets in graphs under binary operations. *European Journal of Pure and Applied Mathematics*, 12(4):1455–1463, 2019.

[14] P. Dankelmann D. Day D. Erwin S. Mukwembi and H. Swart. Domination with exponential decay. *Discrete Math*, 309(19):5877 – 5883, 2009.

[15] C. Natarajan and S.K. Ayyaswamy. Hop domination in graphs-II. *Versita*, 23(2):187–199, 2015.

[16] O. Ore. *Theory of Graphs*, volume 38. 1962.

[17] Y. Pabilona and H. Rara. Total hop dominating sets in the join, corona, and lexicographic product of graphs. *ournal of Algebra and Applied Mathematics*, 2017.

[18] A. Alwardi N. Abhi R. Puttaswamy. Transversal domination in graphs. *Gulf Journal of Mathematics*, 6:41–49, 2018.

[19] A. Alwardi N. Abhi R. Puttaswamy. Transversal Total Domination in Graphs. *preprint at https://www.researchgate.net/publication/325063470*, 2018.

[20] H.A. Ahangar V. Samodivkin and I. Yero. Independent transversal dominating sets in graphs: complexity and structural properties. *Filomat*, 30(2):293–303, 2016.

[21] T. Andreae M. Schughart and Z. Tuza. Clique-transversal sets of line graphs and complements of line graphs. *Discrete Mathematics*, 88(1):11–20, 1991.

[22] D.S. Sevilleno and F.P. Jamil. On independent transversal dominating sets in graphs. *European Journal of Pure and Applied Mathematics*, 14(1):140–163, 2021.