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Graphs and the prime spectrum of unitary commutative rings

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Abstract. In This paper, we study the relationships between graphs and the prime spectrum of unitary commutative rings. It is shown that a graph G equipped with the G-right topology satisfies some spectral properties. In particular we give a necessarily and sufficient condition to obtain a spectral graph.

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1. Introduction

In [4], Hochster proved that an ordered set $(Y, \leq)$ is order-isomorphic to the prime spectrum of a commutative ring with unit equipped with the inclusion if and only if the set Y is equipped with a topology compatible with the order and satisfying the following properties:

i) X is a quasi-compact space.

ii) X is a T_0-space.

iii) Each irreducible closed subset has a generic point.

iv) X has a basis of quasi-compact open subsets.

v) The intersection of two quasi-compact open subsets is quasi-compact.

The above five properties are called spectral properties. Note that a topology compatible with the order is always T_0 [2].

A topology defined on the set Y satisfying the properties $i)$, $iii)$, $iv)$ and $v)$ is called a quasi-spectral topology.

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- The space X is said to be quasi-compact if it satisfies the property of Borel-Lebesgue but it is not necessarily a Hausdorff space.
- A topological space X is a T_0-*space* (or Kolmogorov space) if for every pair of distinct points x and y, there exists a neighborhood containing one of them but not the other; which is equivalent to the following implication $(\overline{\{x\}} = \overline{\{y\}} \Rightarrow x = y)$.
- A closed subset C is irreducible if it is not the union of two proper closed subsets or if the intersection of two nonempty open subsets is nonempty. An element x of C is called a generic point if the closure of the singleton $\{x\}$ is equal to C: $\overline{\{x\}} = C$.

We have the following properties:

1. The quasi-compactness is invariant under continuous map.
2. Each closed subset of a quasi-compact space is quasi-compact.
3. The union of finitely many quasi-compact subsets is quasi-compact.

The intersection of tow quasi-compact open subsets is not necessarily quasi-compact. [1, Example 2.1] confirm this result.

By [5], a spectral set satisfies the following conditions:

(K_1) Each totally ordered family of elements in $(Y, \leq)$ has a supremum and an infimum.

(K_2) For every elements $a < b$ in Y, there exist two consecutive elements $a_1 < b_1$ with $a \leq a_1 < b_1 \leq b$.

Lewis and Ohm showed in [6] that these two conditions are not sufficient to characterize ordered spectral sets. They even added a third independent of (K_1) and (K_2) (still necessary not sufficient):

(H) Let F be a subset of $L = \{] \leftarrow, x] : x \in X\}$ or $R = \{[x, \rightarrow [: x \in X\}$. If $\bigcap_{f \in F} f = \emptyset$, then F contains a finitely many elements with empty intersection. Where $] \leftarrow, x] = \{y \in X | y \leq x\}$ and $[x, \leftarrow, x[= \{y \in X | x \leq y\}$.

Note that the problem of characterization of spectral set is still open.

In this paper we define and characterise spectral graph.

2. Quasi-homeomorphism and spectral properties

According to [3], a continuous mapping $f : X \to Y$ between two topological spaces is a quasi-homeomorphism if the map which associates to each open subset $V \subset Y$ the open subset $U = f^{-1}(V) \subset X$ is a bijective mapping. Equivalently, the map which assigns to each closed subset $G \subset Y$ the closed subset $F = f^{-1}(G) \subset X$ is also a bijective mapping. We have the following properties:

Let $f : X \to Y$ be a quasi-homeomorphism.

1. The composition of two quasi-homeomorphisms is a quasi-homeomorphism.

2. f is open, closed.

3. For every locally closed subset $A \subset X$, we have $A = f^{-1}(f(A))$. We say that every locally closed subset of X is f-saturated.

4. For every $x, y \in X$, we have the following implication:

$$f(x) = f(y) \;\Rightarrow\; \overline{\{x\}} \;=\; \overline{\{y\}}$$

5. If moreover X is a T_0-space, then f is an embedding ($f : X \to f(X)$ is a homeomorphism).

Theorem 2.1. *If $f : (X, T) \to (Y, T')$ is a onto quasi-homeomorphism, then T is quasi-spectral if and only if T' is quasi-spectral.*

Proof. We start by showing the following: Let $f : X \to Y$ be a quasi-homeomorphism and S be a subset of Y.

1. If S is an open set, then S is quasi-compact in Y if and only if, $f^{-1}(S)$ is quasi-compact in X.

2. If S is a closed set, then S is irreducible in Y if and only if, $f^{-1}(S)$ is irreducible in X.

1. Suppose that S is a quasi-compact open subset in Y. Let $(U_i, i \in I)$ be an open covering of $f^{-1}(S)$. The fact that f is a quasi-homeomorphism implies that, for each $i \in I$, there exist an open subset V_i of Y such that $U_i = f^{-1}(V_i)$. Therefore $f^{-1}(S) = f^{-1}(\bigcup_{i\in I} V_i)$ and so $S = \bigcup_{i\in I} V_i$. It follows from the fact that S is quasi-compact in Y, that there exists a finite subset J of I such that $S = \bigcup_{i\in J} V_i$, which gives $f^{-1}(S) = \bigcup_{i\in J} U_i$ and so $f^{-1}(S)$ is quasi-compact in X.

   Conversely, Suppose that $f^{-1}(S)$ is a quasi-compact open subset in X. Let $(V_i, i \in I)$ be an open covering of S. Then $f^{-1}(S) = \bigcup_{i\in I} f^{-1}(V_i)$ and so there exists a finite subset J of I such that

$$f^{-1}(S) = \bigcup_{i\in J} f^{-1}(V_i) = f^{-1}(\bigcup_{i\in J} V_i)$$

   Using the fact that f is a quasi-homeomorphism we obtain $S = \bigcup_{i\in J} V_i$ and so S is quasi-compact in Y.

2. Suppose that S is an irreducible closed subset of Y. Let F and K be two closed subset of X such that $f^{-1}(S) = F \cup K$. Since f is a quasi-homeomorphism there exists two closed subsets F' and K' of Y such that $f^{-1}(F') = F$ and $f^{-1}(K') = K$. Hence $f^{-1}(S) = f^{-1}(F' \cup K')$, which gives $S = F' \cup K'$. From the fact that S is an irreducible closed subset of Y, it follows that $S = F'$ or $F = K'$, this yields $f^{-1}(S) = F$ or $f^{-1}(S) = K$. Therefore $f^{-1}(S)$ is an irreducible closed subset of X.

Conversely, let F' and K' be two closed subset of Y such that $S = F' \cup K'$. Then $f^{-1}(S) = f^{-1}(F') \cup f^{-1}(K')$ and from the fact that $f^{-1}(S)$ is an irreducible closed subset of X it follows that $f^{-1}(S) = f^{-1}(F')$ or $f^{-1}(S) = f^{-1}(K')$. Since f is a quasi-homeomorphism, $S = F'$ or $S = K'$ which implies that S is an irreducible closed subset of Y.

If X is quasi-compact, then since f is onto and continuous, $f(X) = Y$ is quasi-compact. By the above item (1), if Y is quasi-compact, then $f^{-1}(Y) = X$ is quasi-compact.

By the above item (1), (X, T) has a base of quasi-compact open subsets if and only if (Y, T') has a base of quasi-compact open subsets.

By the above item (1) and the fact that f is onto and continuous, the family of quasi-compact open subsets of (X, T) is stable by finite intersection if and only if the family of quasi-compact open subsets of (Y, T') is stable by finite intersection.

By the above item (2) and the fact that f is onto and continuous, every irreducible closed subset of (X, T) has a generic point if and only if every irreducible closed subset of (Y, T') has a generic point.

This ends the proof of the theorem.

3. Spectral graph

Let $G = (V, E)$ be a graph (finite or infinite) and let $u, v \in V$. A path from u to v in G is a sequence of edges e_1, . . . , e_n of E for which there exists a sequence $x_0 = u$, x_1, . . . , x_{n-1}, $x_n = v$ of vertices such that e_i has, for $i = 1, ..., n$, the endpoints x_{i-1} and x_i. We denote by

$$R(u) = \{u\} \cup \{v : \texttt{if there exists a path from } u \texttt{ to } v\}$$

$$L(u) = \{u\} \cup \{v : \texttt{if there exists a path from } v \texttt{ to } u\}.$$

The family $\{R(u) : u \in G\}$ (respectively $\{L(u) : u \in G\}$) forms a base of a topology on G called the G-right $\tau(G^R)$ (respectively G-left $\tau(G^L)$) topology.

Two vertices a and b in a graph G are called adjacent in G if a and b are endpoints of an edge e of G. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism.

Definition 3.1. *[1] The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are homeomorphic if $(G_1, \tau(G_1^R))$ and $(G_2, \tau(G_2^R))$ are homeomorphic.*

Note that, according to [1], if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic, then they are homeomorphic. If moreover every vertex of G_1 and G_2 has a loop, then isomorphic and homeomorphic properties are equivalent.

Let $G = (V, E)$ be a graph equipped with a topology T. We say that T is compatible with the graph structure G, if for all $u \in V$, $\overline{\{u\}} = R(u)$. We remark that the G-left $\tau(G^L)$ topology is the finer topology compatible with the graph structure G.

We define on G the following relation $u \preceq v$ if $u \in R(v)$. It is easy to see that $\preceq$ is reflexive and transitive. Then $\preceq$ is a preorder. We define an equivalence relation on G by $u\mathcal{R}v$ if and only if $R(u) = R(v)$. The quotient set (the set of equivalence classes) is denoted by $G/\mathcal{R}$. $(G/\mathcal{R}, \preceq)$ is a pre-ordered set. Let $G/\widetilde{\mathcal{R}}$ be the universal T_0-space associated to the space $G/\mathcal{R}$ as in Bourbaki [2, Exercise 27 p: 1-104].

Definition 3.2. *G is a spectral graph if there exists a quasi-spectral topology T compatible with the graph structure G.*

Theorem 3.3. *G is a spectral graph if and only if $(G/\widetilde{\mathcal{R}}, \preceq)$ is order-isomorphic to the prime spectrum of a unitary commutative ring equipped with the inclusion.*

Proof. Let projection $q : (G, T) \to G/\mathcal{R}$ be the canonical. Let $\overline{T}$ be the quotient topology on $G/\mathcal{R}$. Let $\varphi : \overline{T} \to T$ be the defined by $\varphi(U) = q^{-1}(U)$, for all $U \in \overline{T}$. First, we show that φ is onto. It suffices to show that $q^{-1}(q(U)) = U$. It is easy to see that $U \subset q^{-1}(q(U))$. Let $x \in q^{-1}(q(U))$. Then $q(x) \in q(U)$ which implies that there exits $y \in U$ such that $q(x) = q(y)$. Therefore $R(x) = R(y)$. Since T is compatible with G, $R(x) \subset U$ which implies that $x \in U$. Thus $q^{-1}(q(U)) = U$. Second, since q is onto, we get $q(q^{-1}(V)) = V$ for all $V \in \overline{T}$. Then φ is injective. Consequently, φ is bijective and so q is an onto quasi-homeomorphism.

By a same method as above we get $\psi : G/\mathcal{R} \to G/\widetilde{\mathcal{R}}$ which associates to each $R(u)$ its class $\widetilde{R}(u) = \{v \in G : \overline{\{R(u)\}} = \overline{\{R(v)\}}\}$ is a quasi-homeomorphism.

Therefore $\psi \circ q : (G, T) \to G/\widetilde{\mathcal{R}}$ is a quasi-homeomorphism.

Hence, By Theorem 2.1 we get (G, T) to be quasi-spectral if and only if $G/\widetilde{\mathcal{R}}$ is quasi-spectral. Since $G/\widetilde{\mathcal{R}}$ is a T_0-space we obtain Theorem 3.3.

Note that a finite graph is spectral.

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