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Connected Grundy Hop Dominating Sequences in Graphs

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Abstract. In this paper, we introduce and do an initial investigation of a variant of Grundy hop domination in a graph called the connected Grundy hop domination. We show that the connected Grundy hop domination number lies between the connected hop domination and Grundy hop domination number of a graph. In particular, we give realization results involving connected hop domination, connected Grundy hop domination, and Grundy hop domination numbers. Moreover, we determine the connected Grundy hop domination numbers of some graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Connected hop domination, closed hop neighborhood sequence, connected Grundy hop dominating sequence, connected Grundy hop domination number

1. Introduction

In 2014, Bresar et al. [6] introduced Grundy domination in a graph and made an initial study of the concept. Subsequent studies on this newly defined parameter can be found in [3], [4], [5], [7], and [12]. The study in [5] specifically gave exact formulas for the Grundy domination numbers of Sierpinski graphs where the authors provided a linear algorithm for determining these numbers in arbitrary interval graphs.

It is without a doubt that the concept of hop domination, just like domination, has ably attracted a lot of researchers to study it. Some of these researchers have actually introduced some variants of the concept by imposing additional properties on the standard definition (see [1], [2], [9], [10], [11], [13]).

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Following the definition of Grundy domination (as a variant of the standard domination concept), Hassan et al. in [8] introduced and studied Grundy hop domination as a variation of hop domination. It was shown that difference of the Grundy hop domination number and the hop domination number can be made arbitrarily large. Values of the Grundy hop domination numbers had also been determined for some graphs under some binary operations.

In this study, the concept of connected Grundy hop domination in a graph will be introduced. Realization results involving connected hop domination, Grundy hop domination, and connected Grundy hop domination numbers are given. Moreover, graphs that attain some specific values for the parameter are characterized.

2. Terminology and Notation

Let G be a simple undirected graph. Two vertices a and b of G are *adjacent*, or *neighbors*, if ab is an edge of G. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u in G. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{u\in X} N_G(u)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

A set $D \subseteq V(G)$ is a *dominating* of G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of a graph G, and let $\hat{S} = \{v_1, v_2, \cdots, v_k\}$. Then S is a *legal closed neighborhood sequence* if $N_G[v_i]\setminus\bigcup_{j=1}^{i-1} N_G[v_j] \neq \varnothing$ for every $i \in \{2, \cdots, k\}$. If, in addition, $\hat{S}$ is a dominating set of G, then S is called a *Grundy dominating sequence*. The maximum length of a Grundy dominating sequence in a graph G is called the *Grundy domination number* of G, and is denoted by $\gamma_{gr}(G)$. We say that vertex v_i *footprints* the vertices from $N_G[v_i] \setminus \cup_{j=1}^{i} N_G[v_j]$, and that v_i is their *footprinter*. Any Grundy dominating sequence S with $|\hat{S}| = \gamma_{gr}(G)$ is called a maximum Grundy dominating sequence or a γ_{gr}-sequence of G. In this case, we call $\hat{S}$ a γ_{gr}-set of G.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G^2(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G^2[u] = N_G^2(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G^2(X) = \bigcup_{u\in X} N_G^2(u)$. The *closed hop neighborhood* of X in G is the set $N_G^2[X] = N_G^2(X) \cup X$.

A set $D \subseteq V(G)$ is a *hop dominating set* of G if $N_G^2[D] = V(G)$, that is, for every $v \in V(G)\setminus D$, there exists $u \in D$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A hop dominating set D is called a connected hop dominating set if $\langle D\rangle$ is connected. The minimum cardinality among all connected hop dominating sets of G, denoted by

$\gamma_{ch}(G)$, is called the *connected hop domination number* of G. Any connected hop dominating set with cardinality equal to $\gamma_{ch}(G)$ is called a γ_{ch}-set.

Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of G and let $\hat{S} = \{v_1, \cdots, v_k\}$. Then S is a *legal closed hop neighborhood sequence* of G if $N_G^2[v_i] \setminus \cup_{j=1}^{i-1} N_G^2[v_j] \neq \varnothing$ for each $i \in \{2, \cdots, k\}$. If, in addition, $\hat{S}$ is a hop dominating set of G, then S is called a *Grundy hop dominating sequence*. The maximum length of a Grundy hop dominating sequence in a graph G, denoted by $\gamma_{gr}^h(G)$, is called the *Grundy hop domination number* of G. We say that vertex v_i *hop-footprints* the vertices from $N_G^2[v_i] \setminus \cup_{j=1}^{i} N_G^2[v_j]$, and that v_i is their *hop-footprinter*. Any Grundy hop dominating sequence S with $|\hat{S}| = \gamma_{gr}^h(G)$ is called a maximum Grundy hop dominating sequence or a γ_{gr}^h-sequence of G. In this case, we call $\hat{S}$ a γ_{gr}^h-set of G.

A Grundy hop dominating sequence S is called a *connected Grundy hop dominating sequence* if $\langle \hat{S} \rangle$ is connected. The maximum length of a connected Grundy hop dominating sequence in a graph G, denoted by $\gamma_{gr}^{ch}(G)$, is called the *connected Grundy hop domination number* of G. We say that vertex v_i *hop-footprints* the vertices from $N_G^2[v_i] \setminus \cup_{j=1}^{i} N_G^2[v_j]$, and that v_i is their *hop-footprinter*. Any connected Grundy hop dominating sequence S with $|\hat{S}| = \gamma_{gr}^{ch}(G)$ is called a maximum connected Grundy hop dominating sequence or a γ_{gr}^{ch}-sequence of G. In this case, we call $\hat{S}$ a γ_{gr}^{ch}-set of G.

A sequence $S = (v_1, v_2, \cdots, v_k)$ of distinct vertices of a graph G is a *co-legal closed neighborhood sequence* in G if $[V(G) \setminus N_G(v_i)] \setminus \cup_{j=1}^{i-1} [V(G) \setminus N_G(v_j)] \neq \varnothing$ for each $i \in \{2, \ldots, k\}$, i.e., S is legal closed neighborhood sequence in $\overline{G}$. A co-legal closed neighborhood sequence $S = (v_1, v_2, \ldots, v_k)$ is a *co-Grundy dominating sequence* if $V(G) = \cup_{i=1}^{k} [V(G) \setminus N_G(v_i)]$, i.e., S is Grundy dominating sequence in $\overline{G}$. The maximum length of a co-Grundy dominating sequence in a graph G is called the *co-Grundy domination number* of G, and is denoted by $\gamma_{cogr}(G)$. Clearly, $\gamma_{cogr}(G) = \gamma_{gr}(\overline{G})$.

The *shadow graph* $S(G)$ of a graph G is constructed by taking two copies of G, say G_1 and G_2, and then joining each vertex $u \in G_1$ to the neighbors of its corresponding vertex $u' \in G_2$.

Let G and H be two graphs. The *join* of G and H, denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

3. Results

Remark 1. *Let G be a connected graph. Then each of the following is true.*

(*i*) *The vertex set $V(G)$ may not form a connected Grundy hop dominating sequence.*

(*ii*) *A proper connected hop dominating set may not form a connected Grundy hop dominating sequence.*

(*iii*) *A Grundy hop dominating sequence need not be a connected Grundy hop dominating sequence.*

To see (i), consider the graph G in Figure 1. Let $S = (v_1, v_2, v_3, v_4, v_5, v_6)$. Then $\hat{S}$ is a connected hop dominating set of G. Observe that $N^2_G[v_6] = \{v_3, v_6\} \subseteq N_G[v_3]$. It follows that $N^2_G[v_6] \setminus \bigcup_{j=1}^{5} N^2_G[v_j] = \varnothing$. Hence, S is not a legal closed hop neighborhood sequence of G. Consequently, S is not connected Grundy hop dominating sequence of G. In fact (as to be shown later), $\gamma^{ch}_{gr}(G) \neq 6$.

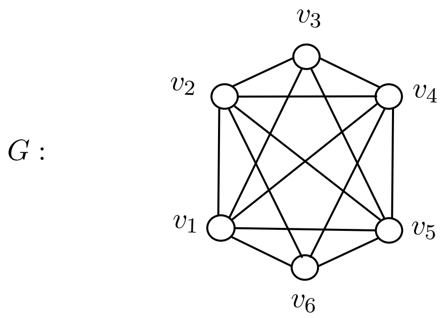

Figure 1: A graph G where vertex set does not form a connected Grundy hop dominating sequence

For (ii), consider the graph $G = C_5 = [x, z, y, w, v, x]$. Clearly, $\hat{S} = \{x, y, z, w\}$ is a connected hop dominating set of G. Observe that $N^2_G[w] \subseteq N^2_G[x] \cup N^2_G[y] \cup N^2_G[z]$. It follows that $N^2_G[w] \setminus [N^2_G[x] \cup N^2_G[y] \cup N^2_G[z]] = \varnothing$. Thus, S is not a legal closed hop neighborhood sequence of G. Therefore, S is not a connected Grundy hop dominating sequence of G.

Finally, for (iii), consider $G = P_5 = [v_1, v_2, v_3, v_4, v_5]$ and let $S' = (v_1, v_3, v_4)$. Clearly, $\hat{S}'$ is a hop dominating set of G. Observe that $v_5 \in N^2_G[v_3] \setminus N^2_G[v_1]$ and $v_2, v_4 \in N_G[v_4] \setminus (N^2_G[v_1] \cup N^2_G[v_3])$. It follows that S' is a legal closed hop neighborhood sequence of G. Thus, S' is a Grundy hop dominating sequence of G. However, $\langle \hat{S} \rangle$ is not connected. Hence, S' is not a connected Grundy hop dominating sequence of G.

Remark 2. *Let G be a connected graph. Then $\gamma_{ch}(G) \leq \gamma^{ch}_{gr}(G) \leq \gamma^{h}_{gr}(G)$ and these bounds are tight. Moreover, both strict inequalities are attainable.*

Note that the first inequality follows from the fact that every connected Grundy hop dominating sequence induces a connected hop dominating set (by definition). Moreover, since every connected Grundy hop dominating sequence is a Grundy hop dominating sequence, the second inequality follows.

To see that the bounds are tight, consider $G = K_4$. Then $\gamma_{ch}(G) = \gamma^{ch}_{gr}(G) = \gamma^{h}_{gr}(G) = 4$. For strict inequalities, consider first the graph G in Figure 2. Let $S_1 = \{s_3, s_4, s_5, s_6\}$ and $S_2 = (s_1, s_2, s_3, s_4, s_5, s_6)$. Then it can be verified that S_1 and S_2 are γ_{ch}- and γ^{ch}_{gr}-sequences of G, respectively. Hence, $\gamma_{ch}(G) = 4 < 6 = \gamma^{ch}_{gr}(G) = 6$.

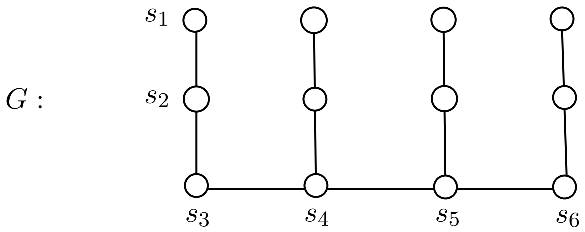

Figure 2: A graph G with $\gamma_{ch}(G) < \gamma_{gr}^{ch}(G)$

Lastly, consider the graph G' in Figure 3. Let $S' = (u_3, u_4, u_5)$ and $S'' = (u_1, u_2, u_6, u_7)$. Then S' and S'' are γ_{gr}^{ch}-sequence and γ_{gr}^{h}-sequence of G', respectively. Hence, $\gamma_{gr}^{ch}(G') = 3$ and $\gamma_{gr}^{h}(G') = 4$, that is, $\gamma_{gr}^{ch}(G') = 3 < 4 = \gamma_{gr}^{h}(G')$.

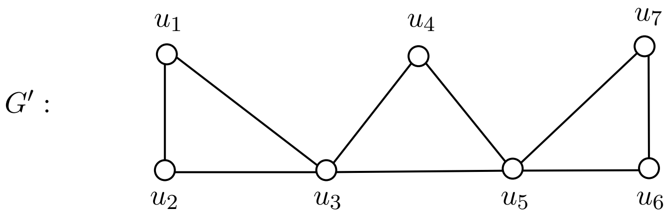

Figure 3: A graph G' with $\gamma_{gr}^{ch}(G') < \gamma_{gr}^{h}(G')$

Proposition 1. *Let G be a connected graph. Then $S = (s_1, s_2, \cdots, s_k)$ is a legal closed hop neighborhood sequence of G with maximum length and $\langle \hat{S} \rangle$ connected if and only if S is a connected Grundy hop dominating sequence of G with $\gamma_{gr}^{ch}(G) = k$.*

Proof. Let $S = (s_1, s_2, \cdots, s_k)$ be a legal closed hop neighborhood sequence of G with maximum length k and $\langle \hat{S} \rangle$ connected. Suppose $\hat{S}$ is not a connected hop dominating set of G. Then there exists $v \in V(G) \setminus N_G^2[\hat{S}]$. This implies that $v \notin N_G^2[u]$ for every $u \in \hat{S}$. Pick $u_0 = s_t \in \hat{S}$ such that $d_G(v, u_0) \leq d_G(v, s_j)$ for all $j \in \{1, 2, \ldots, k\}$. Let $[q_1, q_2, \ldots, q_m]$, where $q_1 = u_0$ and $q_m = v$, be a u_0-v geodesic. Then $m \geq 4$ and $q_4 \notin N_G^2[u]$ for every $u \in \hat{S}$. Let $S^* = (s_1, s_2, \cdots, s_k, q_2)$. Then $\langle \hat{S^*} \rangle$ is connected and $q_4 \in N_G^2[q_2] \setminus \cup_{j=1}^{k} N_G^2[s_i] \neq \varnothing$. It follows that S^* is a legal closed hop neighborhood sequence of G, a contradiction to the maximality of S. Thus, $\hat{S}$ is a connected dominating set of G. Therefore, by assumption, S is a connected Grundy hop dominating sequence of G and $\gamma_{gr}^{ch}(G) = k$.

The converse is clear. □

The next result follows from Proposition 1

Corollary 1. *Let G be a connected graph and let $S = (s_1, s_2, \dots, s_m)$ be a legal closed hop neighborhood sequence of G such that $\langle \hat{S} \rangle$ is connected. Then $|\hat{S}| = m \leq \gamma_{gr}^{ch}(G)$.*

Theorem 1. *[8] Let G be any graph on n ($n \geq 2$) vertices. Then $\gamma_{gr}^{h}(G) = n$ if and only if every component C of G is complete.*

Theorem 2. *Let G be a connected graph on n vertices. Then $1 \leq \gamma_{gr}^{ch}(G) \leq n$. Moreover, each of the following statements holds.*

(*i*) *$\gamma_{gr}^{ch}(G) = 1$ if and only if G is trivial.*

(*ii*) *$\gamma_{gr}^{ch}(G) = 2$ if and only if G a non-trivial graph, has no induced cycles C_3 and C_5, and $\{a, b\}$ is a (connected) hop dominating set for each pair of adjacent vertices $a, b \in V(G)$.*

(*iii*) *$\gamma_{gr}^{ch}(G) = n$ if and only if G is complete.*

Proof. Clearly, $1 \leq \gamma_{gr}^{ch}(G) \leq n$.

(i) Assume that $\gamma_{gr}^{h}(G) = 1$. Suppose on the contrary that G is non-trivial. Then $\gamma_{ch}(G) \geq 2$. By Proposition 2, $\gamma_{gr}^{ch}(G) \geq 2$, a contradiction. Therefore, G is trivial.

The converse is clear.

(ii) Suppose $\gamma_{gr}^{ch}(G) = 2$. Then G is non-trivial by (i). Suppose G has a triangle, say $C_3 = [x, y, z, x]$. Then $\langle \{x, y, z\} \rangle$ is connected and (x, y, z) is a legal closed hop neighborhood sequence of G. By Corollary 1, $\gamma_{gr}^{ch}(G) \geq 3$, a contradiction. Hence, G is triangle-free. Next, suppose that G has an induced cycle $C_5 = [p_1, p_2, \dots, p_5, p_1]$. Then (p_1, p_3, p_2) is a legal closed hop neighborhood sequence and $\langle \{p_1, p_3, p_2\} \rangle$ is connected. Again, by Corollary 1, this implies that $\gamma_{gr}^{ch}(G) \geq 3$, a contradiction. Thus, G does not have an induced cycle of order 5. Now let u and v be adjacent vertices. Then (u, v) is a legal closed hop neighborhood sequence of G. By assumption and Proposition 1, $\{a, b\}$ is a hop dominating set of G.

For the converse, suppose that G satisfies the given conditions. Let $(v_1, v_2, \dots, v_k)$ be a γ_{gr}^{ch}-sequence of G. Suppose further that $k \geq 2$. Let $i \in \{1, 2, \dots, k-1\}$. Since $\langle \hat{S} \rangle$ is connected, there exists $1 \leq j \leq k$, where $j \neq i$, such that $v_i v_j \in E(G)$. If $j < k$, then $N_G^2[v_k] \subseteq N_G^2[v_i] \cup N_G^2[v_j]$ because $\{v_i, v_j\}$ is a hop dominating set of G by assumption. Thus, $N_G^2[v_k] \setminus \cup_{r=1}^{k-1} N_G^2[v_r] = \varnothing$, a contradiction. Therefore, $j = k$ and $v_i v_k \in E(G)$ for all $i \in \{1, 2, \dots, k-1\}$. Moreover, since G is triangle-free, $\langle \hat{S} \rangle$ is a star. Now let $w \in N_G^2[v_2]$. Suppose $w \notin N_G^2[v_1]$. Let $[v_2, z, w]$ be a v_2-w geodesic. Since $\{v_1, v_k\}$ is a hop dominating set of G and $w \in N_G^2[v_1]$, it follows that $w \notin N_G^2[v_k]$. Let $[v_k, y, w]$ be a v_k-w geodesic. Since $v_2 v_k \in E(G)$ and G is tringle-free, $z v_k \notin E(G)$; hence, $y \neq z$. This implies that $[v_k, v_2, z, w, y, v_k]$ is an induced cycle of G of order 5, a contradiction to an assumption. Thus, $w \in N_G^2[v_1]$, implying that $N_G^2[v_2] \subseteq N_G^2[v_1]$. This contradicts the legality property of S. Therefore, $k = 2$, i.e., $\gamma_{gr}^{ch}(G) = 2$.

(iii) Suppose $\gamma_{gr}^{ch}(G) = n$. Then by Remark 2, $\gamma_{gr}^{h}(G) = n$. Since G is connected, it follows that G is complete by Theorem 1.

Conversely, suppose that G is complete. Then $N_G^2[u] = \{u\}$ for each $u \in V(G)$. Let $V(G) = \{a_1, a_2, \ldots, a_n\}$. Then

$$N_G^2[a_i] \setminus \cup_{j=1}^{i-1} N_G^2[a_j] = \{a_i\} \setminus \{a_j : j \neq i\} = \{a_i\} \neq \varnothing \text{ for each } i \in \{2, 3, \ldots, n\}.$$

It follows that $(a_1, a_2, \cdots, a_n)$ is a Grundy hop dominating sequence of G. Since G is connected, it follows that $\gamma_{gr}^{ch}(G) = n$. □

The next results are immediate from Theorem 2.

Corollary 2. *Let T be a non-trivial tree. Then $\gamma_{gr}^{ch}(G) = 2$ if and only if $\{a, b\}$ is a (connected) hop dominating set for each pair of adjacent vertices $a, b \in V(T)$.*

Corollary 3. *Let G be a connected graph on n vertices. Then $\gamma_{gr}^{ch}(G) \leq n - 1$ if and only if G is non-complete.*

Proposition 2. *Let n be any positive integer. Then each of the following holds.*

(*i*) *There exists a connected graph G such that $\gamma_{gr}^{ch}(G) - \gamma_{ch}(G) = n$.*

(*ii*) *There exists a connected graph H such that $\gamma_{gr}^{h}(H) - \gamma_{gr}^{ch}(H) = n$.*

Proof. For (i), consider the graph G given in Figure 4. Let $S_1 = \{u, v_{n+2}\}$ and $S_2 = (v_1, v_2, \cdots, v_{n+2})$. Then S_1 and S_2 are γ_{ch}-set and γ_{gr}^{ch}-sequence of G, respectively. Hence, $\gamma_{ch}(G) = 2$ and $\gamma_{gr}^{ch}(G) = n + 2$. Consequently, $\gamma_{gr}^{ch}(G) - \gamma_{ch}(G) = n + 2 - 2 = n$.

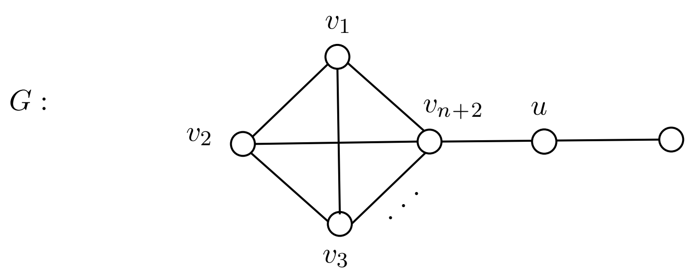

Figure 4: A graph G with $\gamma_{gr}^{ch}(G) - \gamma_{ch}(G) = n$

For (ii), consider the graph H given in Figure 5. Let $S' = (v_1, u, w)$ and $S'' = (v_1, v_2, \ldots, v_{n+2}, u)$. Then S' and S'' are γ_{gr}^{ch}- and γ_{gr}^{h}-sequences of H, respectively. Therefore, $\gamma_{gr}^{ch}(H) = 3$ and $\gamma_{gr}^{h}(H) = n+3$. Consequently, $\gamma_{gr}^{h}(H) - \gamma_{gr}^{ch}(H) = n+3-3 = n$.

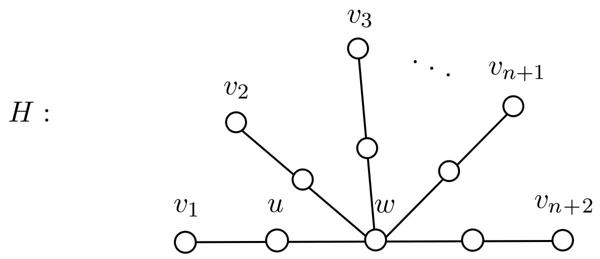

Figure 5: A graph H with $\gamma^h_{gr}(H) - \gamma^{ch}_{gr}(H) = n$

This proves the assertion. □

Corollary 4. *Let G be a connected graph. Then $\gamma^{ch}_{gr}(G) - \gamma_{ch}(G)$ and $\gamma^h_{gr}(G) - \gamma^{ch}_{gr}(G)$ can be made arbitrarily large.*

Next, we give a more general result (than Proposition 2) involving the connected hop domination, connected Grundy hop domination and Grundy hop domination parameters.

Theorem 3. *Let a and b be positive integers such that $3 \leq a \leq b$. Then each of the following holds.*

(i) *There exists a connected graph G such that $\gamma_{ch}(G) = a$ and $\gamma^{ch}_{gr}(G) = b$.*

(ii) *There exists a connected graph G' such that $\gamma^{ch}_{gr}(G') = a$ and $\gamma^h_{gr}(G') = b$.*

Proof. For $a = b$, consider $G = K_a$. Then $\gamma_{ch}(G) = a = \gamma^{ch}_{gr}(G) = \gamma^h_{gr}(G)$. Suppose now that $a < b$.

For (i), let $m = b - a$ and consider the following cases:

Case 1: a is odd.

Consider the graph G given in Figure 6, where $\langle\{y_1, y_2, \cdots, y_{m-1}, w\}\rangle$ is complete. One can verify that $S_1 = (v_1, v_2, \ldots, v_a\}$ and $S_2 = (y_1, y_2, \cdots, y_{m-1}, w, v_a, v_{a-1} \cdots, v_1)$ are γ_{ch}- and γ^{ch}_{gr}-sequences of G, respectively. Hence, $\gamma_{ch}(G) = a$ and $\gamma^{ch}_{gr}(G) = m + a = b$.

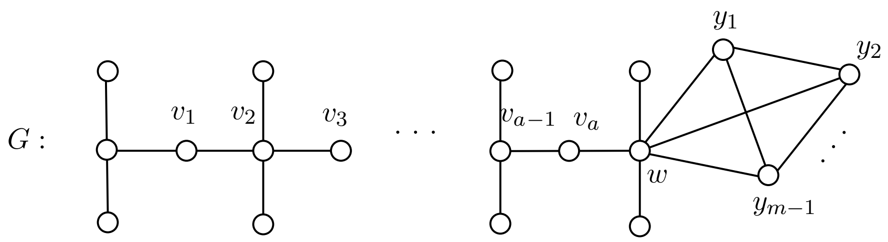

Figure 6: A graph G with $\gamma_{ch}(G) < \gamma_{gr}^{ch}(G)$

Case 2: a is even.

Consider the graph G' given in Figure 7, where $\langle\{y_1, y_2, \cdots, y_{m-1}, w\}\rangle$ is complete. One can verify that $S' = \{v_1, v_2, \ldots, v_a\}$ and $S'' = (y_1, y_2, \ldots, y_{m-1}, w, v_a, v_{a-1} \ldots, v_1)$ are γ_{ch}- and γ_{gr}^{ch}-sequences of G', respectively. Thus, $\gamma_{ch}(G') = a$ and $\gamma_{gr}^{ch}(G') = m + a = b$.

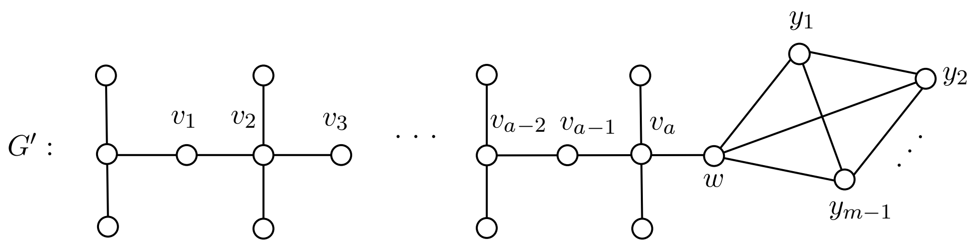

Figure 7: A graph G' with $\gamma_{ch}(G') < \gamma_{gr}^{ch}(G')$

For (ii), let $m = b - a + 1$. Consider the graph H given in Figure 8. One can verify that $C' = \{x_1, x_2, \cdots, x_a)$ and $C'' = (x_1, x_2, \cdots, x_{a-1}, y_1, y_2, \cdots, y_m)$ are γ_{gr}^{ch}-sequence and γ_{gr}^{h}-sequence of H, respectively. Therefore, $\gamma_{gr}^{ch}(H) = a$ and $\gamma_{gr}^{h}(H) = m + a - 1 = b$.

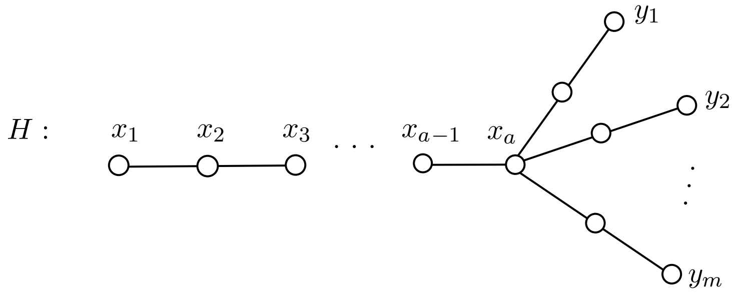

Figure 8: A graph H with $\gamma_{gr}^{ch}(H) < \gamma_{gr}^{h}(H)$

This proves the assertion. □

Proposition 3. *[8] For any positive integer $n \geq 2$,*

$$\gamma_{gr}^{h}(P_n) = \begin{cases} 2 & if\ n = 2,3 \\ n-2 & if\ n \geq 4. \end{cases}$$

Proposition 4. *For any positive integer $n \geq 2$,*

$$\gamma_{gr}^{ch}(P_n) = \begin{cases} 2 & if\ n = 2,3 \\ n-2 & if\ n \geq 4. \end{cases}$$

Proof. Let $P_n = [v_1, v_2, \cdots, v_n]$. Clearly, $\gamma_{gr}^{ch}(P_n) = 2$ for $n = 2, 3$. Next, suppose that $n \geq 4$. Let $S' = (v_1, v_2 \cdots, v_{n-2})$. Clearly, S' is a connected Grundy hop dominating sequence of P_n. Thus, $\gamma_{gr}^{ch}(P_n) \geq n-2$. Since $\gamma_{gr}^{h}(P_n) = n-2$ for all $n \geq 4$, it follows that $\gamma_{gr}^{ch}(P_n) \leq n-2$ for all $n \geq 4$ by Remark 2. Consequently, $\gamma_{gr}^{ch}(P_n) = n-2$ for all $n \geq 4$. □

Proposition 5. *Let G be a connected graph of order n. If $|N_G^2[v]| \geq m$ for every $v \in V(G)$, then $\gamma_{gr}^{ch}(G) \leq n-(m-1)$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose $\gamma_{gr}^{ch}(G) = k$, say $S = (s_1, s_2, \cdots, s_k)$ is a connected Grundy hop dominating sequence of G. Assume $s_1 = v_i$ for some $i \in \{1, \ldots, n\}$. Then $|N_G^2[s_1]| = |N_G^2[v_i]| \geq m$ by assumption. It follows that there are at most $n-m$ remaining vertices of G that could be hop footprinted by the next terms of S. Therefore, $\gamma_{gr}^{ch}(G) = k \leq n-m+|\{v_i\}| = n-m+1 = n-(m-1)$. □

The next result follows immediately form Proposition 5.

Corollary 5. *Let G be a connected graph on n vertices. If $|N_G^2[u]| = 3$ for every $u \in V(G)$, then $\gamma_{gr}^{ch}(G) \leq n-2$.*

Proposition 6. *[8] For any positive integer $n \geq 3$,*

$$\gamma_{gr}^{h}(C_n) = \begin{cases} 3 & if\ n = 3 \\ 2 & if\ n = 4 \\ n-4 & if\ n \geq 6\ and\ even \\ n-2 & if\ n \geq 5\ and\ odd. \end{cases}$$

Proposition 7. *For any positive integer $n \geq 3$,*

$$\gamma_{gr}^{ch}(C_n) = \begin{cases} 2 & if\ n = 4 \\ 3 & if\ n = 3, 5 \\ n-4 & if\ n \geq 6\ and\ even \\ n-3 & if\ n \geq 7\ and\ odd \end{cases}$$

Proof. Let $G = C_n = [v_1, v_2, \cdots, v_n, v_1]$. Clearly, $\gamma_{gr}^{ch}(C_3) = 3 = \gamma_{gr}^{ch}(C_5)$ and $\gamma_{gr}^{ch}(C_4) = 2$. Suppose that $n \geq 6$ and is even. Let $S = (v_1, v_2, \cdots, v_{n-4})$. Then $N_G^2[v_2] \setminus N_G^2[v_1] = \{v_2, v_4, v_n\} \neq \varnothing$ and $v_{i+2} \in N_G^2[v_i] \setminus \cup_{j=1}^{i-1} N_G^2[v_j]$ for all $i \in \{3, 4, \cdots, n-4\}$. It follows that S is a connected Grundy hop dominating sequence. Hence, $\gamma_{gr}^{ch}(C_n) \geq |\hat{S}| = n-4$. Now, since $\gamma_{gr}^{h}(C_n) = n-4$ for even integers $n \geq 6$, it follows that $\gamma_{gr}^{ch}(C_n) \leq n-4$ by Proposition 2. Consequently, $\gamma_{gr}^{ch}(C_n) = n-4$ for all even integers $n \geq 6$.

Next, suppose that $n \geq 7$ and is odd. Let $S = (v_1, v_3, \cdots, v_{n-4}, v_{n-3}, v_{n-5}, \cdots, v_2)$. Then S is a maximum connected Grundy hop dominating sequence of C_n. Hence, $\gamma_{gr}^{ch}(G) = n-3$ for all $n \geq 7$ and odd. □

Lemma 1. *[11] Let G be a non-trivial connected graph and let G_1 and G_2 be two copies of G in the graph $S(G)$. If $w \in V(G_1)$ and $w' \in V(G_2)$ is the corresponding vertex of w, then*

$$N_{S(G)}^2[w] = N_{G_1}^2[w] \cup N_{G_2}^2[w'] = N_{S(G)}^2[w'].$$

In what follows, if G_1 and G_2 are copies of G in the shadow graph $S(G)$, and $D \subseteq V(G_1)$, and $Q \subseteq V(G_2)$, then the sets D' and Q' are given by $D' = \{v' \in V(G_2) : v \in D\}$ and $Q' = \{w \in V(G_1) : w \in Q\}$.

Theorem 4. *Let G be a non-trivial connected graph and let G_1 and G_2 be copies of G in the shadow graph $S(G)$. Then C is a connected hop dominating set of $S(G)$ if and only if one of the following conditions holds:*

(i) *C is a connected hop dominating set in G_1.*

(ii) *C is a connected hop dominating set in G_2.*

(iii) *$C = C_{G_1} \cup C_{G_2}$ such that $C_{G_1} \cup C'_{G_2}$ and $C'_{G_1} \cup C_{G_2}$ are connected hop dominating sets of G_1 and G_2, respectively.*

Proof. Let $C_{G_1} = C \cap V(G_1)$ and $C_{G_2} = C \cap V(G_2)$. If $C_{G_2} = \varnothing$, then $C = C_{G_1}$ is a connected hop dominating set of G_1. If $C_{G_1} = \varnothing$, then $C = C_{G_2}$ is a connected hop dominating set of G_2. Hence, (i) or (ii) holds. Next, suppose $C_{G_1} \neq \varnothing$ and $C_{G_2} \neq \varnothing$. Let $x \in V(G_1) \setminus (C_{G_1} \cup C'_{G_2})$. Then $x \in V(S(G)) \setminus C$. Since C is a hop dominating of $S(G)$, there exists $y \in C$ such that $d_{S(G)}(x, y) = 2$. If $y \in C_{G_1}$, then we are done. Suppose $y \in C_{G_2}$, say $y = z'$, where $z \in V(G_1)$. Then $z \in C'_{G_2}$ and $d_{S(G)}(x, y) = d_{G_1}(x, z) = 2$ by Lemma 1. Therefore, $C_{G_1} \cup C'_{G_2}$ is a hop dominating set of G_1. Clearly, $\langle C_{G_1} \cup C'_{G_2} \rangle$ is connected. Consequently, $C_{G_1} \cup C'_{G_2}$ is a connected hop dominating set of G_1. Similarly, $C'_{G_1} \cup C_{G_2}$ is a connected hop dominating set of G_2. Hence, (iii) holds.

Conversely, suppose (i) holds. Let $a \in V(S(G)) \setminus C$. If $a \in V(G_1)$, then there exists $b \in C$ such that $d_{G_1}(a, b) = d_{S(G)}(a, b) = 2$. Suppose $a \in V(G_2)$, say $a = u'$, where $u \in V(G_1)$. If $u \in C$, then $d_{G_1}(a, u) = d_{S(G)}(a, u) = 2$. If $u \notin C$, then there exists $v \in C$ such that $d_{G_1}(u, v) = 2$. It follows that $d_{S(G)}(a, u) = d_{S(G)}(u', v) = 2$. Therefore, C is a hop dominating set of $S(G)$. Clearly, $\langle C \rangle$ is connected. Consequently, C is a connected hop dominating set of $S(G)$. Similarly, if (ii) holds, then C is a connected hop dominating set of $S(G)$. Next, suppose that (iii) holds. Let $x \in V(S(G)) \setminus C$. Then $x \notin C_{G_1} \cup C_{G_2}$. Suppose $x \in V(G_2) \setminus C_{G_2}$, say $x = y'$, where $y \in V(G_1)$. Then $y \notin C'_{G_2}$. If $y \in C_{G_1}$, then $d_{S(G)}(x, y) = d_{S(G)}(y', y) = 2$. Suppose $y \notin C_{G_1}$. Since $C_{G_1} \cup C'_{G_2}$ is a hop dominating set of G_1, there exists $w \in C_{G_1} \cup C'_{G_2}$ such that $d_{G_1}(w, y) = 2 = d_{S(G)}(w, y)$. If $w \in C_{G_1}$, then $w \in C$ and $d_{S(G)}(w, y') = 2$ by Lemma 1. If $w \in C'_{G_2}$, then $w' \in C_{G_2} \subseteq C$ and $d_{G_2}(w', y') = d_{S(G)}(w', y') = 2$ by Lemma 1. Therefore, C is a hop dominating set of $S(G)$. Clearly, $\langle C \rangle$ is connected. Consequently, C is a connected hop dominating set of $S(G)$. □

The next result follows from Theorem 4.

Corollary 6. *Let G be a non-trivial connected graph and let G_1 and G_2 be copies of G in the shadow graph $S(G)$. Then $\gamma_{ch}(S(G)) = \gamma_{ch}(G)$.*

Theorem 5. *Let G be a non-trivial connected graph and let G_1 and G_2 be copies of G in the shadow graph $S(G)$. If S is a connected Grundy hop dominating sequence of G_1 or G_2, then S is a connected Grundy hop dominating sequence of $S(G)$. In particular, $\gamma^{ch}_{gr}(G) \leq \gamma^{ch}_{gr}(S(G))$.*

Proof. Suppose $S = (v_1, v_2, \ldots, v_k)$ is a connected Grundy hop dominating sequence of G_1. Then $\hat{S}$ is a connected hop dominating set of G_1. Hence, $\hat{S}$ is a connected hop dominating set of $S(G)$ by Theorem 4. Let $i \in \{2, 3, \ldots, k\}$. Since $N^2_{G_1}[v_i] \cap (\cup_{j=1}^{i-1} N^2_{G_2}[v'_j]) = \varnothing$ and $N^2_{G_2}[v'_i] \cap (\cup_{j=1}^{i-1} N^2_{G_1}[v_j]) = \varnothing$, Lemma 1 implies that

$$N^2_{S(G)}[v_i] \setminus \cup_{j=1}^{i-1} N^2_{S(G)}[v_j] = (N^2_{G_1}[v_i] \setminus (\cup_{j=1}^{i-1} N^2_{G_1}[v_j]) \cup (N^2_{G_2}[v'_i] \setminus (\cup_{j=1}^{i-1} N^2_{G_2}[v'_j]).$$

By the legality property of S,

$$\varnothing \neq N^2_{G_1}[v_i] \setminus \cup_{j=1}^{i-1} N^2_{G_1}[v_j] \subseteq N^2_{S(G)}[v_i] \setminus \cup_{j=1}^{i-1} N^2_{S(G)}[v_j].$$

Thus, S is a legal closed hop neighborhood sequence in $S(G)$, showing that S is a connected Grundy hop dominating sequence in $S(G)$. Similarly, S is a connected Grundy hop dominating sequence in $S(G)$ if S is a connected Grundy hop dominating sequence in G_2. Therefore, $\gamma^{ch}_{gr}(G) \leq \gamma^{ch}_{gr}(S(G))$. □

Lemma 2. *Let G be a connected graph of order n. If $|N_G[v]| \geq k$ for every $v \in V(G)$, then $\gamma_{gr}(G) \leq n-(k-1)$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose $\gamma_{gr}(G) = t$, say $S = (s_1, s_2, \cdots, s_t)$ is a Grundy dominating sequence of G. Then $|N_G[s_1]| \geq k$ by assumption. It follows that there are at most $n-k$ remaining vertices of G that could be footprinted by the next terms of S. Therefore, $\gamma^c_{gr}(G) = t \leq n-k+1$. □

Proposition 8. *Let n be a positive integer. Then*

$$\gamma_{gr}(\overline{P}_n) = \begin{cases} 1, & n = 1, \\ 2, & n = 2, 3 \\ 3, & n \geq 4. \end{cases}$$

Proof. Clearly, for $n = 1$ and $n = 2, 3$, $\gamma_{gr}(\overline{P}_n) = 1$ and $\gamma_{gr}(\overline{P}_n) = 2$, respectively. Let $\{v_1, v_2, \cdots, v_n\}$ be a vertex set of $G = \overline{P}_n$. Let $= \{v_2, v_4, v_3\}$. Notice that S is a Grundy dominating sequence of G. Hence, $\gamma_{gr}(G) \geq 3$.
Now, notice that $|N_G[v_i]| = n-2$ for every i, where $i \neq 1, n$. Fix i. If we let v_i to be the first element of a Grundy dominating sequence S', then there are only two remaining vertices which are not in $N_G[v_i]$. Thus, $\gamma_{gr}(G) \leq 3$. Therefore, $\gamma^h_{gr}(G) = 3$. □

Proposition 9. *Let n be a positve integer. Then*

$$\gamma_{gr}(\overline{C}_n) = \begin{cases} 3, & n = 3, \\ 2, & n = 4 \\ 3, & n \geq 5. \end{cases}$$

Proof. Clearly, for $n = 3$ and $n = 4$, $\gamma_{gr}(\overline{C}_n) = 3$ and $\gamma_{gr}(\overline{C}_n) = 2$, respectively. Let $\{v_1, v_2, \cdots, v_n\}$ be a vertex set of $G = \overline{C}_n$. Let $= \{v_1, v_3, v_2\}$. Notice that S is a Grundy dominating sequence of G. Hence, $\gamma_{gr}(G) \geq 3$.
Now, notice that $|N_G[v_i]| = n-2$ for every $i \in \{1, \cdots, n\}$. Fix i. If we let v_i to be the first element of a Grundy dominating sequence S', then there are only two remaining vertices which are not in $N_G[v_i]$. Thus, $\gamma_{gr}(G) \leq 3$. Therefore, $\gamma_{gr}(G) = 3$. □

The next result is the correction of a result found in an earlier paper of the authors in [8].

Theorem 6. *Let G and H be two graphs and let C be a sequence of distinct vertices of $G + H$. Then C is a legal closed hop neighborhood sequence in $G + H$ if and only if one of the following holds:*

(*i*) *C is a co-legal closed neighborhood sequence in G.*

(*ii*) *C is a co-legal closed neighborhood sequence in H.*

(*iii*) *The subsequences C_G and C_H of C, where $\hat{C} = \hat{C}_G \cup \hat{C}_H$, are co-legal closed neighborhood sequences of G and H, respectively.*

Proof. Assume that $C = (a_1, a_2, \cdots, a_m)$ is a legal closed hop neighborhood sequence of $G+H$. If $\hat{C} \subseteq V(G)$, then $N^2_G[a_i] = N^2_{G+H}[a_i] = V(G) \setminus N_G(a_i)$ for each $i \in \{1, 2, \ldots, m\}$. Thus, by the legality condition property of C,

$$[V(G) \setminus N_G(a_i)] \setminus \cup_{j=1}^{i-1}[V(G) \setminus N_G(a_j)] \neq \varnothing \text{ for each } \ i \in \{2, 3, \ldots, m\}.$$

It follows that C is a co-legal closed neighborhood sequence in G. Hence, (*i*) holds. Similarly, if $\hat{C} \subseteq V(H)$, then (*ii*) holds. Next, suppose that $\hat{C} \cap V(G) \neq \varnothing$ and $\hat{C} \cap V(H) \neq \varnothing$. Let C_G and C_H be subsequences of C such that $\hat{C}_G = \hat{C} \cap V(G)$ and $\hat{C}_H = \hat{C} \cap V(H)$. Let $C_G = (a_{n_1}, a_{n_2}, \ldots, a_{n_t})$ and $C_H = (a_{m_1}, a_{m_2}, \ldots, a_{m_r})$. Note that $N^2_{G+H}[a_{n_j}] \subseteq V(G)$ for all $j \in \{1, 2, \ldots, t\}$ and $N^2_{G+H}[a_{m_s}] \subseteq V(H)$ for all $s \in \{1, 2, \ldots, r\}$. Since C is a connected legal closed hop neighborhood sequence in $G+H$, it follows that

$$[V(G) \setminus N_G(a_{n_i})] \setminus \cup_{j=1}^{i-1}[V(G) \setminus N_G(a_{n_j})] = N^2_{G+H}[a_{n_i}] \setminus \cup_{j=1}^{i-1} N^2_{G+H}[a_{n_j}] \neq \varnothing$$

for all $i \in \{2, 3, \ldots, t\}$. Hence, C_G is a co-legal closed neighborhood sequence in G. Similarly, C_H is a co-legal closed neighborhood sequence in H. Therefore, (*iii*) holds.

For the converse, suppose (*i*) or (*ii*) holds. Then C is a legal closed hop neighborhood sequence in $G+H$. Suppose (*iii*) holds. Let $C = (a_1, a_2, \cdots, a_m)$ and let $C_G = (a_{n_1}, a_{n_2}, \ldots, a_{n_t})$ and $C_H = (a_{m_1}, a_{m_2}, \ldots, a_{m_r})$. Let $i \in \{2, 3, \ldots, m\}$. Suppose $a_i \in \hat{C}_G$. Then $a_i = a_{n_k}$ for some $k \in \{1, 2, \ldots, t\}$. Since C_G is a co-legal closed neighborhood sequence in G,

$$\begin{aligned} N^2_{G+H}[a_i] \setminus \cup_{j=1}^{i-1} N^2_{G+H}[a_j] &= N^2_{G+H}[a_{n_k}] \setminus \cup_{p=1}^{k-1} N^2_{G+H}[a_{n_p}] \\ &= [V(G) \setminus N_G(a_{n_k})] \setminus \cup_{p=1}^{k-1}[V(G) \setminus N_G(a_{n_p})] \neq \varnothing. \end{aligned}$$

If $a_i \in \hat{C}_H$, then $a_i = a_{m_q}$ for some $q \in \{1, 2, \ldots, m\}$. Since C_H is a co-legal closed neighborhood sequence in H,

$$\begin{aligned} N^2_{G+H}[a_i] \setminus \cup_{j=1}^{i-1} N^2_{G+H}[a_j] &= N^2_{G+H}[a_{m_q}] \setminus \cup_{r=1}^{q-1} N^2_{G+H}[a_{m_r}] \\ &= [V(H) \setminus N_H(a_{m_q})] \setminus \cup_{r=1}^{q-1}[V(H) \setminus N_H(a_{m_r})] \neq \varnothing. \end{aligned}$$

This shows that C is a legal closed hop neighborhood sequence in $G+H$. □

Theorem 7. *Let G and H be two graphs and let C be a sequence of distinct vertices of $G+H$. Then C is a connected Grundy hop dominating sequence in $G+H$ if and only if the subsequences C_G and C_H of C, where $\hat{C} = \hat{C}_G \cup \hat{C}_H$, are co-Grundy dominating sequences in G and H, respectively.*

Proof. Suppose C is a connected Grundy hop dominating sequence in $G+H$. Since C is a legal closed hop neighborhood sequence and $\hat{C}$ is a connected hop dominating set in $G + H$, it follows that C satisfies (iii) of Theorem 6. Hence, the subsequences C_G and C_H of C, where $\hat{C} = \hat{C}_G \cup \hat{C}_H$, are co-legal closed neighborhood sequences in G and H, respectively. Since $\hat{C}$ is a hop dominating set in $G + H$, $V(G) = \cup_{v \in C_G}[V(G) \setminus N_G(v)]$ and $V(H) = \cup_{w \in C_H}[V(H) \setminus N_H(w)]$. Thus, C_G and and C_H are co-Grundy dominating sets in G and H, respectively.

Conversely, suppose that the subsequences C_G and C_H of C, where $\hat{C} = \hat{C}_G \cup \hat{C}_H$, are co-Grundy dominating sequences in G and H, respectively. By Theorem 6, C is a legal closed hop neighborhood sequence in $G + H$. Since C_G and C_H co-Grundy dominating sequences in G and H, respectively, $\hat{C}$ is a hop dominating set. Moreover, because $\langle \hat{C} \rangle$ is connected, C is a connected Grundy hop dominating sequence in $G + H$. □

The following result follows from Theorem 7, Proposition 8, and Proposition 9.

Corollary 7. *Let G and H be graphs. Then*

$$\gamma_{gr}^{ch}(G + H) = \gamma_{cogr}(G) + \gamma_{cogr}(H).$$

In particular, each of the following holds.

(i) $\gamma_{gr}^{ch}(K_1 + G) = 1 + \gamma_{cogr}(G)$.

(ii) $\gamma_{gr}^{ch}(K_{m,n}) = 2$ *for* $m, n \geq 1$.

(iii) $\gamma_{gr}^{ch}(W_n) = 4$ *for all* $n \geq 5$.

(iv) $\gamma_{gr}^{ch}(F_n) = 4$ *for all* $n \geq 4$.

(v) $\gamma_{gr}^{ch}(P_n + P_m) = 6$ *for all* $n, m \geq 4$.

(vi) $\gamma_{gr}^{ch}(C_n + C_m) = 6$ *for all* $n, m \geq 5$.

(vii) $\gamma_{gr}^{ch}(P_n + C_m) = 6$ *for all* $n \geq 4$ *and* $m \geq 5$.

4. Conclusion

In this study, connected Grundy hop domination numbers of some graphs are determined. Realization results involving connected hop domination, Grundy hop domination, and connected Grundy hop domination numbers are given. For the shadow graph $S(G)$ of a non-trivial graph G, it is conjectured that $\gamma_{gr}^{ch}(S(G)) = \gamma_{gr}^{ch}(G)$. Connected Grundy hop domination can still be studied further.

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