1-Movable Resolving Hop Domination in Graphs

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Abstract. Let $G$ be a connected graph. A set $W \subseteq V(G)$ is a resolving hop dominating set of $G$ if $W$ is a resolving set in $G$ and for every vertex $v \in V(G) \setminus W$ there exists $u \in W$ such that $d_G(u,v) = 2$. A set $S \subseteq V(G)$ is a 1-movable resolving hop dominating set of $G$ if $S$ is a resolving hop dominating set of $G$ and for every $v \in S$, either $S \setminus \{v\}$ is a resolving hop dominating set of $G$ or there exists a vertex $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a resolving hop dominating set of $G$. The 1-movable resolving hop domination number of $G$, denoted by $\gamma_{mRh}^1(G)$ is the smallest cardinality of a 1-movable resolving hop dominating set of $G$. This paper presents the characterization of the 1-movable resolving hop dominating sets in the join, corona and lexicographic product of graphs. Furthermore, this paper determines the exact value or bounds of their corresponding 1-movable resolving hop domination number.

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1. Introduction

Dominating sets in graphs have been studied extensively and there have been many published studies that have introduced different variants of domination in graphs [7, 13]. In 2015, Natarajan and Ayyaswamy [12] studied the concept of hop domination in graphs and the hop domination number.

Movable resolving domination in graphs was studied in [11] and the resolving hop domination sets in graphs was introduced in [10]. Other variations of resolving sets can be found in [2, 3, 6] and resolving dominating sets in [1, 4, 5, 9, 14]. This paper introduces and characterizes the concept of 1-movable resolving hop domination in graphs.

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We consider connected graphs that are finite, simple, and undirected. For elementary Graph Theory concepts, it is recommended that readers refer to [8].

Let $G = (V(G), E(G))$ be a graph. $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ is a neighborhood of $v$. An element $u \in N_G(v)$ is called a neighbor of $v$. $N_G[v] = N_G(v) \cup \{v\}$ is a closed neighborhood of $v$. The degree of $v$, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$.

The distance $d_G(u,v)$ of two vertices $u,v$ in $G$ is the length of a shortest $u$-$v$ path in $G$. The greatest distance between any two vertices in $G$, denoted by $diam(G)$, is called the diameter of $G$.

A set $S \subseteq V(G)$ is a dominating set if every $u \in V(G) \setminus S$ is adjacent to at least one vertex $v \in S$. The domination number of a graph $G$, denoted by $\gamma(G)$, is given by $\gamma(G) = \min \{|S| : S$ is a dominating set of $G\}$.

A set $S \subseteq V(G)$ is a total dominating set if every vertex in graph $G$ is adjacent to some vertex of $S$. The minimum cardinality of a total dominating set in $G$ is the total domination number of $G$, denoted by $\gamma_t(G)$, and we refer to such a set as $\gamma_t$-set of $G$.

A set $S \subseteq V(G)$ is a hop dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The minimum cardinality of a hop dominating set of $G$, denoted by $\gamma_h(G)$, is called the hop domination number of $G$. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a $\gamma_h$-set.

A vertex $v$ in $G$ is a hop neighbor of vertex $u$ in $G$ if $d_G(u,v) = 2$. The set $N_G(u,2) = \{v \in V(G) : d_G(v,u) = 2\}$ is called the open hop neighborhood of $u$. The closed hop neighborhood of $u$ in $G$ is given by $N_G[u,2] = N_G(u,2) \cup \{u\}$. The open hop neighborhood of $X \subseteq V(G)$ is the set $N_G(X,2) = \bigcup_{u \in X} N_G(u,2)$. The closed hop neighborhood of $X$ in $G$ is the set $N_G[X,2] = N_G(X,2) \cup X$.

A set $S \subseteq V(G)$ is a total hop dominating set of $G$ if for every $v \in V(G)$, there exists $u \in S$ such that $d_G(u,v) = 2$. That is, $S$ is a hop dominating set of $G$ and for all $z \in S$, $N_G(z,2) \cap S \neq \emptyset$. The smallest cardinality of a total hop dominating set of $G$, denoted by $\gamma_{th}(G)$, is called the total hop domination number of $G$. Any total hop dominating set with cardinality equal to $\gamma_{th}(G)$ is called a $\gamma_{th}$-set.

A set $S \subseteq V(G)$ is a locating set of $G$ if for every two distinct vertices $u$ and $v$ of $V(G) \setminus S$, $N_G(u) \cap S \neq N_G(v) \cap S$. The locating number of $G$, denoted by $ln(G)$, is the smallest cardinality of a locating set of $G$. A locating set of $G$ of cardinality $ln(G)$ is referred to as $ln$-set of $G$. A set $S \subseteq V(G)$ is a strictly locating set of $G$ if it is a locating set of $G$ and $N_G(u) \cap S \neq S$ for all $u \in V(G) \setminus S$. The strictly locating number of $G$, denoted by $sln(G)$, is the smallest cardinality of a strictly locating set of $G$. A strictly locating set of $G$ of cardinality $sln(G)$ is referred to as a $sln$-set of $G$.

A locating (resp. strictly locating) subset $S$ of $V(G)$ is a 1-movable locating (resp. 1-movable strictly locating) set of $G$ if for every $v \in S$, either $S \setminus \{v\}$ is a locating (resp. strictly locating) set of $G$ or there exists a vertex $u \in ((V(G) \setminus S) \cap N_G(v)))$ such that $(S \setminus \{v\}) \cup \{u\}$ is a locating (resp. strictly locating) set of $G$. The minimum cardinality of a 1-movable locating (resp. 1-movable strictly locating) set of $G$, denoted by $mln(G)$ (resp.
msln($G$) is the 1-movable location number (resp. 1-movable strictly location number) of $G$. Any 1-movable locating (resp. 1-movable strictly locating) set of cardinality $mln(G)$ (resp. $msln(G)$) is referred to as $mln$-set (resp. $msln$-set) of $G$.

A vertex $x$ of a graph $G$ is said to resolve two vertices $u$ and $v$ of $G$ if $d_G(x, u) \neq d_G(x, v)$. For an ordered set $W = \{x_1, \ldots, x_k\} \subseteq V(G)$ and a vertex $v$ in $G$, the $k$-vector $r_G(v/W) = (d_G(v, x_1), d_G(v, x_2), \ldots, d_G(v, x_k))$ is called the representation of $v$ with respect to $W$. The set $W$ is a resolving set for $G$ if and only if no two vertices of $G$ have the same representation with respect to $W$. The metric dimension of $G$, denoted by, $dim(G)$, is the minimum cardinality over all resolving sets of $G$. A resolving set of cardinality $dim(G)$ is called basis.

A set $S \subseteq V(G)$ is a resolving hop dominating set of $G$ if $S$ is both a resolving set and a hop dominating set. The minimum cardinality of a resolving hop dominating set of $G$, denoted by $\gamma_{Rh}(G)$, is called the resolving hop domination number of $G$. Any resolving hop dominating set with cardinality equal to $\gamma_{Rh}(G)$ is called a $\gamma_{Rh}$-set.

A set $S \subseteq V(G)$ is a 1-movable resolving hop dominating set of $G$ if $S$ is a resolving hop dominating set of $G$ and for every $v \in S$, either $S \setminus \{v\}$ is a resolving hop dominating set of $G$ or there exists a vertex $u \in ((V(G) \setminus S) \cap N_G(v))$ such that $(S \setminus \{v\}) \cup \{u\}$ is a resolving hop dominating set of $G$. The 1-movable resolving hop domination number of $G$, denoted by $\gamma_{mRh}^1(G)$ is the smallest cardinality of a 1-movable resolving hop dominating set of $G$. Any 1-movable resolving hop dominating set of cardinality $\gamma_{mRh}^1(G)$ is referred to as a $\gamma_{mRh}^1$-set of $G$.

2. Preliminary Results

**Remark 1.** Every 1-movable resolving hop dominating set of $G$ is a resolving hop dominating set. Thus,

$$2 \leq \gamma_{Rh}(G) \leq \gamma_{mRh}^1(G).$$

**Remark 2.** Every 1-movable resolving hop dominating set of $G$ is a hop dominating set. Thus,

$$2 \leq \gamma_h(G) \leq \gamma_{mRh}^1(G).$$

**Remark 3.** Every 1-movable resolving hop dominating set of $G$ is a resolving set. Thus,

$$1 \leq dim(G) \leq \gamma_{mRh}^1(G).$$

Consider $G = P_5$ where $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ with $deg(v_1) = deg(v_5) = 1$ and $N_G(v_3) = \{v_2, v_4\}$. Let $S_1 = \{v_1\}$, $S_2 = \{v_2, v_3\}$ and $S_3 = V(G)$. Then, $S_1$ is a resolving set of $G$, $S_2$ is a hop dominating set and a resolving set of $G$ and $S_3$ is a 1-movable resolving hop dominating set of $G$. It can be verified that $dim(G) = 1$, $\gamma_h(G) = 2$, $\gamma_{Rh}(G) = 2$ and $\gamma_{mRh}^1(G) = 5$. Hence for $G = P_5$, Remarks 1, 2 and 3 holds.
Proposition 1. Let $G$ be a nontrivial connected graph. Then $G$ admits a 1-movable resolving hop dominating set if and only if $\gamma(G) \neq 1$.

Proof: Suppose $G$ has a 1-movable resolving hop dominating set $S$. Suppose further that $\gamma(G) = 1$. Let $A = \{x \in V(G) : \{x\} \text{ is a dominating set of } G\}$. Then $A \neq \emptyset$ since $\gamma(G) = 1$. Since $S$ is a hop dominating set, $A \subseteq S$. Let $x \in A$. Then $S \setminus \{x\}$ and $(S \setminus \{x\}) \cup \{y\}$ for each $y \in V(G) \setminus S$ are not hop dominating sets of $G$. Thus, $S$ is not a 1-movable resolving hop dominating set, a contradiction.

Conversely, suppose that $\gamma(G) \neq 1$. Let $S = V(G)$. Then $S$ is a resolving hop dominating set of $G$. For each $x \in S$, $S \setminus \{x\}$ is a resolving set of $G$. Also, since $\{x\}$ is not a dominating set, there exists $y \in (S \setminus \{x\}) \cap N_G(x,2)$. Hence, $S \setminus \{x\}$ is a hop dominating set of $G$. Therefore, $S \setminus \{x\}$ is a resolving hop dominating set of $G$ for each $x \in S$. Accordingly, $S$ is a 1-movable resolving hop dominating set of $G$.

As a consequence of Proposition 1 the next result follows.

Corollary 1. A graph $G$ does not admit a 1-movable resolving hop dominating set if and only if $G = K_1 + H$ for any graph $H$.

Proposition 2. Let $G$ be a connected graph and $S$ a 1-movable resolving hop dominating set of $G$. Then for all $z \in S$, $N_G(z,2) \cap S \neq \emptyset$ and for each $x \in V(G) \setminus S$, $|N_G(x,2) \cap S| \geq 1$ and there exists $w \in (V(G) \setminus S) \cap N_G(x,2) \cap N_G(v)$ whenever $N_G(x,2) \cap S = \{v\}$.

Proof: Let $S$ be a 1-movable resolving hop dominating set of $G$ and $z \in S$. Suppose $N_G(z,2) \cap S = \emptyset$. Then $S \setminus \{z\}$ and $(S \setminus \{z\}) \cup \{u\}$ where $u \in (V(G) \setminus S) \cap N_G(z)$ are not hop dominating sets of $G$ since $z$ has no hop neighbor in both sets, a contradiction. Thus, $N_G(z,2) \cap S \neq \emptyset$. Now, let $x \in V(G) \setminus S$. Since $S$ is hop dominating, $N_G(x,2) \cap S \neq \emptyset$. Suppose $|N_G(x,2) \cap S| = 1$. Let $v \in N_G(x,2) \cap S$. Then $S \setminus \{v\}$ is not hop dominating, since $x$ has no hop neighbor in $S \setminus \{v\}$. It follows that $(S \setminus \{v\}) \cup \{w\}$ for some $w \in (V(G) \setminus S) \cap N_G(v)$ is a resolving hop dominating set of $G$. Hence, $x$ must be a hop neighbor of $w$ and so $w \in (V(G) \setminus S) \cap N_G(x,2) \cap N_G(v)$.

As a consequence of Proposition 2, the next corollary follows.

Corollary 2. Every 1-movable resolving hop dominating set is a total hop dominating set. Moreover, $\gamma_{th}(G) \leq \gamma_{mRh}^1(G)$.

3. On 1-Movable Resolving Hop Domination in the Join of Graphs

Let $A$ and $B$ be sets which are not necessarily disjoint. The disjoint union of $A$ and $B$, denoted by $A \dot{\cup} B$, is the set obtained by taking the union of $A$ and $B$ treating each element in $A$ as distinct from each element in $B$. The union $G_1 \cup G_2$ of graphs $G_1$ and $G_2$ with disjoint vertex-sets $V(G_1)$ and $V(G_2)$, respectively, is the graph $G$ with $V(G) = V(G_1) \dot{\cup} V(G_2)$ and $E(G) = E(G_1) \dot{\cup} E(G_2)$. The join of two graphs $G$ and $H$, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \dot{\cup} V(H)$ and edge-set $E(G + H) = E(G) \dot{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.
Theorem 1. [10] Let $G$ and $H$ be nontrivial connected graphs. A set $W \subseteq V(G + H)$ is a resolving hop dominating set of $G + H$ if and only if $W = W_G \cup W_H$ where $W_G$ and $W_H$ are strictly locating sets of $G$ and $H$, respectively.

As an illustration, consider the graph $P_3 + P_3$ in Figure 1. It is easy to verify that $\text{sln}(P_3) = 2$, and by Theorem 1, the set of shaded vertices is a resolving hop dominating set of $P_3 + P_3$. It follows that $\gamma_{Rh}(P_3 + P_3) = 4$.

![Graph $P_3 + P_3$ with $\gamma_{Rh}(P_3 + P_3) = 4$](image)

Figure 1: Graph $P_3 + P_3$ with $\gamma_{Rh}(P_3 + P_3) = 4$

Theorem 2. Let $G$ and $H$ be connected graphs with $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. A set $W \subseteq V(G + H)$ is a 1-movable resolving hop dominating set of $G + H$ if and only if $W = W_G \cup W_H$ where $W_G \subseteq V(G)$ and $W_H \subseteq V(H)$ are 1-movable strictly locating sets of $G$ and $H$, respectively, and one of the following statements holds:

(i) For each $u \in W_G$, $W_G \setminus \{u\}$ and $W_H \setminus \{v\}$ are strictly locating sets of $G$ and $H$, respectively, for some $v \in V(H) \setminus W_H$;

(ii) For each $q \in W_H$, $W_H \setminus \{q\}$ and $W_G \cup \{b\}$ are strictly locating sets of $H$ and $G$, respectively, for some $b \in V(G) \setminus W_G$.

Proof: Suppose that $W \subseteq V(G + H)$ is a 1-movable resolving hop dominating set of $G + H$. Then $W$ is resolving hop dominating. By Theorem 1, $W = W_G \cup W_H$ where $W_G \subseteq V(G)$ and $W_H \subseteq V(H)$ are strictly locating sets of $G$ and $H$, respectively. Moreover, since $G$ and $H$ are connected graphs with $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$, $W_G \neq \emptyset$ and $W_H \neq \emptyset$. Let $x \in W_G$. By assumption, $W \setminus \{x\} = (W_G \setminus \{x\}) \cup W_H$ or $(W \setminus \{x\}) \cup \{w\} = (W_G \setminus \{x\} \cup \{u\}) \cup W_H$ for some $u \in N_G(x) \cap (V(G) \setminus W_G)$ or $(W \setminus \{x\}) \cup \{z\} = (W_G \setminus \{x\}) \cup (W_H \cup \{z\})$ for some $z \in V(H) \setminus W_H$ is a resolving hop dominating set of $G + H$. Thus, by Theorem 1, $W_G \setminus \{x\} \cup (W_G \setminus \{x\}) \cup \{w\}$ is a strictly locating set of $G$. This implies that $W_G$ is a 1-movable strictly locating set of $G$. Similarly, $W_H$ is a 1-movable strictly locating set of $H$.

Now, let $u \in W_G$. Since $W$ is a 1-movable resolving hop dominating set, $W \setminus \{u\} = (W_G \setminus \{u\}) \cup W_H$ or $(W \setminus \{u\}) \cup \{r\} = (W_G \setminus \{u\}) \cup \{r\} \cup W_H$ for some $r \in N_G(u) \cap (V(G) \setminus W_G)$ or $(W \setminus \{u\}) \cup \{v\} = (W_G \setminus \{u\}) \cup (W_H \cup \{v\})$ for some $v \in V(H) \setminus W_H$ is a resolving hop dominating set of $G + H$. It follows from Theorem 1 that $W_G \setminus \{u\}$ and $W_H \cup \{v\}$ are strictly locating sets of $G$ and $H$, respectively. Thus, (i) holds. Similarly, (ii) holds.

For the converse, suppose that $W_G$ and $W_H$ are 1-movable strictly locating sets of $G$ and $H$, respectively. Suppose (i) holds. Then $W = W_G \cup W_H$ is a resolving hop
It can be easily verified that \( \ln \) vertices is a resolving hop dominating set of \( G \).

Theorem 1, \( W \setminus \{u\} = (W_G \setminus \{u\}) \cup W_H \) or \( (W \setminus \{u\}) \cup \{w\} = [(W_G \setminus \{u\}) \cup \{w\}] \cup W_H \) for some \( w \in N_G(u) \cap (V(G) \setminus W_H) \) or \( W \setminus \{u\} \cup \{z\} = (W_G \setminus \{u\}) \cup (W_H \cup \{z\}) \) for some \( z \in V(H \setminus W_H) \) is a resolving hop dominating set of \( G + H \). Now, suppose that \( u \in W_H \).

Since \( W_G \) and \( W_H \) are 1-movable strictly locating sets of \( G \) and \( H \), respectively, it follows from Theorem 1 that \( W \setminus \{u\} = (W_H \setminus \{u\}) \cup W_G \) or \( (W \setminus \{u\}) \cup \{y\} = [(W_H \setminus \{u\}) \cup \{y\}] \cup W_G \) for some \( y \in N_H(u) \cap (V(H) \setminus W_H) \) is a resolving hop dominating set of \( G + H \). Therefore, \( W \) is a 1-movable resolving hop dominating set of \( G + H \). Similarly, \( W \) is a 1-movable resolving hop dominating set of \( G + H \) if (ii) holds.

Corollary 3. Let \( G \) and \( H \) be nontrivial connected graphs with \( \gamma(G) \neq 1 \) and \( \gamma(H) \neq 1 \). If \( G \) and \( H \) have 1-movable strictly locating sets, then

\[
\gamma_{\text{mRh}}^1(G + H) \leq \text{msln}(G) + \text{msln}(H).
\]

Proof: Suppose \( G \) and \( H \) have 1-movable strictly locating sets. Let \( W_G \) and \( W_H \) be \( \text{msln} \)-sets of \( G \) and \( H \), respectively. Then \( W = W_G \cup W_H \) is a 1-movable resolving hop dominating set of \( G + H \) by Theorem 2. Thus,

\[
\gamma_{\text{mRh}}^1(G + H) \leq |W| = |W_G| + |W_H| = \text{msln}(G) + \text{msln}(H). \tag*{\square}
\]

4. On 1-Movable Resolving Hop Domination in the Corona of Graphs

The corona of two graphs \( G \) and \( H \), denoted by \( G \circ H \), is the graph obtained by taking one copy of \( G \) of order \( n \) and \( n \) copies of \( H \), and then joining every vertex of the \( i \)th copy of \( H \) to the \( i \)th vertex of \( G \). For \( v \in V(G) \), denote by \( H^v \) the copy of \( H \) whose vertices are attached one by one to the vertex \( v \). Subsequently, denote by \( v + H^v \) the subgraph of the corona \( G \circ H \) corresponding to the join \( \langle \{v\} \rangle + H^v, v \in V(G) \).

Theorem 3. \cite{10} Let \( G \) and \( H \) be nontrivial connected graphs. Then \( W \subseteq V(G \circ H) \) is a resolving hop dominating set of \( G \circ H \) if and only if \( W \cap V(H^v) \neq \emptyset \) for every \( v \in V(G) \) and \( W = A \cup B \cup D \) where \( A \subseteq V(G) \),

\[
B = \cup \{B_v : v \in V(G) \cap N_G(A) \text{ and } B_v \text{ is a locating set of } H^v\}
\]

and

\[
D = \cup \{D_u : u \in V(G) \setminus N_G(A) \text{ and } D_u \text{ is a strictly locating set of } H^u\}.
\]

As an illustration, consider the graph \( P_3 \circ P_4 \) in Figure 2 and let \( G = P_3 \) and \( H = P_4 \). It can be easily verified that \( lnh(P_3) = slnh(P_4) = 2 \) and by Theorem 3, the set of shaded vertices is a resolving hop dominating set of \( P_3 \circ P_4 \). It can be verified that \( \gamma_{\text{Rh}}(P_3 \circ P_4) = 6 \).
Suppose that \( x \in G \) is a resolving hop dominating set of \( B \). Let \( x \in H \). Theorem 4.

\[
W = A \cup \left( \bigcup_{v \in N_G(A)} B_v \right) \cup \left( \bigcup_{u \in V(G) \setminus N_G(A)} D_u \right)
\]

where \( A \subseteq V(G) \), \( B_v \subseteq V(H^v) \) for all \( v \in V(G) \cap N_G(A) \) and \( D_u \subseteq V(H^u) \) for all \( u \in V(G) \setminus N_G(A) \) are 1-moveable locating and 1-moveable strictly locating sets of \( H^v \) and \( H^u \), respectively.

**Proof:** Suppose that \( W \subseteq V(G \circ H) \) is a 1-moveable resolving hop dominating set of \( G \circ H \). Then \( W \) is a resolving hop dominating set. By Theorem 3, \( W \cap V(H^v) \neq \emptyset \) and \( W \cap V(H^u) \) is a locating set of \( H^u \) for all \( v \in V(G) \). Let \( A = W \cap V(G) \), \( B_v = W \cap V(H^v) \) for all \( v \in V(G) \cap N_G(A) \) and \( D_u = W \cap V(H^u) \) for all \( u \in V(G) \setminus N_G(A) \). By Theorem 3, \( B_v \) is a locating set of \( H^v \) and \( D_u \) is a strictly locating set of \( H^u \). Let \( x \in B_v \). Since \( W \) is a 1-moveable resolving hop dominating set and \( x \in W \), either \( W \setminus \{x\} \) is a resolving hop dominating set of \( G \circ H \) or there exists \( y \in (V(G \circ H) \setminus W) \cap N_{G \circ H}(x) \) such that \( (W \setminus \{x\}) \cup \{y\} \) is a resolving hop dominating set of \( G \circ H \). Note that

\[
W \setminus \{x\} = (B_v \setminus \{x\}) \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} D_u^* \right) \cup A
\]

and \( (W \setminus \{x\}) \cup \{y\} \) is equal to \( (B_v \setminus \{x\}) \cup \{y\}) \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} D_u^* \right) \cup A \) if \( y \in V(H^v) \setminus B_v \) or equal to \( (B_v \setminus \{x\}) \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} D_u^* \right) \cup (A \cup \{y\}) \) if \( y = v \in V(G) \setminus A \). Hence, either \( B_v \setminus \{x\} \) is a locating set of \( H^v \) or \( (B_v \setminus \{x\}) \cup \{y\} \) for some \( y \in (V(H^v) \setminus B_v) \cap N_{H^v}(x) \) is a locating set of \( H^v \). Thus, \( B_v \) is a movable locating set of \( H^v \). The proof that \( D_u \) is a 1-moveable strictly locating set of \( H^u \) is similar.

For the converse, suppose that \( W \) is a set described above. Then by Theorem 3, \( W \) is a resolving hop dominating set. Let \( x \in W \) and let \( v \in V(G) \) such that \( x \in V((v) + H^v) \). Suppose that \( x \neq v \). Consider the following cases.
Case 1. $v \in V(G) \cap N_G(A)$

Then $x \in B_v$ and $W \setminus \{x\} = (B_v \setminus \{x\}) \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} D_u \right) \cup A$ or $(W \setminus \{x\}) \cup \{y\}$ for some $y \in (V(G) \cap H) \setminus N_{G \circ H}(x)$ is a resolving hop dominating set by Theorem 3.

Case 2. $v \in V(G) \setminus N_G(A)$

Then $x \in D_v$ and $W \setminus \{x\} = (D_v \setminus \{x\}) \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} B_u \right) \cup A$ or $(W \setminus \{x\}) \cup \{y\}$ is a resolving hop dominating set by Theorem 3.

Therefore $W$ is a 1-movable resolving hop dominating set of $G \circ H$.

Corollary 4. Let $G$ and $H$ be nontrivial connected graphs where $|V(G)| = p$. Then

$$\gamma_{mRh}(G \circ H) \leq \min \{ p(msln(H)), \gamma_l(G) + p(mln(H)) \}.$$}

Proof: Let $W \subseteq V(G \circ H)$ be a 1-movable resolving hop dominating set of $G \circ H$. Then $W \cap V(H^u) \neq \emptyset$ and $W \cap V(H^v)$ is a 1-movable locating set for each $v \in V(G)$ and

$$W = A \cup \left( \bigcup_{v \in N_G(A)} B_v \right) \cup \left( \bigcup_{u \in V(G) \setminus N_G(A)} D_u \right),$$

where $A \subseteq V(G)$ and $B_v$ and $D_u$ satisfy the given properties in Theorem 4. Consider the following cases for set $A$.

Case 1. $A = \emptyset$

Then $N_G(A) = \emptyset$. Let $D_u = W \cap V(H^u)$ for each $u \in V(G)$.

Thus, $W = \left( \bigcup_{u \in V(G)} D_u \right)$ is a 1-movable resolving hop dominating set of $G \circ H$ by Theorem 4. Implying that,

$$\gamma_{mRh}^{1}(G \circ H) \leq |W| = |V(G)||D_u| \leq p(msln(H)).$$

Case 2. $A$ is a $\gamma_l$-set of $G$

Then $N_G(A) = V(G)$. Let $B_v = W \cap V(H^v)$ be an $mln$-set of $H^v$ for each $v \in V(G)$.

Hence, $W = A \cup \left( \bigcup_{v \in V(G)} B_v \right)$ is a 1-movable resolving hop dominating set of $G \circ H$ by Theorem 4. It follows that

$$\gamma_{mRh}^{1}(G \circ H) \leq |W| = |A| + |V(G)||B_v| = \gamma_l(G) + p(mln(H)).$$

Therefore,

$$\gamma_{mRh}^{1}(G \circ H) \leq \min \{ p(msln(H)), \gamma_l(G) + p(mln(H)) \}.$$
5. On 1-Movable Resolving Hop Domination in the Lexicographic Product of Graphs

The **lexicographic product** of two graphs $G$ and $H$, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ such that $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

**Theorem 5.** [10] Let $G$ and $H$ be nontrivial connected graphs with $\Delta(H) \leq |V(H)| - 2$. Then $W = \bigcup_{x \in S}([x] \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a resolving hop dominating set of $G[H]$ if and only if

- (i) $S = V(G)$;
- (ii) $T_x$ is a locating set for every $x \in V(G)$;
- (iii) $T_x$ or $T_y$ is a strictly locating set of $H$ whenever $x$ and $y$ are adjacent vertices of $G$ with $N_G[x] = N_G[y]$;
- (iv) $T_x$ or $T_y$ is a (locating) dominating set of $H$ whenever $x$ and $y$ are nonadjacent vertices of $G$ with $N_G(x) = N_G(y)$; and
- (v) $T_x$ is a strictly locating set of $H$ for each $x \in S \setminus N_G(S, 2)$.

The set of shaded vertices in the lexicographic product $P_3[P_4]$ in Figure 3 where $G = P_3$ and $H = P_4$ satisfies the conditions in Theorem 5 and thus it is a resolving hop dominating set of $G[H]$. In fact, the set of vertices that are not shaded is also a resolving hop dominating set of $G[H]$.

![Figure 3: Resolving hop dominating sets of $P_3[P_4]$](image)

**Theorem 6.** Let $G$ and $H$ be nontrivial connected graphs with $\Delta(H) \leq |V(H)| - 2$. Then $W = \bigcup_{x \in S}([x] \times T_x)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 1-movable resolving hop dominating set of $G[H]$ if and only if the following conditions hold:

- (i) $S = V(G)$.
- (ii) $T_x$ is a 1-movable locating set for each $x \in S$. 
(iii) $T_x \setminus \{a\}$ or $T_y$ is a strictly locating set of $H$ whenever $x$ and $y$ are adjacent vertices of $G$ with $N_G[x] = N_G[y]$ and for each $a \in T_x$.

(iv) $T_x \setminus \{a\}$ or $T_x \setminus \{a\} \cup \{b\}$ or $T_y$ is a (locating) dominating set of $H$ whenever $x$ and $y$ are nonadjacent vertices of $G$ with $N_G(x) = N_G(y)$ and for each $a \in T_x$ and for some $b \in N_H(a)$.

(v) $T_x \setminus \{a\}$ or $T_x \setminus \{a\} \cup \{b\}$ is a strictly locating set of $H$ for each $x \in S \setminus N_G(S,2)$ and for each $a \in T_x$ and for some $b \in N_H(a)$.

Proof: Suppose $W$ is a 1-movable resolving hop dominating set of $G[H]$. Then by Theorem 5, $S = V(G)$ and $T_x$ is a locating set of $H$ for each $x \in V(G)$. Let $a \in T_x$. Then $(x, a) \in W$. Since $W$ is a 1-movable resolving hop dominating set, either

$$W \setminus \{(x, a)\} = \bigcup_{v \in S \setminus \{x\}} ([v] \times T_v) \cup [x] \times (T_x \setminus \{a\})$$

or

$$W \setminus \{(x, a)\} \cup \{(x, b)\} = \bigcup_{z \in S \setminus \{x\}} ([z] \times T_z) \cup [x] \times (T_x \setminus \{a\} \cup \{b\})$$

for some $b \in N_H(a) \cap (V(H) \setminus T_x)$ or

$$W \setminus \{(x, a)\} \cup \{(y, u)\} = \bigcup_{p \in S \setminus \{(x, y)\}} ([p] \times T_p) \cup [x] \times (T_x \setminus \{a\})$$

$$\cup \{(y) \times (T_y \cup \{a\})\}$$

for some $y \in V(G) \cap N_G(x)$ and $u \in V(H) \setminus T_y$ is a resolving hop dominating set of $G[H]$.

By Theorem 5, $T_x \setminus \{a\} \setminus \{(x, a)\} \cup \{b\}$ is a locating set of $H$ for each $a \in T_x$ and for some $b \in N_H(a) \cap (V(H) \setminus T_x)$. Hence, $T_x$ is a 1-movable locating set of $H$ for each $x \in V(G)$ or $T_x \setminus \{a\}$ is locating and (ii) holds. Suppose (iii) does not hold. Then there exist $p \in V(H) \setminus (T_x \setminus \{a\})$ and $q \in V(H) \setminus T_y$ such that $N_H(p) \cap (T_x \setminus \{a\}) = T_x \setminus \{a\}$ and $N_H(q) \cap T_y = T_y$ for some adjacent vertices $x$ and $y$ of $G$ with $N_G[x] = N_G[y]$ and for some $a \in T_x$. Hence, both $W \setminus \{(x, a)\}$ and $(W \setminus \{(x, a)\}) \cup \{(y, b)\}$ are not resolving sets, a contradiction. Thus, (iii) holds.

Statement (iv) is proved similarly. If (v) does not hold, then $W \setminus \{(x, a)\}$ and $(W \setminus \{(x, a)\} \cup \{(y, b)\})$ are not hop dominating sets of $G[H]$ for all $y \in N_G(x)$ and $b \in V(H) \setminus T_x$ or $x = y$ and $b \in N_H(a)$. This is a contradiction to $W$ being a 1-movable resolving hop dominating set of $G[H]$. Hence, (v) holds.

For the converse, suppose that $W$ satisfies properties (i) to (v). By Theorem 5, $W$ is a resolving hop dominating set of $G[H]$. Let $x \in V(G)$ and $a \in T_x$. Then $(x, a) \in W$ and

$$W \setminus \{(x, a)\} = \bigcup_{v \in S \setminus \{x\}} ([v] \times T_v) \cup [x] \times (T_x \setminus \{a\})$$
and for some \( b \in N_H(a) \cap (V(H) \setminus T_x) \),

\[
(W \setminus \{(x, a)\}) \cup \{(x, b)\} = \bigcup_{z \in S \setminus \{x\}} \{(z) \times T_z\} \cup \{(x) \times ((T_x \setminus \{a\}) \cup \{b\})\}
\]

and

\[
(W \setminus \{(x, a)\}) \cup \{(y, q)\} = \bigcup_{p \in S \setminus \{(x, y)\}} \{(p) \times T_p\} \cup \{(x) \times (T_x \setminus \{a\})\} \\
\cup \{(y) \times (T_y \cup \{q\})\}
\]

for some \( y \in V(G) \cap N_G(x) \) and \( q \in V(H) \setminus T_y \).

By (i) to (v) and Theorem 5, for every \((x, a) \in W\) either \(W \setminus \{(x, a)\}\) is a resolving hop dominating set of \(G[\overline{H}]\) or there exists \((y, b) \in N_G[(\overline{H}) \setminus W] \) such that \(W \setminus \{(x, a)\} \cup \{(y, b)\}\) is a resolving hop dominating set of \(G[H]\). Therefore, \(W\) is a 1-movable resolving hop dominating set of \(G[H]\) .

Corollary 5. Let \( G \) be a nontrivial connected totally point determining graph with \( \gamma(G) \neq 1 \) and \( H \) be a nontrivial connected graph with \( \Delta(H) \leq |V(H)| - 2 \). Then

\[
\gamma_{1mR}(G[\overline{H}]) = |V(G)|\text{mln}(H).
\]

Proof: Let \( S = V(G) \) and let \( R_x \) be an \( \text{mln}\)-set of \( H \) for each \( x \in S \). Since \( \gamma_G \neq 1 \), \( x \in N_G(S, 2) \) for each \( x \in S \). By Theorem 6, \( W = \bigcup_{x \in S} \{(x) \times R_x\}\) is a 1-movable resolving hop dominating set of \(G[H]\). Thus,

\[
\gamma_{1mR}(G[H]) \leq |W| = |V(G)||R_x| = |V(G)|\text{mln}(H).
\]

Now, if \( W_0 = \bigcup_{x \in S_0} \{(x) \times T_x\}\) is a \( \gamma_{1mR}\)-set of \(G[H]\) then \( S_0 = V(G) \) and \( T_x \) is a 1-movable locating set of \( H \) for each \( x \in V(G) \) by Theorem 6. Hence,

\[
\gamma_{1mR}(G[H]) = |W_0| = |V(G)||T_x| \geq |V(G)|\text{mln}(H).
\]

Therefore, \( \gamma_{1mR}(G[H]) = |V(G)|\text{mln}(H)\).

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