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Restrained 2-Resolving Hop Domination in Graphs

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Abstract. Let G be a connected graph. A set $S \subseteq V(G)$ is a restrained 2-resolving hop dominating set of G if S is a 2-resolving hop dominating set of G and $S = V(G)$ or $\langle V(G)\backslash S\rangle$ has no isolated vertex. The restrained 2-resolving hop domination number of G, denoted by $\gamma_{r2Rh}(G)$ is the smallest cardinality of a restrained 2-resolving hop dominating set of G. This study aims to combine the concept of hop domination with the restrained 2-resolving sets of graphs. The main results generated in this study include the characterization of restrained 2-resolving hop dominating sets in the join, corona, edge corona and lexicographic product of graphs, as well as their corresponding bounds or exact values.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Restrained 2-resolving hop dominating set, restrained 2-resolving hop domination number, join, corona, edge corona, lexicographic product

1. Introduction

The concept of domination in graphs is one of the most studied problems and one of the fastest growing areas in graph theory. This was formally studied by Claude Berge [1] in 1958 and Oystein Ore in 1962. In 2015, Natarajan and Ayyaswamy introduced and studied the concept of hop domination [14].

On the other hand, in 1975 using the term locating set, the concept of resolving sets for a connected graph was first introduced by Slater [17]. These concepts were studied much earlier in the context of the coin-weighing problem. Later that year, Harary and Melter introduced independently these concepts, but with different terminologies [10]. The term metric dimension was used by Harary and Melter instead of locating number.

Recently, 2-resolving hop dominating sets in graphs was studied in [11]. Moreover, other variations of 2-resolving sets in graphs were also studied in [4–6, 8, 12, 13], respectively.

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2. Terminology and Notation

In this study, we consider finite, simple and connected graphs. For basic graph-theoretic concepts, we then refer readers to [2] and [3]. The following concepts are found in [2], [14] and [16].

Let G be a connected graph. A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u,v)=2$. The set $N_G(u,2)=\{v\in V(G): d_G(v,u)=2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G[u,2]=N_G(u,2)\cup\{u\}$. The *open hop neighborhood* of $X\subseteq V(G)$ is the set $N_G(X,2)=\bigcup_{u\in X} N_G(u,2)$. The *closed hop neighborhood* of X in G is the set $N_G[X,2]=N_G(X,2)\cup X$.

A set $S\subseteq V(G)$ is a *hop dominating set* of G if $N_G[S,2]=V(G)$, that is, for every $v\in V(G)\backslash S$, there exists $u\in S$ such that $d_G(u,v)=2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

For an ordered set of vertices $W=\{w_1,w_2,...,w_k\}\subseteq V(G)$ and a vertex v in G, we refer to the k-vector (ordered k-tuple)

$$r_G(v/W)=(d_G(v,w_1),d_G(v,w_2),...,d_G(v,w_k))$$

as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if distinct vertices have distinct representations with respect to W. Hence, if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r_G(v/W): v\in V(G)\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality is called a *minimum resolving set* or a *basis*, and the cardinality of a basis for G is the *dimension* $\dim(G)$ of G. An ordered set of vertices $W=\{w_1,...,w_k\}$ is a *k-resolving set* for G if, for any distinct vertices $u,v\in V(G)$, the (metric) representations $r_G(u/W)$ and $r_G(v/W)$ of u and v, respectively, differ in at least k positions. If $k=1$, then the k-resolving set is called a *resolving set* for G. If $k=2$, then the k-resolving set is called a 2-resolving set for G. If G has a k-resolving set, the minimum cardinality $\dim_k(G)$ of a k-resolving set is called the *k-metric dimension* of G.

A set $S\subseteq V(G)$ is a *restrained 2-resolving hop dominating set* of G if S is a 2-resolving hop dominating set of G and $S=V(G)$ or $\langle V(G)\backslash S\rangle$ has no isolated vertex. The *restrained 2-resolving hop domination number* of G, denoted by $\gamma_{r2Rh}(G)$ is the smallest cardinality of a restrained 2-resolving hop dominating set of G. Any restrained 2-resolving hop dominating set of cardinality $\gamma_{r2Rh}(G)$ is referred to as a γ_{r2Rh}-set of G.

Definition 1. [6] Let G be any nontrivial connected graph and $S\subseteq V(G)$. A set $S\subset V(G)$ is a 2-*locating set* of G if it satisfies the following conditions:

(i) $\big|[\big(N_G(x)\backslash N_G(y)\big)\cap S]\cup[\big(N_G(y)\backslash N_G(x)\big)\cap S]\big|\geq 2$, for all $x,y\in V(G)\backslash S$ with $x\neq y$.

(ii) $\big(N_G(v)\backslash N_G(w)\big)\cap S\neq\varnothing$ or $\big(N_G(w)\backslash N_G[v]\big)\cap S\neq\varnothing$, for all $v\in S$ and for all $w\in V(G)\backslash S$.

The 2-*locating number* of G, denoted by $ln_2(G)$, is the smallest cardinality of a 2-locating set of G. A 2-locating set of G of cardinality $ln_2(G)$ is referred to as an ln_2-set of G.

Definition 2. [15] A set $D \subseteq V(G)$ is a point-wise non-dominating set of G if for each $v \in V(G)\backslash D$, there exists $u \in D$ such that $v \notin N_G(u)$. The smallest cardinality of a point-wise non-dominating set of G, denoted by $pnd(G)$, is called the point-wise non-domination number of G. Any point-wise non-dominating set D of G with $|D| = pnd(G)$, is called a *pnd*-set of G.

Definition 3. [11] A 2-locating set $S \subseteq V(G)$ which is point-wise non-dominating is called a 2-*locating point-wise non-dominating set* in G. The minimum cardinality of a 2-locating point-wise non-dominating set in G, denoted by $ln_2^{pnd}(G)$ is called the 2-*locating point-wise non-domination number* of G. Any 2-locating point-wise non-dominating set of cardinality $ln_2^{pnd}(G)$ is then referred to as a ln_2^{pnd}-set in G.

Definition 4. A set $S \subseteq V(G)$ is a *restrained* 2-*locating point-wise non-dominating set* in G if S is a 2-locating point-wise non-dominating set in G and $S = V(G)$ or $\langle V(G)\backslash S\rangle$ has no isolated vertex. The *restrained* 2-*locating point-wise non-dominating number* of G, denoted by $rln_2^{pnd}(G)$, is the smallest cardinality of a restrained 2-locating point-wise non-dominating set in G. A restrained 2-locating point-wise non-dominating set of cardinality $rln_2^{pnd}(G)$ is then referred to as an rln_2^{pnd}-set in G.

Definition 5. [6] Let G be any nontrivial connected graph and $S \subseteq V(G)$. S is a $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *set* in G if S is 2-locating and $|N_G(y) \cap S| \leq |S| - 2$ ($|N_G(y) \cap S| \leq |S| - 1$, respectively), for all $y \in V(G)$. The $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *number* of G, denoted by $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively), is the smallest cardinality of a $(2,2)$-locating ($(2,1)$-locating, respectively) set in G. A $(2,2)$-locating ($(2,1)$-locating, respectively) set in G of cardinality $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively) is referred to as an $ln_{(2,2)}$-*set* ($ln_{(2,1)}$-*set*, respectively) in G.

Definition 6. [11] A (2,2)-locating ((2,1)-locating, respectively) set $S \subseteq V(G)$ which is a point-wise non-dominating is called a (2,2)-*locating point-wise non-dominating* ((2,1)-*locating point-wise non-dominating*, respectively) *set* in G. The minimum cardinality of a (2,2)-locating point-wise non-dominating ((2,1)-locating point-wise non-dominating, respectively) set in G, denoted by $ln_{(2,2)}^{pnd}(G)$ ($ln_{(2,1)}^{pnd}(G)$,respectively) is called the (2,2)-*locating point-wise non-domination* ((2,1)-*locating point-wise non-domination*) *number* of G. Any (2,2)-*locating point-wise non-dominating* ((2,1)-*locating point-wise non-dominating*, respectively) set of cardinality $ln_{(2,2)}^{pnd}(G)$ ($ln_{(2,1)}^{pnd}(G)$, respectively) is then referred to as a $ln_{(2,2)}^{pnd}$-set ($ln_{(2,1)}^{pnd}$-set) in G.

Definition 7. A set $S \subseteq V(G)$ is a *restrained* $(2,2)$-*locating point-wise non-dominating* ($(2,1)$-*locating point-wise non-dominating*, respectively) in G if S is a $(2,2)$-locating point-wise non-dominating ($(2,1)$-*locating point-wise non-dominating*, respectively) set in G and $S = V(G)$ or $\langle V(G)\backslash S\rangle$ has no isolated vertex. The *restrained* $(2,2)$-*locating point-wise non-domination* ($(2,1)$-*locating point-wise non-domination*, respectively) *number* of G, denoted by $rln_{(2,2)}^{pnd}(G)$ ($rln_{(2,1)}^{pnd}(G)$, *respectively*), is the smallest cardinality of a restrained $(2,2)$-locating point-wise non-dominating ($(2,1)$-locating point-wise non-dominating, respectively) set in G. A restrained $(2,2)$-locating point-wise non-dominating ($(2,1)$-locating point-wise non-dominating, respectively) set of cardinality $rln^{pnd}_{(2,2)}(G)$ ($rln^{pnd}_{(2,1)}(G)$, respectively) is then referred to as an $rln^{pnd}_{(2,2)}(G)$ ($rln^{pnd}_{(2,1)}(G)$, respectively)-set in G.

Definition 8. A restrained 2-resolving set $S \subseteq V(G)$ which is point-wise non-dominating is called a restrained 2-*resolving point-wise non-dominating set* in G. The minimum cardinality of a restrained 2-resolving point-wise non-dominating set in G, denoted by $rdim_{2_{pnd}}(G)$ is called the restrained 2-*resolving point-wise non-domination number* of G. Any r2R-pointwise non-dominating set of cardinality $rdim_{2_{pnd}}(G)$ is then referred to as a $rdim_{2_{pnd}}$-set in G.

Proposition 1. [9] Let G be a connected graph of order $n \geq 2$. Then $\dim_2(G) = 2$ if and only if $G \cong P_n$.

Remark 1. [11] For a path P_n on n vertices, $ln^{pnd}_2(P_n) = \begin{cases} 3, & n = 3 \\ \lceil \frac{n+1}{2} \rceil, & n \geq 4 \end{cases}$

3. Preliminary Results

Remark 2. Every nontrivial connected graph G admits a restrained 2-resolving hop dominating set. Indeed, the vertex set $V(G)$ of G is a restrained 2-resolving hop dominating set.

Theorem 1. If $S \subseteq V(G)$ is a restrained 2-resolving hop dominating set in G, then S is a restrained 2-resolving point-wise non-dominating set in G.

Proof. Suppose S is a restrained 2-resolving hop dominating set in G. Let $v \in V(G) \backslash S$. Since S is hop dominating set, there exists $z \in S$ such that $d_G(v, z) = 2$. Hence, $v \notin N_G(z)$. This shows that S is a point-wise non-dominating set of G. Thus, S is a restrained 2-resolving point-wise non-dominating set in G. □

The next result follows from [5].

Remark 3. Let G be any nontrivial connected graph. Then $2 \leq rln^{pnd}_2(G) \leq |V(G)|$. Moreover,

(*i*) $rln^{pnd}_2(G) = 2$ if and only if $G = K_2$.

(*ii*) If G is a connected graph with $2 \leq |V(G)| \leq 4$, then $rln^{pnd}_2(G) = |V(G)|$.

Proposition 2. Let G be any nontrivial connected graph. Then for any positive integers n and k, we have

(i) $rln_2^{pnd}(P_n) = \begin{cases} n, & \text{if } 2 \leq n \leq 7; \\ \dfrac{3n+2k}{5}, & \text{if } n = k(\text{mod } 5),\ 3 \leq k \leq 7. \end{cases}$

(ii) $rln_2^{pnd}(C_n) = \begin{cases} n, & \text{if } n = 3, 4; \\ \dfrac{3n+2k}{5}, & \text{if } n = k(\text{mod } 5),\ 0 \leq k \leq 4. \end{cases}$

(iii) For all $n \geq 4$, $rln_{(2,2)}^{pnd}(P_n) = \begin{cases} n, & \text{if } 4 \leq n \leq 7; \\ \dfrac{3n+2k}{5}, & \text{if } n = k(\text{mod } 5),\ 3 \leq k \leq 7. \end{cases}$

For all $n \geq 6$, $rln_{(2,2)}^{pnd}(C_n) = \begin{cases} n, & \text{if } n = 4; \\ \dfrac{3n+2k}{5}, & \text{if } n = k(\text{mod } 5),\ 0 \leq k \leq 4. \end{cases}$

(iv) For all $n \geq 2$, $rln_{(2,1)}^{pnd}(P_n) = \begin{cases} n, & \text{if } 2 \leq n \leq 7; \\ \dfrac{3n+2k}{5}, & \text{if } n = k(\text{mod } 5),\ 3 \leq k \leq 7. \end{cases}$

For all $n \geq 3$, $rln_{(2,1)}^{pnd}(C_n) = \begin{cases} n, & \text{if } n = 3, 4; \\ \dfrac{3n+2k}{5}, & \text{if } n = k(\text{mod } 5),\ 0 \leq k \leq 4. \end{cases}$

Proof. (i) Let $P_n = [v_1, v_2, \ldots, v_n]$ and S be an rln_2^{pnd}- set of P_n. The case where $n \leq 7$ can be easily verified by Remark 1. Next, let $n \geq 8$ and $n \equiv k(\text{mod } 5)$ where $3 \leq k \leq 7$. Then $n = 5r + k$. Hence, $r = \dfrac{n-k}{5}$. Then the set

$$S = \{v_1, v_2, v_3, v_6, v_7, v_8, v_{11}, v_{12}, v_{13}, \ldots, v_{5r+1}, v_{5r+2}, \ldots, v_{5r+k}\}$$

is an rln_2^{pnd}- set of P_n. Therefore, $|S| = 5r + k - 2r = \dfrac{3n+2k}{5}$.

The proofs of (ii), (iii) and (iv) are similar to (i). □

Theorem 2. Let G be a connected graph. Then $2 \leq rdim_{2_{pnd}}(G) \leq |V(G)|$. Moreover,

(i) $rdim_{2_{pnd}}(G) = 2$ if and only if G is a path P_n except $n = 3$.

(ii) If G is a cycle C_n for $n \neq 4$, then $rdim_{2_{pnd}}(C_n) = 3$.

Proof. (i) Suppose $\text{rdim}_{2_{pnd}}(G) = 2$. Note that every restrained 2-resolving point-wise non-dominating set is a 2-resolving point-wise non-dominating set in G, that is $\dim_{2_{pnd}}(G) = 2$. Hence, by Proposition 1, $G = P_n$. Since $\text{rdim}_{2_{pnd}}(P_3) = 3$, $G = P_n$ except $n = 3$.

Conversely, if $G = P_n = [v_1, v_2, \ldots, v_n]$, then $S = \{v_1, v_n\}$ is a restrained 2-resolving point-wise non-dominating set of G. Hence, $\text{rdim}_{2_{pnd}}(G) = 2$.

(ii) Suppose $G = C_n = [v_1, v_2, \ldots, v_n]$. Let S be the $\text{rdim}_{2_{pnd}}$-set of C_n. By (i), $\text{rdim}_{2_{pnd}}(C_n) > 2$. Thus, $S = \{v_1, v_2, v_3\}$ is a restrained 2-resolving point-wise non-dominating set of G. Hence, $\text{rdim}_{2_{pnd}}(C_n) = 3$. □

Remark 4. For any connected graph G of order $n \geq 2$, $2 \leq \gamma_{r2Rh}(G) \leq n$. Moreover, $\gamma_{r2Rh}(P_2) = 2$ and $\gamma_{r2Rh}(K_n) = n$.

Example 1. (i) For complete graph K_n on $n \geq 2$ vertices, $\gamma_{r2Rh}(K_n) = n$.

(ii) For complete bipartite graph $K_{m,n}$ on $m + n$ vertices where $m, n \geq 1$,

$$\gamma_{r2Rh}(K_{m,n}) = m + n.$$

(iii) For star graph $K_{1,n}$ on $n + 1$ vertices where $n \geq 1$, $\gamma_{r2Rh}(K_{1,n}) = n + 1$.

The next results follow from [14] and by definition of restrained 2-resolving hop dominating set.

Proposition 3. (i) For a path P_n on n vertices

$$\gamma_{r2Rh}(P_n) = \begin{cases} 2, & \text{if } n = 2, 4; \\ 3, & \text{if } n = 3, 5; \\ 4, & \text{if } n = 6; \\ \dfrac{n+2s}{3}, & \text{if } n \equiv s (\text{mod } 6) \text{ where } 0 \leq s \leq 2 \text{ and } n > 6; \\ \dfrac{n+6-s}{3}, & \text{if } n \equiv s (\text{mod } 6) \text{ where } s = 3, 4 \text{ and } n > 8; \\ \dfrac{n+4}{3}, & \text{if } n \equiv 5 (\text{mod } 6) \text{where } n > 10. \end{cases}$$

(ii) For a cycle C_n on n vertices

$$\gamma_{r2Rh}(C_n) = \begin{cases} 3, & \text{if } n = 3, 5, 6; \\ 4, & \text{if } n = 4; \\ \dfrac{n+2s}{3}, & \text{if } n \equiv s (\text{mod } 6) \text{ where } 0 \leq s \leq 2 \text{ and } n > 6; \\ \dfrac{n+6-s}{3}, & \text{if } n \equiv s (\text{mod } 6) \text{ where } 3 \leq s \leq 5 \text{ and } n > 8. \end{cases}$$

Next, we show that every pair of positive integers are realizable as 2-resolving hop domination number and restrained 2-resolving hop domination number. Thus, as a consequence, the difference $\gamma_{r2Rh} - \gamma_{2Rh}$ can be made arbitrarily large.

Remark 5. Every restrained 2-resolving hop dominating set of G is a 2-resolving hop dominating set of G. Thus, $\gamma_{2Rh}(G) \leq \gamma_{r2Rh}(G)$.

Theorem 3. Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a nontrivial connected graph H such that $\gamma_{2Rh}(H) = a$ and $\gamma_{r2Rh}(H) = b$.

Proof. Suppose $2 \leq a = b$. Consider graph H_1 in Figure 1. Hence, $S = \{x_1, x_2, x_3 \ldots, x_a\}$ is both γ_{2Rh} and a γ_{r2Rh}-set of H_1. Thus, $2 \leq \gamma_{2Rh}(H_1) = a = b = \gamma_{r2Rh}(H_1)$.

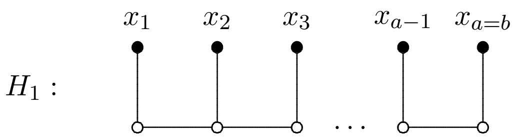

Figure 1

Suppose $2 < a < b$. Consider the graph H_2 in Figure 2. Then $S = \{x_1, x_2, \ldots, x_a\}$ is a γ_{2Rh}-set of H_2 and $X = S \cup \{y_1, y_2, \ldots, y_{b-a}\}$ is a γ_{r2Rh}-set of H_2. Hence $\gamma_{2Rh}(H_2) = a$ and $\gamma_{r2Rh}(H_2) = |X| = |S| + (b - a) = a + b - a = b$.

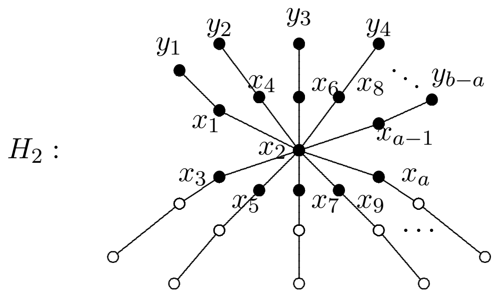

Figure 2

We now characterize the restrained 2-resolving hop dominating sets in some graphs under some binary operations.

4. Restrained 2-Resolving Hop Dominating Sets in the Join of Graphs

This section presents characterizations on the restrained 2-resolving hop dominating sets in the join of graphs.

Theorem 4. [7] Let G be a connected graph of order greater than 3 and let $K_1 = \{v\}$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving set in $K_1 + G$ if and only if either $v \notin S$ and S is a $(2, 2)$-locating set in G or $S = \{v\} \cup T$ where T is a $(2, 1)$-locating set in G.

Theorem 5. [11] Let G be a connected graph and let $K_1 = \{x\}$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving hop dominating set in $K_1 + G$ if and only if $S = \{x\} \cup T$ where T is a $(2, 1)$-locating point-wise non-dominating set in G.

Theorem 6. Let G be a connected graph and let $K_1 = \{x\}$. Then $S \subseteq V(K_1 + G)$ is a restrained 2-resolving hop dominating set in $K_1 + G$ if and only if $S = \{x\} \cup T$ where T is a restrained $(2, 1)$-locating point-wise non-dominating set in G.

Proof. Let $S \subseteq V(K_1 + G)$ be a restrained 2-resolving hop dominating set in $K_1 + G$. Then S is a restrained 2-resolving set in $K_1 + G$. Since S is a hop dominating set, $x \in S$. Hence, $S = \{x\} \cup T$ for $T \subseteq V(G)$. Then by Theorem 5, T is a (2,1)-locating point-wise non-dominating set in G. Now, since $\langle V(K_1 + G)\backslash S\rangle = \langle V(G)\backslash T\rangle$, and S is a restrained 2-resolving hop dominating set in $K_1 + G$, then it follows that $T = V(G)$ or $\langle V(G)\backslash T\rangle$ has no isolated vertex. Therefore, T is a restrained $(2,1)$-locating point-wise non-dominating set in G.

Conversely, assume that $S = \{x\} \cup T$, where T is a restrained (2,1)-locating point-wise non-dominating set in G. By Theorem 5, S is a 2-resolving hop dominating set in $K_1 + G$. Next, since $\langle V(K_1 + G)\backslash S\rangle = \langle V(G)\backslash T\rangle$ and T is a restrained (2,1)-locating point-wise non-dominating set in G, it follows that S is a restrained 2-resolving hop dominating set in $K_1 + G$. □

As a consequence of Theorem 6 the next result follows.

Corollary 1. Let G be connected nontrivial graph. Then $\gamma_{r2Rh}(K_1+G) = rln^{pnd}_{(2,1)}(G)+1$.

Example 2. For a fan $F_n = P_n + K_1$ on $n + 1$ vertices

$$\gamma_{r2Rh}(F_n) = rln^{pnd}_{(2,1)}(P_n) + 1 = \begin{cases} n+1, & \text{if } 2 \leq n \leq 7; \\ \dfrac{3n+2k}{5} + 1, & \text{if } n = k (\text{mod } 5),\ 3 \leq k \leq 7. \end{cases}$$

Example 3. For a wheel $W_n = C_n + 1$ on $n + 1$ vertices

$$\gamma_{r2Rh}(W_n) = rln^{pnd}_{(2,1)}(C_n) + 1 = \begin{cases} n+1, & \text{if } n = 3, 4; \\ \dfrac{3n+2k}{5} + 1, & \text{if } n = k (\text{mod } 5),\ 0 \leq k \leq 4. \end{cases}$$

Theorem 7. [11] Let G and H be any two graphs. A set $S \subseteq V(G + H)$ is a 2-resolving hop dominating set in $G + H$ if and only if $S = S_G \cup S_H$ where $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating point-wise non-dominating sets in G and H, respectively, where S_G or S_H is a $(2,2)$-locating point-wise non-dominating set or S_G and S_H are $(2,1)$-locating point-wise non-dominating sets of G and H, respectively.

Theorem 8. [8] Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is a restrained 2-resolving set in $G+H$ if and only if $S_G = V(G)\cap S$ and $S_H = V(H)\cap S$ where $S = S_G \cup S_H$ are 2-locating set in G and H, respectively where S_G or S_H is a $(2,2)$-locating or S_G and S_H are $(2,1)$-locating sets and one of the following holds:

(i) $S_G = V(G)$ and S_H is a restrained 2-locating set in H;

(ii) $S_H = V(H)$ and S_G is a restrained 2-locating set in G;

(iii) $S_G \neq V(G)$ and $S_H \neq V(H)$.

Theorem 9. Let G and H be any two graphs. A set $S \subseteq V(G + H)$ is a restrained 2-resolving hop dominating set in $G+H$ if and only if $S_G = V(G)\cap S$ and $S_H = V(H)\cap S$

are 2-locating pointwise non-dominating sets in G and H, respectively where S_G or S_H is a $(2,2)$-locating point-wise non-dominating set or S_G and S_H are $(2,1)$-locating point-wise non-dominating sets and one of the following holds:

(i) $S_G = V(G)$ and S_H is a restrained 2-locating point-wise non-dominating set in H;

(ii) $S_H = V(H)$ and S_G is a restrained 2-locating point-wise non-dominating set in G; and

(iii) $S_G \neq V(G)$ and $S_H \neq V(H)$.

Proof. Suppose that $S \subseteq V(G+H)$ is a restrained 2-resolving hop dominating set in $G+H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ where $S = S_G \cup S_H$. Now, since S is a 2-resolving hop dominating set by Theorem 7, S_G and S_H are 2-locating point-wise non-dominating sets in G and H, respectively, where S_G or S_H is a $(2,2)$-locating point-wise non-dominating set or S_G and S_H $(2,1)$-locating point-wise non-dominating sets of G and H, respectively. Suppose $S_G = V(G)$. Let $S_H \neq V(H)$. Since S is restrained 2-resolving hop dominating, $S = V(G+H)$ or $\langle V(G+H)\backslash S\rangle = \langle V(H)\backslash S_H\rangle$ has no isolated vertex. Hence, $S_H = V(H)$ or $\langle V(H)\backslash S_H\rangle$ has no isolated vertex. Thus, it follows that S_H is a restrained 2-locating point-wise non-dominating set of H and so (i) holds. Next, suppose that $S_G \neq V(G)$. If $S_H \neq V(H)$, then (iii) holds. On the other hand, if $S_H = V(H)$, then $\langle V(G)\backslash S_G\rangle$ has no isolated vertex and so (ii) holds.

Conversely, suppose that $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ are 2-locating point-wise non-dominating sets of G and H, respectively, and (i), (ii) and (iii) hold. By Theorem 7, S is a 2-resolving hop dominating set of $G+H$. If (i) holds, then $S = V(G+H)$ or $\langle V(G+H)\backslash S\rangle = \langle V(H)\backslash S_H\rangle$ has no isolated vertex since S_H is restrained 2-resolving hop dominating. Similarly, if (ii) holds, then $S = V(G+H)$ or $\langle V(G+H)\backslash S\rangle = \langle V(G)\backslash S_G\rangle$ has no isolated vertex since S_G is restrained 2-resolving hop dominating set. Therefore, it follows that S is a restrained 2-resolving hop dominating set of $G+H$. □

As a consequence of Theorem 9 the next result follows.

Corollary 2. Let G and H be nontrivial connected graphs. Then

$$\gamma_{r2Rh}(G+H) = \begin{cases} m+n, \text{ if } rln_2^{pnd}(G) = m \text{ and } rln_2^{pnd}(H) = n \\ \min\{ln_{(2,2)}^{pnd}(G) + ln_2^{pnd}(H), ln_2^{pnd}(G) + ln_{(2,2)}^{pnd}(H), \\ ln_{(2,1)}^{pnd}(G) + ln_{(2,1)}^{pnd}(H)\}, \text{otherwise.} \end{cases}$$

Example 4. For any nontrivial connected graph G and H of order n and m, respectively;

(i) $\gamma_{r2Rh}(G+H) = m+n$ if G and H are complete;

(*ii*)

$$\gamma_{r2Rh}(G+H) = \begin{cases} \left(\frac{n}{2}+1\right) + \left(\frac{m}{2}+1\right), & \text{if } n, m \text{ are even} \\ \left(\frac{n}{2}+1\right) + \lceil\frac{m}{2}\rceil, & \text{if } n \text{ is even}, m \text{ is odd} \\ \lceil\frac{n}{2}\rceil + \left(\frac{m}{2}+1\right), & \text{if } n \text{ is odd}, m \text{ is even} \\ \lceil\frac{n}{2}\rceil + \lceil\frac{m}{2}\rceil, & \text{if } n, m \text{ are odd.} \end{cases}$$

where $G = P_n$ and $H = P_m$ and $n, m \geq 4$.

(*iii*)

$$\gamma_{r2Rh}(G+H) = \begin{cases} \left(\frac{n}{2}\right) + \left(\frac{m}{2}\right), & \text{if } n, m \text{ are even} \\ \left(\frac{n}{2}\right) + \lceil\frac{m}{2}\rceil, & \text{if } n \text{ is even}, m \text{ is odd} \\ \lceil\frac{n}{2}\rceil + \left(\frac{m}{2}\right), & \text{if } n \text{ is odd}, m \text{ is even} \\ \lceil\frac{n}{2}\rceil + \lceil\frac{m}{2}\rceil, & \text{if } n, m \text{ are odd.} \end{cases}$$

where $G = C_n$ and $H = C_m$ and $n, m \geq 5$.

5. Restrained 2-Resolving Hop Dominating Sets in the Corona of Graphs

This section presents characterizations on the restrained 2-resolving hop dominating sets in the corona of graphs.

Remark 6. [7] Let $v \in V(G)$. For every $x, y \in V(H^v), d_{G\circ H}(x,w) = d_{G\circ H}(y,w)$ and $d_{G\circ H}(v,w) + 1 = d_{G\circ H}(x,w)$ for every $w \in V(G \circ H)\backslash V(H^v)$.

Theorem 10. [11] Let G and H be nontrivial connected graphs. A set $S \subseteq V(G \circ H)$ is a 2-resolving hop dominating set of $G \circ H$ if and only if

$$S = A \cup \left(\bigcup_{v \in V(G)\cap N_G(A)} S_v \right) \cup \left(\bigcup_{w \in V(G)\backslash N_G(A)} D_w \right)$$

where

(*i*) $A \subseteq V(G)$ such that for each $w \in V(G)\backslash A$, there exists $x \in A$ with $d_G(w,x) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $V(H^y) \cap S \neq \varnothing$;

(*ii*) $S_v \subseteq V(H^v)$ is a 2-locating set of H^v for all $v \in V(G) \cap N_G(A)$; and

(*iii*) $D_w \subseteq V(H^w)$ is a 2-locating point-wise non-dominating set of H^w for all $w \in V(G)\backslash N_G(A)$.

Theorem 11. Let G and H be nontrivial connected graphs. A set $S \subseteq V(G \circ H)$ is a restrained 2-resolving hop dominating set of $G \circ H$ if and only if

$$S = A \cup \left(\bigcup_{v \in (V(G) \setminus A) \cap N_G(A)} S_v \right) \cup \left(\bigcup_{w \in (V(G) \setminus A) \setminus N_G(A)} D_w \right) \cup \left(\bigcup_{u \in A \cap N_G(A)} E_u \right) \cup \left(\bigcup_{j \in A \setminus N_G(A)} F_j \right)$$

where

(i) $A \subseteq V(G)$ such that for each $w \in V(G) \setminus A$, there exists $x \in A$ with $d_G(w, x) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $V(H^y) \cap S \neq \varnothing$;

(ii) S_v is a 2-locating set of H^v for all $v \in (V(G) \setminus A) \cap N_G(A)$;

(iii) D_w is a 2-locating point-wise non- dominating set of H^w for all $w \in (V(G) \setminus A) \setminus N_G(A)$;

(iv) E_u is a restrained 2-locating set of H^u for all $u \in A \cap N_G(A)$;

(v) F_j is a restrained 2-locating point-wise non-dominating set of H^j for all $j \in A \setminus N_G(A)$.

Proof. Suppose $S \subseteq V(G \circ H)$ be a restrained 2-resolving hop dominating set of $G \circ H$. Let $A = S \cap V(G)$, $S_v = S \cap V(H^v)$ for each $v \in (V(G) \setminus A) \cap N_G(A)$, $D_w = S \cap V(H^w)$ for each $w \in (V(G) \setminus A) \setminus N_G(A)$, $E_u = S \cap V(H^u)$ for each $u \in A \cap N_G(A)$ and $F_j = S \cap V(H^j)$ for each $j \in A \setminus N_G(A)$. Then

$$S = A \cup \left(\bigcup_{v \in (V(G) \setminus A) \cap N_G(A)} S_v \right) \cup \left(\bigcup_{w \in (V(G) \setminus A) \setminus N_G(A)} D_w \right) \cup \left(\bigcup_{u \in A \cap N_G(A)} E_u \right) \cup \left(\bigcup_{j \in A \setminus N_G(A)} F_j \right).$$

Since S is a 2-resolving hop dominating set, (i), (ii) and (iii) follow immediately from Theorem 10.

Next, let $u \in A \cap N_G(A)$. If $E_u = V(H^u)$, then E_u is a restrained 2-locating. Suppose that $E_u \neq V(H^u)$. Then $V(G \circ H) \neq S$. Now, since $V(H^u) \setminus E_u \subseteq V(G \circ H) \setminus S$ and S is a restrained 2-resolving, it follows that $\langle V(H^u) \setminus E_u \rangle$ has no isolated vertex. Thus, E_u is a restrained 2-locating set of H^u. Hence, (iv) follows.

Finally, suppose $j \in A \setminus N_G(A)$. Since S is a restrained 2-resolving hop dominating set and $F_j \subseteq S$, F_j is a restrained 2-locating point-wise non-dominating set of H^j. Thus, (v) follows.

Conversely, let S be the set as described and satisfies the given conditions. By Theorem 10, S is 2-resolving hop dominating set. Furthermore, because (i), (ii), (iii), (iv) and (v) hold, S is a restrained 2-resolving hop dominating set in $G \circ H$. □

As a consequence of Theorem 11 the next results follow.

Corollary 3. Let G and H be nontrivial connected graphs and $|V(G)| = n$. Then

(i) $\gamma_{r2Rh}(G \circ H) \leq n(1 + rln_2(H))$.

(*ii*) $\gamma_{r2Rh}(G \circ H) \leq n(ln_2^{pnd}(H))$.

Proof. (*i*) Let $A = V(G)$, E be an rln_2-set of H and $E_u \subseteq V(H^u)$ be an rln_2-set of H^u with $\langle E_u \rangle \cong \langle E \rangle$ for each $u \in V(G)$. Then $S = A \cup \left(\bigcup_{u \in V(G)} E_w \right)$ is a restrained 2-resolving hop dominating set of $G \circ H$ by Theorem 11. Hence,

$$\gamma_{r2Rh}(G \circ H) \leq |S| = |V(G)| + \sum_{w \in V(G)} |E_u| = |V(G)| + |V(G)| \cdot |E| = n(1 + rln_2(H)).$$

(*ii*) Let $A = \varnothing$, D be a ln_2^{pnd}-set of H and $D_w \subseteq V(H^w)$ be a ln_2^{pnd}-set of H^w with $\langle D_w \rangle \cong \langle D \rangle$ for each $w \in V(G)$. Then $S = A \cup \left(\bigcup_{w \in V(G)} D_w \right)$ is a restrained 2-resolving hop dominating set of $G \circ H$ by Theorem 11. Hence,

$$\gamma_{r2Rh}(G \circ H) \leq |S| = |A| + \sum_{w \in V(G)} |D_w| = |V(G)| \cdot |D| = n(ln_2^{pnd}(H)).$$

□

Corollary 4. Let G and H be nontrivial connected graphs where $|V(G)| = n$ and $ln_2^{pnd}(H) = ln_2(H)$. Then $\gamma_{r2Rh}(G \circ H) = n(ln_2^{pnd}(H))$.

Proof. We have $\gamma_{r2Rh}(G \circ H) \leq n(ln_2^{pnd}(H))$ by Corollary 3 (*ii*). Since $ln_2^{pnd}(H) = ln_2(H)$, then by Remark 5 and Corollary 5 in [11], we have $\gamma_{r2Rh}(G \circ H) \geq \gamma_{2Rh}(G \circ H) = n(ln_2^{pnd}(H))$. Therefore, $\gamma_{r2Rh}(G \circ H) = n(ln_2^{pnd}(H))$. □

Example 5. For any nontrivial connected graph G of order n,

(*i*) $\gamma_{r2Rh}(G \circ H) \leq 4n$ if $H = P_3$;

(*ii*) $\gamma_{r2Rh}(G \circ H) = n \cdot \left(\left\lceil \frac{m+1}{2} \right\rceil \right)$ if $H = P_m$ and $m \geq 4$;

(i*iii*) $\gamma_{r2Rh}(G \circ H) = n \cdot \left(\left\lceil \frac{m}{2} \right\rceil \right)$ if $H = C_m$ and $m \geq 5$.

6. Restrained 2-Resolving Hop Dominating Sets in the Edge Corona of Graphs

This section presents characterizations on the 2-resolving hop dominating sets and restrained 2-resolving hop dominating sets in the edge corona of graphs.

Remark 7. Let $uv \in E(G)$. For every $x, y \in V(H^{uv})$, $d_{G \diamond H}(x, w) = d_{G \diamond H}(y, w)$, $d_{G \diamond H}(u, w) = d_{G \diamond H}(x, w)$, and $d_{G \diamond H}(v, w) + 1 = d_{G \diamond H}(x, w)$ for every $w \in V(G \diamond H) \setminus V(H^{uv})$.

Remark 8. Let G and H be nontrivial connected graphs, $C \subseteq V(G \diamond H)$ and $S_{uv} = V(H^{uv}) \cap C$ where $uv \in E(G)$. For each $x \in V(H^{uv}) \backslash S_{uv}$ and $z \in S_{uv}$,

$$d_{G \diamond H}(x, z) = \begin{cases} 1 & \text{if } z \in N_{H^{uv}}(x) \\ 2 & \text{otherwise.} \end{cases}$$

Definition 9. A leaf $l(G)$ of a graph G is a set of vertices v in G with $deg_G(v) = 1$.

Theorem 12. Let $G \neq P_2$ and H be any nontrivial connected graphs. A set $C \subseteq V(G \diamond H)$ is a 2-resolving hop dominating set of $G \diamond H$ if and only if

$$C = A \cup \left(\bigcup_{uv \in E(G)} S_{uv} \right)$$

where

(i) $A \subseteq V(G)$;

(ii) $S_{uv} \subseteq V(H^{uv})$ is a 2-locating set of H^{uv} for all $uv \in E(G)$ or if uv is a pendant edge, then S_{uv} is a $(2,1)$-locating set of H^{uv} whenever $l(\langle \{u, v\} \rangle) \subseteq A$ and S_{uv} is a $(2,2)$-locating set of H^{uv} otherwise.

Proof. Suppose that $C \subseteq V(G \diamond H)$ is a 2-resolving hop dominating set of $G \diamond H$. Let $A = V(G) \cap C$ and $S_{uv} = C \cap V(H^{uv})$ for all $uv \in E(G)$. Then $C = A \cup \left(\bigcup_{uv \in E(G)} S_{uv} \right)$ where $A \subseteq V(G)$ and $S_{uv} \subseteq V(H^{uv})$. Now, suppose that $S_{uv} = \varnothing$ for some $uv \in E(G)$ where $v \in V(G) \cap N_G(A)$ or $u \in V(G) \cap N_G(A)$. Let $x, y \in V(H^{uv})$. Then $r_{G \diamond H}(x/C) = r_{G \diamond H}(y/C)$ which is a contradiction to the assumption of C. Thus, $S_{uv} \neq \varnothing$. Next, we claim that S_{uv} is a 2-locating set in H^{uv} for each $uv \in E(G)$. Let $a, b \in V(H^{uv}) \backslash S_{uv}$ where $a \neq b$ or $[a \in S_{uv}$ and $b \notin S_{uv}]$. Since C is a 2-resolving set in $G \diamond H$, $r_{G \diamond H}(a/C)$ and $r_{G \diamond H}(b/C)$ differ in at least 2 positions. By Remark 7, $r_{H^{uv}}(a/S_{uv})$ and $r_{H^{uv}}(b/S_{uv})$ must differ in at least 2 positions. By definition of $G \diamond H$, there exists at least two vertices say $p, q \in V(H^{uv}) \cap S_{uv}$ such that either $p, q \in N_{H^{uv}}(a) \backslash N_{H^{uv}}(b)$ or $p, q \in N_{H^{uv}}(b) \backslash N_{H^{uv}}(a)$ or $p \in N_{H^{uv}}(a) \backslash N_{H^{uv}}(b)$ and $q \in N_{H^{uv}}(b) \backslash N_{H^{uv}}(a)$. Similarly, if $a \in S_{uv}$ and $b \in V(H^{uv}) \backslash S_{uv}$, then there exists a vertex $s \in V(H^{uv}) \cap S_{uv}$ such that $s \in N_{H^{uv}}(a) \backslash N_{H^{uv}}(b)$ or $s \in N_{H^{uv}}(b) \backslash N_{H^{uv}}(a)$. Thus, it follows that S_{uv} is a 2-locating set of H^{uv}. Next, suppose that uv is a pendant edge and suppose u is an end-vertex. Then $\langle v \rangle + H^{uv}$ is a subgraph $G \diamond H$. Since $S_{uv} = C \cap V(H^{uv}) \subseteq C$ and C is a 2-resolving set it follows by Theorem 4, S_{uv} is a $(2,1)$-locating set of H^{uv} whenever $u \in C$ and S_{uv} is a $(2,2)$-locating set of H^{uv} otherwise.

Conversely, let C be the set as described and satisfies the given conditions. Let $x, y \in V(G \diamond H)$ with $x \neq y$. Then it can be easily verify that $r_{G \diamond H}(x/C)$ and $r_{G \diamond H}(y/C)$ differ in at least two positions for all $x, y \in V(G)$ or $x \in V(H^{uv})$ and $y \in V(G)$ for all edge $uv \in E(G)$ or $x \in V(H^{pq})$ and $y \in V(H^{ab})$ such that $pq \neq ab$ for some $pq, ab \in E(G)$.

Hence, consider only the following cases:
Case 1: $x, y \in V(H^{uv})\backslash S_{uv}$ or $x \in V(H^{uv})\backslash S_{uv}$ and $y \in S_{uv}$ for some edge $uv \in E(G)$.

Now, since S_{uv} is a 2-locating set, $r_{H^{uv}}(x/S_{uv})$ and $r_{H^{uv}}(y/S_{uv})$ differ in at least two positions. Then by definition of $G \diamond H$, $r_{G\diamond H}(x/C)$ and $r_{G\diamond H}(y/C)$ differ in at least two positions.
Case 2: $x \in V(H^{uv})\backslash S_{uv}$ or $x \in S_{uv}$ and $y = u$ for some pendant edge $uv \in E(G)$ and u is an end-vertex

Since S_{uv} is a $(2,2)$-locating set, there exists $a, b \in S_{uv}\backslash N_{H^{uv}}(x)$ but $a, b \in N_{G\diamond H}(y)$. Thus, it follows that $r_{G\diamond H}(x/C)$ and $r_{G\diamond H}(y/C)$ differ in a^{th} and b^{th} positions.

Therefore, C is a 2-resolving set in $G \diamond H$.
Next, we claim that C is a hop dominating set. Let $x \in V(G)\backslash A$. Since G is a connected graph and $G \neq P_2$, there exist $y, q \in V(G)$ such that $y \in N_G(x) \cap N_G(q)$. Now, since $S_{yq} \neq \varnothing$, a vertex $z \in S_{yq} \cap N_{G\diamond H}(x, 2)$ exists. On the other hand, if $x \in V(H^{uv})\backslash S_{uv}$, then there exists $y \in N_G(u) \cup N_G(v)$ such that $N_{G\diamond H}(x, 2) \cap S_{vy} \neq \varnothing$ or $N_{G\diamond H}(x, 2) \cap S_{uy} \neq \varnothing$. Thus, C is a hop dominating set in $G \diamond H$.

Accordingly, C is a 2-resolving hop dominating set in $G \diamond H$. □

As a consequence of Theorem 12 the next result follows.

Corollary 5. Let $G \neq P_2$ be any nontrivial connected graph of size m and H a nontrivial connected graph. Then the following statements hold.

(*i*) If G is a graph with no pendant edges, then $\gamma_{2Rh}(G \diamond H) = m \cdot ln_2(H)$.

(*ii*) If G is a graph with $k \geq 1$ pendant edges, then

$$\gamma_{2Rh}(G\diamond H) = \min\{(m-k)ln_2(H)+k\cdot ln_{(2,1)}(H)+k, (m-k)ln_2(H)+k\cdot ln_{(2,2)}(H)\}$$

and $\gamma_{2Rh}(G \diamond H) = (m-k)ln_2(H) + k \cdot ln_{(2,2)}(H)$ whenever $ln_{(2,2)}(H) = ln_{(2,1)}(H)$.

Theorem 13. Let $G \neq P_2$ and H be any nontrivial connected graphs. A set $S \subseteq V(G\diamond H)$ is a restrained 2-resolving hop dominating set of $G \diamond H$ if and only if

$$C = A \cup \left(\bigcup_{uv\in E(G)} S_{uv} \right)$$

is a 2-resolving hop dominating set and

(*i*) $\langle V(G)\backslash A\rangle$ has no isolated vertex whenever $S_{uv} = V(H^{uv})$; and

(*ii*) S_{uv} is a restrained 2-locating set of H^{uv} for all $uv \in E(G)$ if $u \in A$ and $v \in A$.

Proof. Suppose C is a restrained 2-resolving hop dominating set in $G \diamond H$. Then C is a 2-resolving hop dominating set in $G \diamond H$. By Theorem 12, S_{uv} is a 2-locating set in H^{uv} for all $uv \in E(G)$. Let $A = V(G) \cap C$ and $S_{uv} = C \cap V(H^{uv})$ for all $uv \in E(G)$. Then $C = A \cup \left(\bigcup_{uv\in E(G)} S_{uv} \right)$ where $A \subseteq V(G)$ and $S_{uv} \subseteq V(H^{uv})$ for each $uv \in E(G)$.

Now, suppose $S_{uv} = V(H^{uv})$. Since C is a restrained 2-resolving hop dominating set, then $\langle V(G)\backslash A\rangle$ must contain no isolated vertex. Thus, (i) holds. Next, let $u, v \in A$. If $S_{uv} = V(H^{uv})$, then S_{uv} is a restrained 2-locating set of H^{uv}. Suppose $S_{uv} \neq V(H^{uv})$. Since $V(H^{uv})\backslash S_{uv} \subseteq V(G \diamond H)\backslash C$ and C is a restrained 2- resolving hop dominating set in $G \diamond H$, it follows $\langle V(H^{uv})\backslash S_{uv}\rangle$ must have no isolated vertex. Hence, S_{uv} is a restrained 2-locating set in H^{uv}. Hence, (ii) holds.

Conversely, let C be a 2-resolving hop dominating set as described and satisfies the given conditions. Suppose $V(H^{uv}) = S_{uv}$ for all $uv \in E(G)$. Then $\langle V(G \diamond H)\backslash C\rangle = \langle V(G)\backslash A\rangle$. By (i), $\langle V(G \diamond H)\backslash C\rangle$ has no isolated vertex. Next, suppose $V(H^{uv}) \neq S_{uv}$ for some $uv \in E(G)$. If u or v is not an element of A, then $\langle V(H^{uv})\backslash S_{uv}\rangle + \langle\{u, v\}\rangle$ has no isolated vertex. On the other hand, if $u, v \in A$, then $V(H^{uv})\backslash S_{uv}$ has no isolated vertex by (ii). Thus, it follows that $\langle V(G \diamond H)\backslash C\rangle$ has no isolated vertex. Therefore, C is a restrained 2-resolving hop dominating set in $G \diamond H$. □

Corollary 6. Let G and H be a nontrivial connected graph. Then

$$\gamma_{r2Rh}(G \diamond H) = \gamma_{2Rh}(G \diamond H).$$

7. Restrained 2-Resolving Hop Dominating Sets in the Lexicographic Product of Graphs

This section presents characterizations on the restrained 2-resolving hop dominating sets in the lexicographic product of graphs.

Theorem 14. [11] Let G and H be nontrivial connected graphs. Then $W = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 2-resolving hop dominating set in $G[H]$ if and only if

- (i) $S = V(G)$;
- (ii) T_x is a 2-locating set in H for every $x \in V(G)$;
- (iii) T_x or T_y is a $(2, 1)$-locating set or one of T_x and T_y is a $(2, 2)$-locating set in H whenever $x, y \in EQ_1(G)$;
- (iv) T_x and T_y are $(2 - locating)$ dominating sets in H or one of T_x and T_y is a 2-dominating set whenever $x, y \in EQ_2(G)$.
- (v) T_x is a 2-locating point-wise non-dominating set in H for every $x \in S$ with $|N_G(x, 2)\cap S| = 0$.

Theorem 15. Let G and H be nontrivial connected graphs. Then $W = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a restrained 2-resolving hop dominating set in $G[H]$ if and only if it is a 2-resolving hop dominating set and T_x is a restrained 2-locating point-wise non-dominating set for each x with $T_y = V(H)$ for all $y \in N_G(x)$.

Proof. Let $W = \bigcup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, be a restrained 2-resolving hop dominating set in $G[H]$. Then W is a 2-resolving hop dominating set in $G[H]$. By Theorem 14, (i)-(iv) hold and T_x is a 2-locating point-wise non-dominating set in H for every $x \in S$ with $|N_G(x,2) \cap S| = 0$. Since $V(H) \backslash T_x \subseteq V(G[H]) \backslash W$ and W is a restrained 2-resolving hop dominating set, it follows that $\langle V(H) \backslash T_x \rangle$ has no isolated vertex. Hence, T_x is a restrained 2-locating point-wise non-dominating set of H.

For the converse, let W be a 2-resolving hop dominating set as described and satisfies the given conditions. Suppose that $V(G[H]) = W$. Then W is a restrained 2-resolving hop dominating set of $G[H]$. Suppose that $V(G[H]) \neq W$. Let $(x,v) \in V(G[H]) \backslash W$. If $T_y \neq V(H)$, for all $y \in N_G(x)$, then $\langle V(G[H]) \backslash W \rangle$ has no isolated vertex. If $T_y = V(H)$, for some $y \in N_G(x)$, then T_x is a restrained 2-locating point-wise non-dominating set. Thus, $\langle V(H) \backslash T_x \rangle$ has no isolated vertex. Hence, $\langle V(G[H]) \backslash W \rangle$ has no isolated vertex. Therefore, W is a restrained 2-resolving hop dominating set in $G[H]$. □

The following results follow from Theorem 15.

Corollary 7. Let G and H be nontrivial connected graphs such that G is not free-equidistant.. Then,

$$\gamma_{r2Rh}(G[H]) \leq n \cdot ln_{(2,1)}(H) + m \cdot \gamma_{2L}(H) + p \cdot rln_2^{pnd}(H),$$

where $n + m + p = |V(G)|$ with $|EQ_1(G)| = n, |EQ_2(G)| = m$ and $|fr(G)| = p$.

Corollary 8. Let G and H be any nontrivial connected graph and G is a free-equidistant. Then

$$\gamma_{r2Rh}(G[H]) = \begin{cases} |V(G)| \cdot ln_2^{pnd}(H), \text{ if } ln_2^{pnd}(H) \neq V(H) \\ |V(G)| \cdot rln_2^{pnd}(H), \text{otherwise.} \end{cases}$$

Example 6. For any nontrivial connected graph G of order $n \geq 3$,

(i) $\gamma_{r2Rh}(G[H]) = n \cdot \left(\left\lceil \frac{m+1}{2} \right\rceil \right)$ if $H = P_m$;

(ii) $\gamma_{r2Rh}(G[H]) = n \cdot \left(\left\lceil \frac{m}{2} \right\rceil \right)$ if $H = C_m$.

(iii) $\gamma_{r2Rh}(G[H]) = n \cdot \ln_2^{pnd}(H)$ if $G = K_n$.

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