



The Homotopy Analysis Method to Solve the Nonlinear System of Volterra Integral Equations and Applying the Genetic Algorithm to Enhance the Solutions

Rasha F. Ahmed¹, Waleed Al-Hayani^{1,*}, Abbas Y. Al-Bayati²

¹ *Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Mosul, Iraq*

² *Department of Mathematics, College of Basic Education, University of Telafer, Mosul, Iraq*

Abstract. This paper presents the application of the Homotopy Analysis Method (HAM) for solving nonlinear system of Volterra integral equations used to obtain a reasonably approximate solution. The HAM contains the auxiliary parameter h which provides a simple way to adjust and control the convergence region of the solution series. The results show that the HAM is a very effective method as well. The results were compared with the solutions obtained by developing a homotopy analysis method using the genetic algorithm (HAM-GA), considering the residual error function as a fitness function of the genetic algorithm.

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1. Introduction

The integral equations can be used to describe a wide variety of problems in science and engineering, such as the population dynamics, the spread of diseases, some Dirichlet problems in potential theory, electrostatics, mathematical modeling of radioactive equilibrium, particle transport problems of astrophysics and reactor theory, radiative energy and/or heat transfer problems, other general heat transfer problems, oscillation of strings and membranes, the problem of momentum representation in quantum mechanics, etc. However, various additional challenging problems in mechanics, astrophysics, chemistry, and mathematics can also be represented in terms of Volterra integral equations. In

*Corresponding author.

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Email addresses: rasha.20csp146@student.uomosul.edu.iq (Rasha F. Ahmed),

waleedalhayani@uomosul.edu.iq (Waleed Al-Hayani),

profabbasalbayati@yahoo.com (Abbas Y. Al-Bayati)

addition, there are several real-world problems where urges naturally develop. A differential equation, an integral equation, an integro-differential equation, or a system of these equations together can be used to describe problems that either arise naturally (as in population dynamics or many biological applications) or are brought about by a control system [9–11]. A numerical method is frequently required because systems of integral and/or integro-differential equations, particularly nonlinear systems of Volterra integral equations or with variable coefficients, are typically challenging to solve analytically. Therefore, there are many research papers to solve such systems using several analytical methods (see [4, 5, 7, 15, 29]), etc. Many researchers [3, 36, 37] have tried to solve the Volterra integral equation using different analytical and numerical methods. Recently, Younis and Al-Hayani [50] have been used Adomian decomposition method (ADM) for solving fuzzy system of Volterra integro-differential equations. The researchers [39, 40, 46–49] have been applied different analytical methods such as optimal homotopy asymptotic method (OHAM), homotopy perturbation method (HPM) and Padé approximation technique to solve the mathematical models as (porous rotating disk electrodes, ECE reactions at rotating disk electrodes, magnetohydrodynamic, etc.). So, in this paper, we consider the nonlinear system of Volterra Integral Equations (NLSVIEs) [2]:

$$U(x) = F(x) + \int_{f(x)}^{g(x)} K(x, t) R[U(t)] dt, \tag{1}$$

where

$$\begin{aligned} U(x) &= [u_1(x), u_2(x), \dots, u_n(x)]^T, \\ F(x) &= [f_1(x), f_2(x), \dots, f_n(x)]^T, \\ K(x, t) &= [k_{ij}(x, t)], \quad i, j = 1, 2, \dots, n \end{aligned} \tag{2}$$

Consider the i th equation of Eq 1

$$u_i(x) = f_i(x) + \int_{f(x)}^{g(x)} \sum_{j=1}^n [K_{i,j}(x, t) R(u_j(t))] dt, \quad i = 1, 2, \dots, n \tag{3}$$

where the unknown functions $u_i(x) \in C[a, b], x \in [a, b]$, the functions $f_i(x) \in C[a, b], f, g \in C[a, b], a \leq f(x) \leq g(x) \leq b$. The non-negative kernel functions $K_{i,j}(x, t) \in C[a, b] \times C[a, b], i = 1, 2, \dots, n$ and $R : C[a, b] \rightarrow C[a, b]$, are nonlinear operator. Therefore, R , the operator, is assumed to satisfy the Lipschitz condition, $\|R(v_1) - R(v_2)\| \leq s \|v_1 - v_2\|$ for every $v_1, v_2 \in C[a, b]$ and some $s > 0$. As the norm of the function, we take the supremum norm $\|v\| = \sup_{x \in [a, b]} |v(x)|$, in particular $\|K\| = \sup_{(x,t) \in [a,b] \times [a,b]} |K(x, t)|$ and $\|F\| = \sup_{x \in [a, b]} |F(x)|$. Shijun Liao invented the HAM [24–27, 41] by which many types of equations can be resolved [24]. The method by Shijun Liao is mainly derived from topology, and can approach to the exact solution. Some examples of using this method for solving integral equations well discussed [17–19]. However, the researcher was able to prove the convergence of HAM [31]. This method has been successfully applied to solve many types of

linear and nonlinear problems [1, 13, 14, 16, 30, 32, 33, 38, 45], In 2020, researchers were able to solve Two-Dimensional nonlinear Volterra-Fredholm fuzzy integral equations by HAM [15]. Also, others were able to solve Mixed Volterra-Fredholm integro-differential equations using same method [16].

In the last years, the researchers [35, 43] have been applied the HAM to solve the mathematical models as (porous rotating disk electrodes and ECE reactions at rotating disk electrodes). Finally, Liao [28] published an explanation of how to avoid “small denominator problems” by the HAM.

The main objective of the present study is to improve the results obtained from HAM application to the system of Volterra integral equations using the genetic algorithm by considering the residual error function as a fitness function and finding the best values for the constants. In particular, this study demonstrates that for reasonable assumptions, the analyzed equations have both existent and unique solutions. Also, a comparison of the solutions with that obtained by developing a HAM using the genetic algorithm was performed considering the residual error function as a fitness function of the genetic algorithm, as shown in section 4 and listed in several examples.

2. Fundamental of the HAM

The HAM is applied for solving the operator equations

$$N[u(x)] = 0, \quad x \in \Omega \tag{4}$$

where N refers to general operator (linear or non-linear), while $u(x)$ is an unknown function. First, we define the homotopy operator \mathbb{H} as follows:

$$\mathbb{H}(\phi, q) = (1 - q)\mathcal{L}[\phi(x; q) - u_0(x)] + qh\mathcal{N}[\phi(x; q)], \tag{5}$$

where $q \in [0, 1]$ is an embedding parameter, h is a non-zero auxiliary function (denotes the convergence control parameter [23, 26, 27, 44]), \mathcal{L} is an auxiliary linear operator which has the property $\mathcal{L}(0) = 0$, $u_0(x)$ is an initial guess of $u(x)$ and $\phi(x; q)$ is an unknown function. The zero-order deformation equation of the operator $\mathbb{H}(\phi, q) = 0$ is considered as follow:

$$(1 - q)\mathcal{L}[\phi(x; q) - u_0(x)] = qh\mathcal{N}[\phi(x; q)], \tag{6}$$

Obviously, when $q=0$ we have $\mathcal{L}[\phi(x; q) - u_0(x)] = 0$, that implies $\phi(x; 0) = u_0(x)$. While, when $q = 1$, we have $\mathcal{N}[\phi(x; q)] = 0$, which leads to $\phi(x; 1) = u(x)$, where $u(x)$ is the solution of Equation 4. in this case, the variation of parameter q from zero to one coincides with the variation of problem from the initial guess $u_0(x)$ to the non-trivial solution $u(x)$. Using the Maclaurin series for expanding the function $\phi(x; q)$ with respect the parameter q , we get:

$$\phi(x; q) = \phi(x; 0) + \sum_{m=1}^{\infty} \frac{1}{m!} \left. \frac{\partial^m \phi(x; q)}{\partial q^m} \right|_{q=0}, \tag{7}$$

By considering

$$u_m = \frac{1}{m!} \left. \frac{\partial^m \phi(x; q)}{\partial q^m} \right|_{q=0}, \quad m = 1, 2, 3, \tag{8}$$

Equation 7 can be formulated as follow:

$$\phi(x; q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m, \tag{9}$$

If the series in Equation 9 converge at $q = 1$ then we obtain the solution

$$u(x) = \sum_{m=0}^{\infty} u_m(x). \tag{10}$$

we differentiate both sides of Equation 10 m times with respect to the embedding parameter q in order to determine functions u_m . Then, we divide the received finding by $m!$ and then setting $q = 0$. In this way, we deduce the so-called m_{th} order deformation equation ($m > 0$):

$$\mathcal{L}[u_m(x) - \chi_m u_{m-1}(x)] = h \mathbf{R}_m(u_{m-1}) \tag{11}$$

where

$$\vec{u}_{m-1} = \{u_0(x), u_1(x), \dots, u_{m-1}(x)\}.$$

and

$$\chi_m = \begin{cases} 0, & m = 1 \\ 1, & m > 1 \end{cases}, \tag{12}$$

and

$$\mathbf{R}_m(\vec{u}_{m-1}(x)) = \frac{1}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial q^{m-1}} \mathcal{N} \left(\sum_{i=0}^{\infty} u_i(x) q^i \right) \right) \Big|_{q=0}, \tag{13}$$

In case of inability to find the sum of the series in Equation 10, the partial sum was determined of this series.

$$\hat{u}_n(x) = \sum_{m=0}^n u_m(x), \tag{14}$$

is the approximate solution of the considered equation.

Choosing convenient amount of the parameter h has a great influence on the region of convergence of the series in Equation 10 and the convergence rate as well [27, 34, 42]. One of the methods for selecting the value of convergence control parameter is Genetic Algorithm (GA).

3. Description of the Method

Consider the operators \mathcal{L} and \mathcal{N} are defined as follows

$$\mathcal{L}(u_i) = u_i, \quad \mathcal{N}(u_i) = u_i(x) - f_i(x) - \int_{f(x)}^{g(x)} \left[\sum_{j=1}^n K_{ij}(x, t) R(u_i(t)) \right] dt, \quad i = 1, 2, \dots, n, \tag{15}$$

Let $u_{i,0} \in C[a, b]$, $i = 1, 2, \dots, n$. In this case, by considering the HAM, we obtain the following formula for function $u_{i,m}$, $i = 1, 2, \dots, n$, $m \geq 1$:

$$u_{i,m} = \chi_m u_{i,m-1}(x) + h \mathbf{R}_m(\vec{u}_{i,m-1}(x)) \tag{16}$$

where χ_m and \mathbf{R}_m are defined as in Equations 12 and 13 respectively.

Using the definitions of the respective operators, we obtain

$$\begin{aligned} \mathbf{R}_{i,m}(\vec{u}_{i,m-1}, x) &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \mathcal{N} \left(\sum_{k=0}^1 u_{i,k}(x) q^k \right) \Bigg|_{q=0}, \tag{17} \\ &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[\sum_{k=1}^1 u_{i,k}(x) q^k - f_i(x) - \int_{f(x)}^{g(x)} \left(\sum_{j=1}^n K_{ij}(x, t) R \left(\sum_{k=1}^1 u_{j,k}(t) q^k \right) \right) \right] \Bigg|_{q=0}, \\ &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[\sum_{k=1}^1 u_{i,k}(x) q^k - f_i(x) - \sum_{k=1}^1 \left[\int_{f(x)}^{g(x)} \sum_{j=1}^n K_{ij}(x, t) R(u_{i,k}(t)) q^k dt \right] \right] \Bigg|_{q=0}, \\ &= \frac{1}{(m-1)!} \left((m-1)! u_{i,m-1}(x) - (1 - \chi_m) f_i(x) - \int_{f(x)}^{g(x)} \sum_{j=1}^n \left[K_{ij}(x, t) (m-1)! R(u_{i,m-1}(t)) \right] \right), \\ &= u_{i,m-1}(x) - \frac{1 - \chi_m}{(m-1)!} f_i(x) - \int_{f(x)}^{g(x)} \left[\sum_{j=1}^n K_{ij}(x, t) R(u_{i,m-1}(t)) \right] dt. \end{aligned}$$

Utilizing the above relation and Equation 3 we get the following formula:

$$u_{1,i} = h \left(u_{0,i} - f_i(x) - \int_{f(x)}^{g(x)} \left[\sum_{j=1}^n K_{i,j}(x, t) R(u_{0,i}(t)) \right] dt \right).$$

And form $m = 2$:

$$u_{i,m}(x) = (1 + h) u_{i,m-1}(x) + \frac{h}{(m-1)!} \int_{f(x)}^{g(x)} \left[\sum_{j=1}^n K_{i,j}(x,t) \left(\frac{\partial^{m-1}}{\partial q^{m-1}} R \left(\sum_{i=0}^1 u_{i,m-1}(t) q^i \right) \right) \right] dt. \tag{18}$$

In literature, one can find the expression $\frac{\partial^{m-1}}{\partial q^{m-1}} R \left(\sum_{i=0}^1 u_{i,m-1}(t) q^i \right)$, calculated for various nonlinear operators R . Most of these results were collected in monograph [27]. Now, we turn to prove under proper assumptions Equation 6 which has a unique solution.

Theorem 1. Consider the system of VIEs as in Equation 3, such that $\|K_{ij}(x, t)\| \leq M_{ij}$, and $\|M\| = \max_{i,j} M_{ij}$, $i, j = 1, 2, \dots, n$. If the following condition is satisfied

$$\|M\| \|b - a\| < 1. \tag{19}$$

Then Equation 3 possess at most one solution.

Proof. Suppose that there exist two solutions $u_{i,1}$ and $u_{i,2}$, we have

$$\begin{aligned} \|u_{i,1} - u_{i,2}\| &= \left\| \int_{f(x)}^{g(x)} \left[\sum_{j=1}^n K_{ij}(x,t) (R(u_{j,1}(t)) - R(u_{j,2}(t))) \right] dt \right\| \\ &\leq \|M\| \int_{f(x)}^{g(x)} \sum_{j=1}^n \|R(u_{j,1}) - R(u_{j,2})\| dt \\ &\leq \|M\| \int_{f(x)}^{g(x)} \sum_{j=1}^n \|R(u_{j,1}) - R(u_{j,2})\| dt \leq \|M\| \|b - a\| \|u_{i,1} - u_{i,2}\|. \end{aligned}$$

Hence, we obtain

$$(1 - \|M\| \|b - a\|) \|u_{i,1} - u_{i,2}\| = 0. \tag{20}$$

So, if the condition 19 is satisfied, then the quality $u_{i,1} = u_{i,2}$, $i = 1, 2, \dots, n$ must be true. we now proceed to prove the theorem to ensure that the sum of the given series is the solution to the equation discussed.

Theorem 2. Let the functions $u_{i,m}$, $i = 1, 2, \dots, n$, $m = 0, 1, 2, \dots$, be defined in Equations 17 and 18. then, if $(s < 1)$ and the series

$$u_i(x) = \sum_{m=0}^{\infty} u_{i,m}(x), \quad i = 1, 2, \dots, n, \tag{21}$$

is convergent, then the sum of this series is the solution of Equation 3.

Proof. Let the series in Equation 10 is convergent. Then, from the necessary condition for the convergent series, we conclude that for any $x \in [a, b]$:

$$\lim_{m \rightarrow \infty} u_{i,m}(x) = 0, \quad i = 1, 2, \dots, n.$$

Now, let $T_{i,m}$ has the form:

$$T_{i,m} = \frac{1}{m!} \left(\frac{\partial^m}{\partial q^m} R \left(\sum_{k=0}^{\infty} u_{i,k}(x) q^k \right) \right) \Big|_{q=0}, \quad i = 1, 2, \dots, n.$$

If R is the contraction mapping and ($s < 1$), the series in Equation 21 converges to $u_i(x)$, then the series $\sum_{m=0}^{\infty} T_{i,m}$ converges to $R(u_i(x))$ (see [12]). Utilizing the definition of the operator \mathcal{L} we can write

$$\begin{aligned} \sum_{m=1}^{\infty} \mathcal{L}(u_{i,m}(x) - \chi_m u_{i,m-1}(x)) &= \sum_{m=1}^{\infty} (u_{i,m}(x) - \chi_m u_{i,m-1}(x)), \\ &= u_{i,1}(x) + (u_{i,2}(x) - u_{i,1}(x)) + (u_{i,3}(x) - u_{i,2}(x)) + \dots + (u_{i,n}(x) - u_{i,n-1}(x)) \\ &= u_{i,n}(x). \end{aligned}$$

Hence

$$\sum_{m=1}^{\infty} \mathcal{L}(u_{i,m}(x) - \chi_m u_{i,m-1}(x)) = \lim_{n \rightarrow \infty} u_{i,n}(x) = 0$$

From Equation 11 we get

$$h \sum_{m=1}^{\infty} R_{i,m}(\vec{u}_{i,m-1}, x) = \sum_{m=1}^{\infty} \mathcal{L}(u_{i,m}(x) - \chi_m u_{i,m-1}(x)), \quad i = 1, 2, \dots, n.$$

and since $h \neq 0$ thus we have

$$\sum_{m=1}^{\infty} R_{i,m}(\vec{u}_{i,m-1}, x) = 0, \quad i = 1, 2, \dots, n.$$

As a result of some transformations, we successively get

$$\begin{aligned} 0 &= \sum_{m=1}^{\infty} R_{i,m}(\vec{u}_{i,m-1}, x), \\ &= \sum_{m=1}^{\infty} \left(\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \sum_{k=1}^{\infty} u_{i,k}(x) q^k - f_i(x) \right. \\ &\quad \left. \int_{f(x)}^{g(x)} \left[\sum_{j=1}^n K_{i,j}(x, t) R \left(\sum_{k=1}^{\infty} u_{i,k}(t) q^k \right) \right] dt \Big|_{q=0} \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{m=1}^1 \left(u_{i,m-1}(x) \frac{1}{(m-1)!} \chi_m f_i(x) \int_{f(x)}^{g(x)} \left[\sum_{j=1}^n K_{ij}(x,t) \left[\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} R \left(\sum_{k=1}^1 u_{i,k}(t) q^k \right) \right] \right] \Big|_{q=0} dt \right) \\
 &= \sum_{m=1}^1 \left(u_{i,m-1}(x) \frac{1}{(m-1)!} \chi_m f_i(x) \int_{f(x)}^{g(x)} \left[\sum_{j=1}^n K_{ij}(x,t) T_{i,m-1}(t) \right] \Big|_{q=0} dt \right), \\
 &= \sum_{m=1}^1 u_{i,m-1}(x) f_i(x) \int_{f(x)}^{g(x)} \left[\sum_{j=1}^n K_{ij}(x,t) \sum_{m=1}^1 T_{i,m-1}(t) \right] dt, \\
 &= u_i(x) f_i(x) \int_{f(x)}^{g(x)} \left[\sum_{j=1}^n K_{ij}(x,t) R(u_i(t)) \right] dt.
 \end{aligned}$$

4. Genetic Algorithm (GA)

The genetic algorithm is a random search technique used to optimize difficult problems and solve nonlinear systems of equations. Instead of employing deterministic principles, GA makes the use of probabilistic transition rules and manages a population of alternative solutions known as individuals or iteratively evolving chromosomes [22]. The iterations of algorithm are known as "generations". A fitness function and genetic operators like reproduction, crossover, and mutation are used to mimic how solutions evolve [21]. The initial population of a genetic algorithm, as shown in Figure 1, is often random. A chromosome, which is a binary string or a real-valued number, is typically used to represent this population (mating pool). The objective function, which provides each person a corresponding number called its fitness, measures and evaluates the individual performance. The objective function, which provides each person a corresponding number called its fitness, measures and evaluates the individual performance. Each chromosome's fitness is evaluated, as well as a survival of the fittest approach is used. The fitness of each chromosome is evaluated in this work using the residual error value. A genetic algorithm primarily performs three operations: reproduction, crossover, and mutation. Figure 1 describes the GA operation sequences in detail.

4.1. Basic Steps of Genetic Algorithm [20]

Step 1: Use a population of random solutions to initialize the parameter, including crossover rate, mutation rate, number of clusters, and generations. Discover the coding mode.

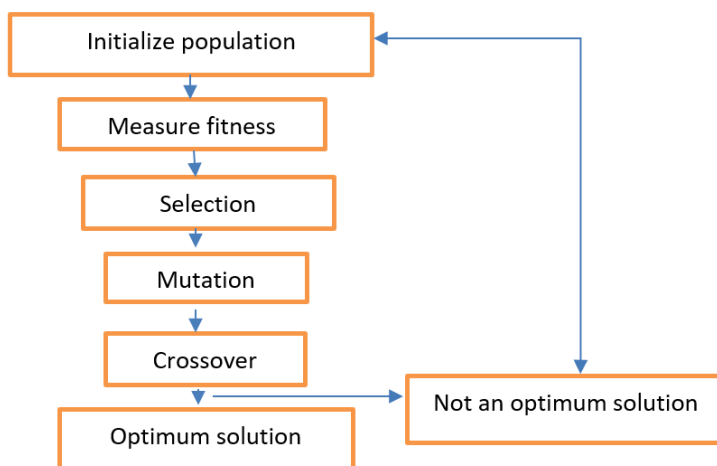


Figure 1: Flowchart of genetic algorithm.

Step 2: Calculate and assess the fitness function's value.

Step 3: Continue with the crossover and mutation process to create the new cluster.

Step 4: Repetition of Step 2 is necessary to get the best result.

Here, we use the genetic algorithm to improve the results of using the HAM in the following ways:

Algorithm 1 Genetic Algorithm for the best parameters h and λ_1, λ_2 in system of volterra integral equations

Input:

- Set number of variables (var) ,
- Set upper and lower limit for each variable (ub, lb),
- Set size of population (a),
- Set rate of crossover (rc),
- Set rate of mutation (rm),
- Set number of iterations ($Max_iteration$),
- Set fitness function name

Output:

solution λ_1, λ_2 and h

Initialization

- 1: Generate individual feasible solutions randomly with limit boundary
- 2: Save them in the population p ;
- 3: Find fitness value for each population F

Loop until the terminal condition

- 4: **for** $i = 1$ to $Max_iteration$ **do** , Elitism based selection using Rank selection
 - 5: $fitness =$ Sort individual descending to find **minimum** fitness value.
 - 6: Select the best rc solutions in pop_1 and save them in pop_t according to $fitness$;
 - 7: **end for**
-

Crossover

- 8: number of crossover $nc = (\alpha \ ne)/2$
- 9: **for** $j = 1$ to nc **do**
- 10: randomly select two solutions X_A and X_B from Pop_i ;
- 11: generate X_C and X_D by **Arithmetic crossover** to X_A and X_B ;
- 12: save X_C and X_D to Pop_t ;
- 13: **end for**

Mutation

- 14: **for** $j = 1$ to ne **do**
- 15: select a solution X_j from Pop_t ;
- 16: mutate **Random Resetting** of X_j under the rate rm and generate a new solution X_j^θ ;
- 17: **if then** X_j^θ is unfeasible
- 18: update X_j^θ with a feasible solution by repairing X_j^θ ;
- 19: **end if**
- 20: update X_j with X_j^θ in Pop_t ;
- 21: **end for**

Updating

- 22: update $pop_i + 1 = Pop_i + pop_t$;

Returning the best solution

- 23: return the best solution X in Pop

5. Applications and numerical results

In this section, the HAM applied to obtain an approximate-exact solution for NLSVIEs is displayed in the following three problems. To show the high accuracy of the solution results compared with the exact solution, the maximum absolute errors (MAE) are defined as:

$$k \ k_1 = k \ y_{Exact}(x_i) \ \phi_n(x_i) \ k_1$$

Moreover, giving the maximum residual error (MRE), the computations associated with the problems were performed using the Maple 18 package with a precision of 20 digits.

Problem 1. *Let us consider the following NLSVIEs [6]*

$$\begin{cases} u_1(x) = f_1(x) + \lambda_1 \int_0^x (u_1^2(t) + u_2^3(t)) dt, \\ u_2(x) = f_2(x) + \lambda_2 \int_0^x (u_1^3(t) - u_2^2(t)) dt, \end{cases} \tag{22}$$

where

$$\begin{cases} f_1(x) = \left(\frac{1}{10}x^{10} - \frac{1}{5}x^5 \right) \lambda_1 + x^2 \\ f_2(x) = x^3, \end{cases} \tag{23}$$

with the exact solutions

$$u_{Exact1}(x) = x^2, \quad u_{Exact2}(x) = x^3. \tag{24}$$

To solve 22 and 23 by means of the standard HAM, we choose the initial approximation

$$u_{1,0}(x) = f_1(x), \quad u_{2,0}(x) = f_2(x), \tag{25}$$

and the linear operator

$$L[\phi_1(x, q)] = \phi_1(x, q), \quad L[\phi_2(x, q)] = \phi_2(x, q). \tag{26}$$

Furthermore, the system 22 suggests that we define the non-linear operator as

$$\begin{cases} N_1[\phi_1(x, q), \phi_2(x, q)] = \phi_1(x, q) - f_1(x) - \lambda_1 \int_0^x (\phi_1^2(t, q) + \phi_2^3(t, q)) dt, \\ N_2[\phi_1(x, q), \phi_2(x, q)] = \phi_2(x, q) - f_2(x) - \lambda_2 \int_0^x (\phi_1^3(t, q) - \phi_2^2(t, q)) dt, \end{cases} \tag{27}$$

Using the above definition, we construct the zeroth-order deformation equation as in 6 and 7 and the m th-order deformation equation for $m \geq 1$ which is

$$\begin{cases} L[u_{1,m}(x) - \chi_m u_{1,m-1}(x)] = h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \\ L[u_{2,m}(x) - \chi_m u_{2,m-1}(x)] = h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \end{cases} \tag{28}$$

where

$$\begin{cases} R_{1,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}) = u_{1,m-1}(x) - f_1(x) - \lambda_1 \int_0^x (u_{1,m-1}^2(t) + u_{2,m-1}^3(t)) dt, \\ R_{2,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}) = u_{2,m-1}(x) - f_2(x) - \lambda_2 \int_0^x (u_{1,m-1}^3(t) - u_{2,m-1}^2(t)) dt, \end{cases} \tag{29}$$

Now, for $m \geq 1$, the solutions of the m th-order deformation Equation 29 are

$$\begin{cases} u_{1,m}(x) = \chi_m u_{1,m-1}(x) + h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \\ u_{2,m}(x) = \chi_m u_{2,m-1}(x) + h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \end{cases} \tag{30}$$

Now, we successively obtain

$$u_{1,1}(x) = \frac{1}{2100} h \lambda_1^3 x^{21} - \frac{1}{400} h \lambda_1^3 x^{16} + \frac{1}{65} h \lambda_1^2 x^{13} - \frac{1}{275} h \lambda_1^3 x^{11} + \frac{1}{20} h \lambda_1^2 x^8 - \frac{1}{5} h \lambda_1 x^5 - \frac{1}{10} h \lambda_1 x^{10}$$

$$u_{2,1}(x) = \frac{1}{31000} h \lambda_1^3 \lambda_2 x^{31} + \frac{3}{13000} h \lambda_1^3 \lambda_2 x^{26} - \frac{3}{2300} h \lambda_1^2 \lambda_2 x^{23} + \frac{1}{1750} h \lambda_1^3 \lambda_2 x^{21} - \frac{1}{150} h \lambda_1^2 \lambda_2 x^{18} + \frac{1}{50} h \lambda_1 \lambda_2 x^{15} + \frac{1}{2000} h \lambda_1^3 \lambda_2 x^{16} - \frac{3}{325} h \lambda_1^2 \lambda_2 x^{13} + \frac{3}{50} h \lambda_1 \lambda_2 x^{10}$$

⋮

Thus, the approximate solution in a series form is given by

$$u_1(x) = u_{1,0}(x) + \sum_{i=1}^6 u_{1,i}(x), \quad u_2(x) = u_{2,0}(x) + \sum_{i=1}^6 u_{2,i}(x),$$

This series has the closed form as $m! - 1$

$$u_1(x) = x^2, \quad u_2(x) = x^3.$$

which are the exact solutions of the Problem 1.

Tables 1 and 2 show a comparison of the numerical results with the errors applying the standard HAM ($m = 6$) and the numerical results applying the HAM developed by genetic algorithm HAM-GA with the exact solutions 24 within the interval $0 \leq x \leq 1$ for various values of λ_1, λ_2 and h . Tables 3 and 4, we list the MAE and the MRE by the HAM and HAM-GA on the interval $[0, 1]$ for various values of λ_1, λ_2 and h . Table 5 gives the errors on the interval h -curves $[-1.2, -0.8]$ when $\lambda_1 = \lambda_2 = 1$.

Table 1: Numerical results when $\lambda_1 = \lambda_2 = 1$, for (Problem 1)

x	i	$u_{Exacti}(x)$	HAM $h=-1$	AE	HAM-GA $h=-0.98213$	AE
0.1	1	0.010	0.0100000000	2.113E-24	0.0099999999	3.899E-15
	2	0.001	0.0010000000	5.828E-26	0.0009999999	3.113E-18
0.3	1	0.090	0.0899999999	7.314E-15	0.0899999999	4.325E-12
	2	0.027	0.0269999999	1.689E-15	0.0269999999	4.316E-13
0.5	1	0.250	0.2499999997	2.022E-10	0.2499999982	1.763E-09
	2	0.125	0.1249999998	1.166E-10	0.1249999992	7.596E-10
0.7	1	0.490	0.4899997951	2.048E-07	0.4899995366	4.633E-07
	2	0.343	0.3429998134	1.865E-07	0.3429996197	3.802E-07
0.9	1	0.810	0.8099446465	5.535E-05	0.8099235084	7.649E-05
	2	0.729	0.7289423171	5.768E-05	0.7289244930	7.550E-05

Table 2: Numerical results when $\lambda_1 = 0.5, \lambda_2 = 1.33838729$ for (Problem 1)

x	i	$u_{Exacti}(x)$	HAM $h=-1$	AE	HAM-GA $h=-0.98213$	AE
0.1	1	0.010	0.0100000000	3.302E-26	0.0099999999	1.884E-15
	2	0.001	0.0010000000	2.305E-27	0.0009999999	2.043E-18
0.3	1	0.090	0.0899999999	1.152E-16	0.0899999999	1.040E-12
	2	0.027	0.0269999999	5.883E-17	0.0269999999	1.718E-13
0.5	1	0.250	0.2499999999	3.432E-12	0.2499999999	1.340E-10
	2	0.125	0.1249999999	3.983E-12	0.1249999999	9.760E-11
0.7	1	0.490	0.4899999995	4.232E-09	0.4899999979	2.034E-08
	2	0.343	0.3429999990	9.081E-09	0.3429999975	2.470E-08
0.9	1	0.810	0.8099999004	9.954E-07	0.809996725	3.274E-06
	2	0.729	0.728994536	5.463E-06	0.728993970	6.029E-06

Table 3: MAE and MRE of HAM and HAM-GA for (Problem 1)

m	i	HAM $\lambda_1 = \lambda_2 = 1, h = 1$		HAM-GA $\lambda_1 = \lambda_2 = 1, h = 0.98213$	
		MAE	MRE	MAE	MRE
2	1	5.877E-02	1.308E-02	6.308E-02	1.434E-02
	2	6.413E-02	1.935E-02	6.298E-02	1.858E-02
3	1	2.574E-02	6.445E-03	2.698E-02	6.684E-03
	2	1.335E-02	3.409E-03	1.513E-02	3.957E-03
4	1	5.897E-03	1.379E-03	6.974E-03	1.651E-03
	2	6.426E-03	1.897E-03	6.839E-03	1.990E-03
5	1	2.390E-03	6.002E-04	2.672E-03	6.635E-04
	2	1.947E-03	5.564E-04	2.273E-03	6.527E-04
6	1	6.738E-04	1.679E-04	8.337E-04	2.078E-04
	2	7.095E-04	2.088E-04	8.304E-04	2.429E-04

Table 4: MAE and MRE of HAM-GA for (Problem 1)

m	i	HAM-GA $\lambda_1 = 0.5,$ $\lambda_2 = 1.33838, h = 1$		HAM-GA $\lambda_1 = 0.5,$ $\lambda_2 = 1.33838, h = 0.98213$	
		MAE	MRE	MAE	MRE
2	1	1.551E-02	3.520E-03	1.792E-02	4.297E-03
	2	4.800E-02	1.655E-02	4.714E-02	1.596E-02
3	1	6.088E-03	1.764E-03	6.465E-03	1.833E-03
	2	1.283E-04	3.189E-04	1.772E-03	3.028E-04
4	1	3.162E-04	5.617E-05	6.297E-04	1.475E-04
	2	2.101E-03	6.938E-04	2.040E-03	6.561E-04
5	1	3.177E-04	9.251E-05	3.286E-04	9.307E-05
	2	1.741E-06	1.322E-05	1.426E-04	3.460E-05
6	1	4.388E-06	2.567E-07	3.172E-05	7.968E-06
	2	9.404E-05	3.052E-05	9.224E-05	2.888E-05

Table 5: Errors on the interval h -curves for (Problem 1) when $\lambda_1 = \lambda_2 = 1$

x	i	$h=-1.2$	$h=-0.98213$	$h=-1$	$h=-0.8$
-0.9	1	2.260E-03	1.646E-05	3.642E-05	5.252E-06
	2	3.258E-03	5.517E-05	9.355E-05	2.752E-06
-0.7	1	9.179E-05	5.215E-08	2.007E-07	1.512E-06
	2	7.466E-05	9.922E-08	3.115E-07	2.430E-06
-0.5	1	4.730E-06	2.689E-12	2.133E-10	1.005E-06
	2	1.023E-06	3.624E-12	1.693E-10	2.430E-07
-0.3	1	1.895E-07	9.567E-14	7.451E-15	1.354E-07
	2	3.866E-09	4.992E-14	2.062E-15	2.089E-09
-0.1	1	6.448E-10	3.398E-15	2.115E-24	6.368E-10
	2	5.785E-14	2.886E-18	6.186E-26	3.829E-14
0.1	1	6.352E-10	3.899E-15	2.113E-24	6.432E-10
	2	5.736E-14	3.113E-18	5.828E-26	3.849E-14
0.3	1	1.267E-10	4.325E-12	7.314E-15	1.778E-07
	2	3.074E-09	4.316E-13	1.689E-15	2.412E-09
0.5	1	7.503E-07	1.763E-09	2.022E-10	3.644E-06
	2	3.825E-07	7.596E-10	1.166E-10	4.909E-07
0.7	1	2.699E-06	4.633E-07	2.048E-07	5.064E-05
	2	9.241E-06	3.802E-07	1.865E-07	2.237E-05
0.9	1	1.872E-04	7.649E-05	5.535E-05	8.640E-04
	2	2.275E-04	7.550E-05	5.768E-05	6.502E-04

In Figures 2 and 3, an illustration shows how well the exact solution matches with the approximate solution using the genetic algorithm $h=-0.98213$ with $\lambda_1 = \lambda_2 = 1$. In Figures 4 and 5 present the h -curves for (Problem 1) when $\lambda_1 = \lambda_2 = 1$. We present the error residual $ER(u_i(x))$, $i = 1, 2$ of HAM-GA when $\lambda_1 = 0.5$, $\lambda_2 = 1.33838$, $h = -0.98213$ in Figures 6 and 7. The CPU time = 1.188 Seconds.

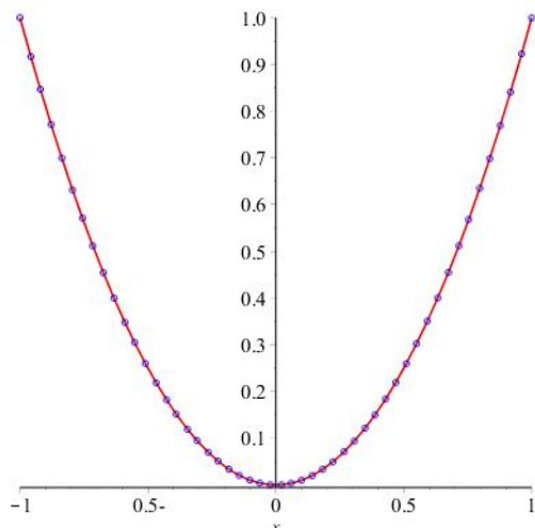


Figure 2: Line : $u_{Exact1}(x)$, o : HAM $GA_1(x)$

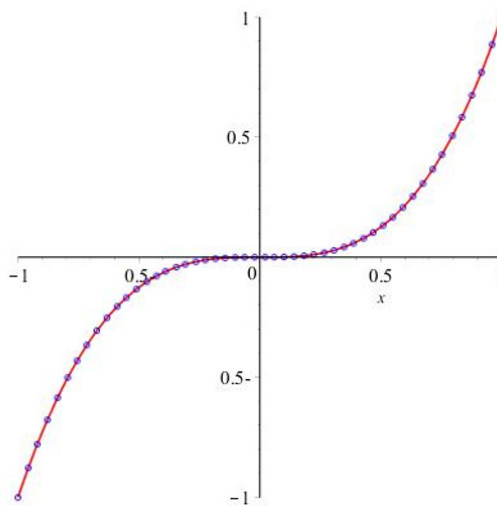


Figure 3: Line : $u_{Exact2}(x)$, o : HAM $GA_2(x)$

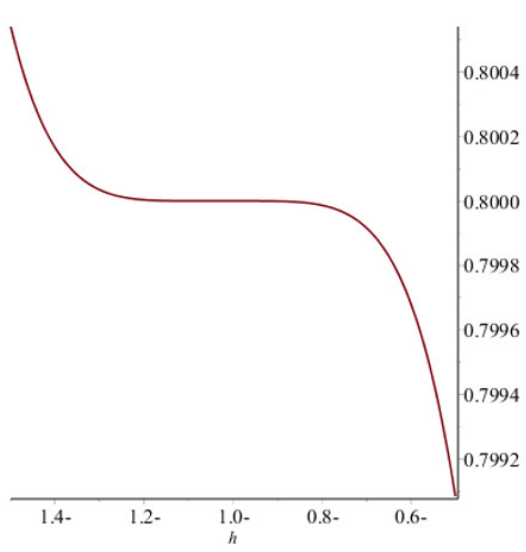


Figure 4: h -curve for $u_1'(0.4, h)$

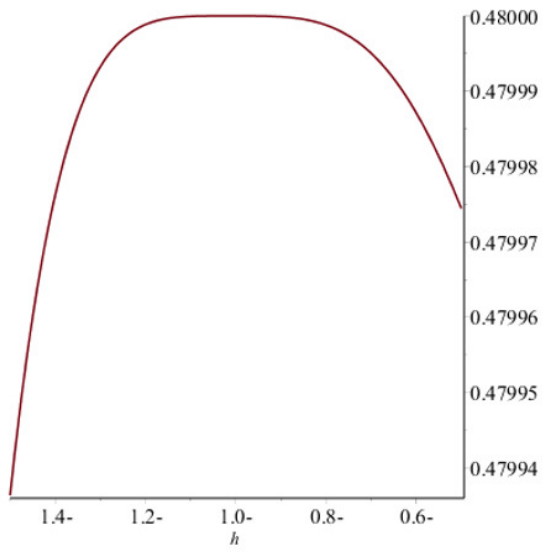


Figure 5: h -curve for $u_2'(0.4, h)$

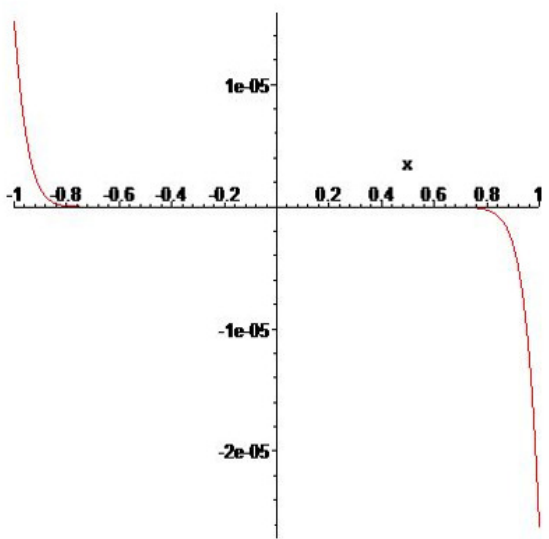


Figure 6: $ER(u_1(x))$ of HAM-GA

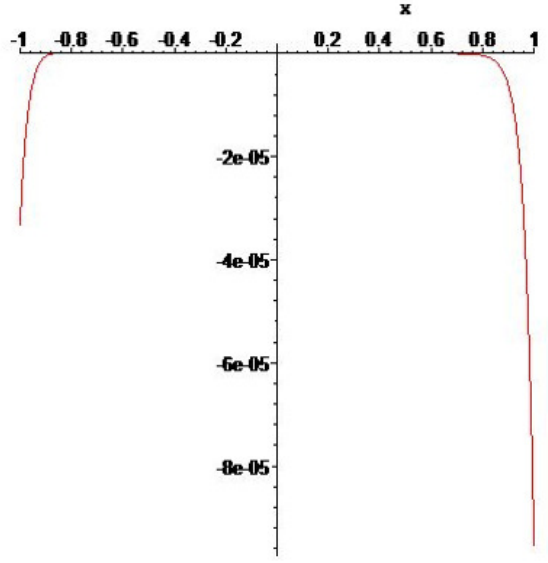


Figure 7: $ER(u_2(x))$ of HAM-GA

Problem 2. Let us consider the following NLSVIEs [8]

$$\begin{cases} u_1(x) = f_1(x) + \lambda_1 \int_0^x (u_1^2(t) + u_2^2(t)) dt, \\ u_2(x) = f_2(x) + \lambda_2 \int_0^x u_1(t)u_2(t)dt, \end{cases} \quad (31)$$

where

$$\begin{cases} f_1(x) = \sin(x) & \lambda_1 x, \\ f_2(x) = \cos(x) & \frac{1}{2}\lambda_2 \sin^2(x), \end{cases} \tag{32}$$

with the exact solutions

$$u_{Exact1}(x) = \sin(x), \quad u_{Exact2}(x) = \cos(x), \tag{33}$$

To solve 31 and 32 by means of the standard HAM, we choose the initial approximation

$$u_{1,0}(x) = f_1(x), \quad u_{2,0}(x) = f_2(x), \tag{34}$$

and the linear operator

$$L[\phi_1(x, q)] = \phi_1(x, q), \quad L[\phi_2(x, q)] = \phi_2(x, q), \tag{35}$$

Furthermore, the system 31 suggests that we define the non-linear operator as

$$\begin{cases} N_1[\phi_1(x, q), \phi_2(x, q)] = \phi_1(x, q) - f_1(x) - \lambda_1 \int_0^x (\phi_1^2(t, q) + \phi_2^2(t, q)) dt, \\ N_2[\phi_1(x, q), \phi_2(x, q)] = \phi_2(x, q) - f_2(x) - \lambda_2 \int_0^x (\phi_1(t, q) \phi_2(t, q)) dt, \end{cases} \tag{36}$$

Using the above definition, we construct the zeroth-order deformation equation as in 6 and 7 and the *m*th-order deformation equation for $m \geq 1$ is

$$\begin{cases} L[u_{1,m}(x) - \chi_m u_{1,m-1}(x)] = h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \\ L[u_{2,m}(x) - \chi_m u_{2,m-1}(x)] = h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \end{cases} \tag{37}$$

where

$$\begin{cases} R_{1,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}) = u_{1,m-1}(x) - f_1(x) - \lambda_1 \int_0^x (u_{1,m-1}^2(t) + u_{2,m-1}^2(t)) dt, \\ R_{2,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}) = u_{2,m-1}(x) - f_2(x) - \lambda_2 \int_0^x (u_{1,m-1}(t) u_{2,m-1}(t)) dt, \end{cases} \tag{38}$$

Now, for $m \geq 1$, the solutions of the *m*th-order deformation Equation 38 are

$$\begin{cases} u_{1,m}(x) = \chi_m u_{1,m-1}(x) + h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \\ u_{2,m}(x) = \chi_m u_{2,m-1}(x) + h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \end{cases} \tag{39}$$

we now successively obtain

$$\begin{aligned} u_{1,1}(x) &= \frac{1}{3}h\lambda_1^3 x^3 - 2h\lambda_1^2 x \cos x + 2h\lambda_1^2 \sin x - h\lambda_1 x + \frac{1}{16}h\lambda_1\lambda_2^2 \sin^3 x \cos x \\ &\quad + \frac{3}{32}h\lambda_1\lambda_2^2 \sin x \cos x - \frac{3}{32}h\lambda_1\lambda_2^2 x + \frac{1}{3}h\lambda_1\lambda_2 \sin^3 x \end{aligned}$$

$$u_{2,1}(x) = -h\lambda_1\lambda_2 + \frac{1}{3}h\lambda_2^2 + \frac{1}{4}h\lambda_1\lambda_2^2 x \sin x \cos x - \frac{1}{8}h\lambda_1\lambda_2^2 x^2$$

$$\begin{aligned} & \frac{1}{8}h\lambda_1\lambda_2^2\sin^2 x + h\lambda_1\lambda_2x\sin x + h\lambda_1\lambda_2\cos x \\ & \frac{1}{6}h\lambda_2^2\sin^2 x\cos x \quad \frac{1}{3}h\lambda_2^2\cos x \quad \frac{1}{2}h\lambda_2\sin^2 x \\ & \vdots \end{aligned}$$

Thus, the approximate solution in a series form is given by

$$u_1(x) = u_{1,0}(x) + \sum_{i=1}^5 u_{1,i}(x), \quad u_2(x) = u_{2,0}(x) + \sum_{i=1}^5 u_{2,i}(x),$$

This series has the closed form as $m ! - 1$

$$u_1(x) = \sin x, \quad u_2(x) = \cos x.$$

which are the exact solutions of the Problem 2.

Tables 6, 7 show a comparison of the numerical results with the errors applying the standard HAM ($m = 5$) and the numerical results applying the HAM developed by genetic algorithm HAM-GA with the exact solutions 33 within the interval $0 \leq x \leq 0.5$ for various values of λ_1, λ_2 and h . Tables 8, 9, we list the MAE and the MRE by the HAM and HAM-GA on the interval $[0, 0.5]$ for the various values of λ_1, λ_2 and h . Table 10 gives the errors on the interval h -curves $[-1.2, -0.8]$ when $\lambda_1 = \lambda_2 = 1$.

Table 6: Numerical results when $\lambda_1 = \lambda_2 = 1$, for (Problem 2)

x	i	$u_{Exacti}(x)$	HAM $h = -1$	AE	HAM-GA $h=-0.98401$	AE
0.1	1	0.099833416	0.099829110	4.306E-06	0.099828113	5.303E-06
	2	0.995004165	0.995003990	1.745E-07	0.995002111	2.054E-06
0.3	1	0.295520206	0.294524648	9.955E-04	0.294500710	1.019E-03
	2	0.955336489	0.955219396	1.170E-04	0.955085971	2.505E-04
0.5	1	0.479425538	0.467847859	1.157E-02	0.467776042	1.164E-02
	2	0.877582561	0.875456892	2.125E-03	0.874603732	2.978E-03

Table 7: Numerical results when $\lambda_1 = 0.5, \lambda_2 = 0.5$, for (Problem 2)

x	i	$u_{Exacti}(x)$	HAM $h = -1$	AE	HAM-GA $h=-0.98401$	AE
0.1	1	0.099833416	0.099833280	1.356E-07	0.099833132	2.841E-07
	2	0.995004165	0.995004155	9.445E-09	0.995004017	1.473E-07
0.3	1	0.295520206	0.295486811	3.339E-05	0.295479223	4.098E-05
	2	0.955336489	0.955329938	6.550E-06	0.955320666	1.582E-05
0.5	1	0.479425538	0.478989428	4.361E-04	0.478926206	4.993E-04
	2	0.877582561	0.877455542	1.270E-04	0.877391282	1.912E-04

Table 8: MAE and MRE of HAM and HAM-GA for (Problem 2)

		HAM $\lambda_1 = \lambda_2 = 1, h = 1$		HAM-GA $\lambda_1 = \lambda_2 = 1, h = 0.98401$	
m	i	MAE	MRE	MAE	MRE
2	1	7.494E-02	1.545E-02	8.174E-02	1.899E-02
	2	1.170E-01	5.382E-02	1.170E-01	5.295E-02
3	1	7.716E-02	3.207E-02	7.720E-02	3.158E-02
	2	1.534E-02	2.581E-03	1.857E-02	4.154E-03
4	1	1.114E-02	3.207E-03	1.425E-02	4.529E-03
	2	1.577E-02	6.432E-03	1.583E-02	6.287E-03
5	1	1.157E-02	4.556E-03	1.164E-02	4.514E-03
	2	2.125E-03	4.892E-04	2.978E-03	8.526E-04
6	1	1.642E-03	5.243E-04	2.413E-03	8.310E-04
	2	2.204E-03	8.691E-04	2.232E-03	8.547E-04

Table 9: MAE and MRE of HAM-GA for (Problem 2)

		HAM-GA $\lambda_1 = 0.5,$ $\lambda_2 = 0.5, h = 1$		HAM-GA $\lambda_1 = 0.5,$ $\lambda_2 = 0.5, h = 0.98401$	
m	i	MAE	MRE	MAE	MRE
2	1	2.411E-02	8.656E-03	2.772E-02	1.058E-02
	2	3.019E-02	1.452E-02	3.063E-02	1.451E-02
3	1	1.051E-02	4.590E-03	1.100E-02	4.749E-03
	2	2.735E-03	9.328E-04	3.613E-03	1.351E-03
4	1	1.221E-03	4.665E-04	1.671E-03	6.618E-04
	2	1.110E-03	4.649E-04	1.207E-03	4.968E-04
5	1	4.361E-04	1.779E-04	4.993E-04	2.020E-04
	2	1.270E-04	4.611E-05	1.912E-04	7.297E-05
6	1	5.861E-05	2.268E-05	9.015E-05	3.552E-06
	2	4.302E-05	1.721E-05	5.203E-05	2.049E-06

Table 10: Errors on the interval h -curves for (Problem 2) when $\lambda_1 = \lambda_2 = 1$

x	i	$h=-1.2$	$h=-0.98213$	$h=-1$	$h=-0.8$
-0.5	1	3.319E-02	1.164E-02	1.157E-02	2.387E-02
	2	1.640E-02	2.978E-03	2.125E-03	1.287E-02
-0.3	1	6.576E-03	1.019E-03	9.955E-04	4.423E-03
	2	3.711E-03	2.505E-04	1.170E-04	2.504E-03
-0.1	1	3.672E-04	5.303E-06	4.306E-06	2.826E-04
	2	2.132E-04	2.054E-06	1.745E-07	1.520E-04
0.1	1	3.672E-04	5.303E-06	4.306E-06	2.826E-04
	2	2.132E-04	2.054E-06	1.745E-07	1.520E-04
0.3	1	6.576E-03	1.019E-03	9.955E-04	4.423E-03
	2	3.711E-03	2.505E-04	1.170E-04	2.504E-03
0.5	1	3.319E-02	1.164E-02	1.157E-02	2.387E-02
	2	1.640E-02	2.978E-03	2.125E-03	1.287E-02

In Figures 8 and 9, an illustration shows how well the exact solution matches with the approximate solution using the genetic algorithm $h=-0.98401$ with $\lambda_1 = \lambda_2 = 1$. In Figures 10 and 11 the h -curves present for (Problem 2) when $\lambda_1 = \lambda_2 = 1$. We present the error residual $ER(u_i(x))$, $i = 1, 2$ of HAM-GA when $\lambda_1 = \lambda_2 = 0.5$, $h = -0.98401$ in Figures 12 and 13. The CPU time = 2.828 Seconds.

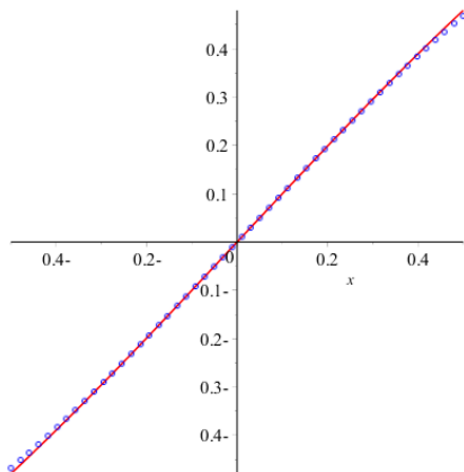


Figure 8: Line : $u_{Exact1}(x)$, o : HAM GA₁(x)

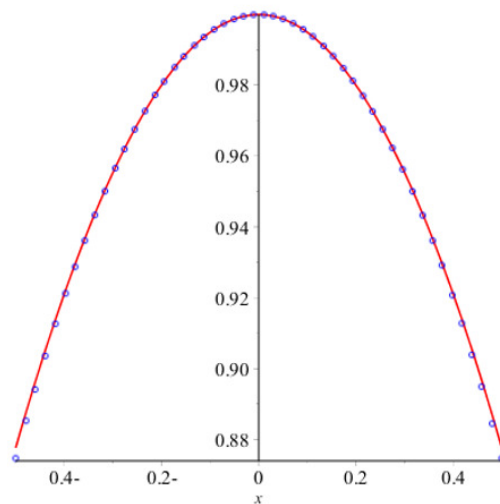


Figure 9: Line : $u_{Exact2}(x)$, o : HAM GA₂(x)

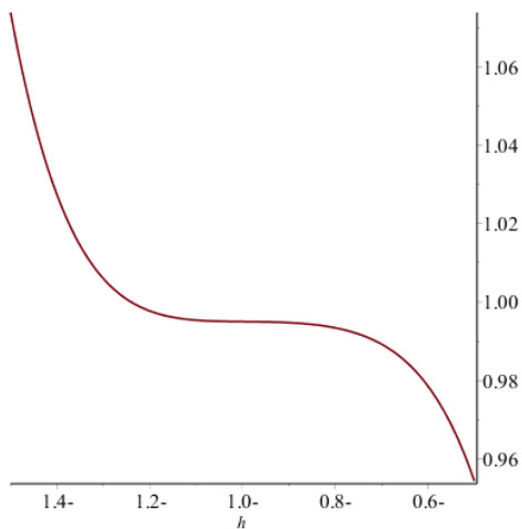


Figure 10: h -curve for $u'_1(0.1, h)$

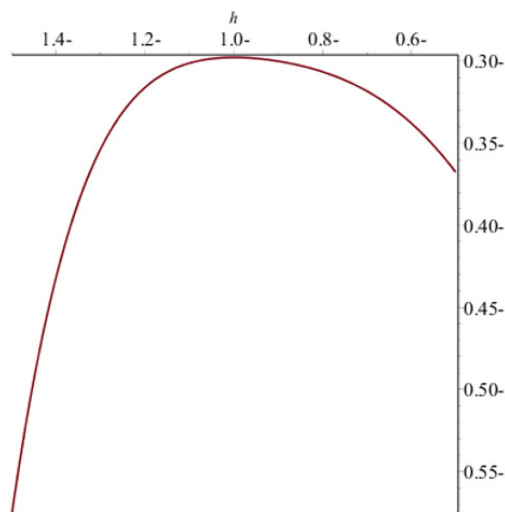


Figure 11: h -curve for $u'_2(0.4, h)$

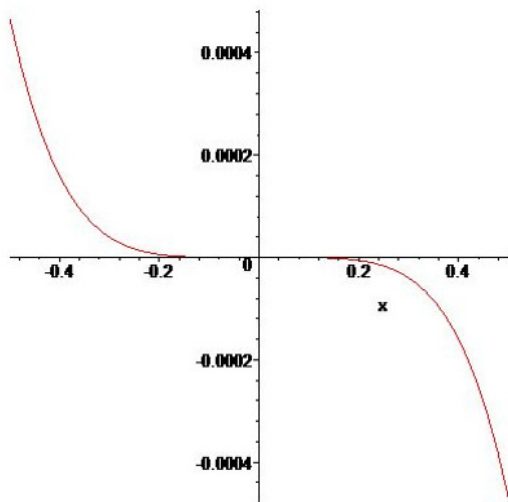


Figure 12: $ER(u_1(x))$ of HAM-GA

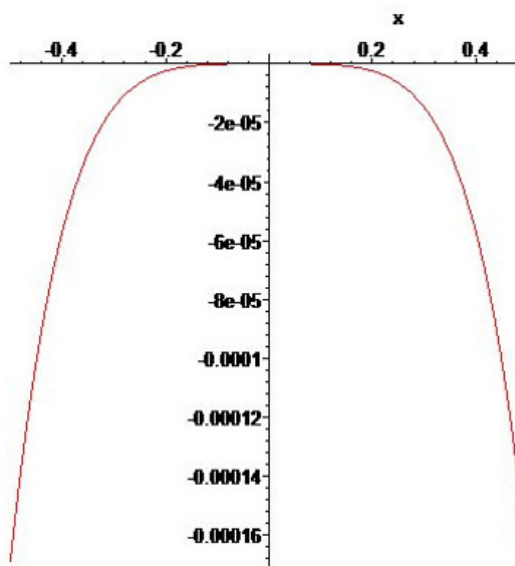


Figure 13: $ER(u_2(x))$ of HAM-GA

Problem 3. Finally, let us consider the following NLSVIEs [8]

$$\begin{cases} u_1(x) = f_1(x) + 4 \int_0^x u_1(t)u_2(t)dt, \\ u_2(x) = f_2(x) + \int_0^x tu_1(t)u_3^2(t)dt, \\ u_3(x) = f_3(x) - \frac{1}{3} \int_0^x tu_2(t)u_3(t)dt, \end{cases} \quad (40)$$

where

$$\begin{cases} f_1(x) = 2 \ln(x)x^2 + x^2 + \ln(x), \\ f_2(x) = \frac{1}{6} \ln(x)x^6 + \frac{1}{36}x^6 + x, \\ f_3(x) = \frac{1}{15}x^5 + x^2, \end{cases} \quad (41)$$

with the exact solutions

$$u_{Exact1}(x) = \ln(x), \quad u_{Exact2}(x) = x, \quad u_{Exact3}(x) = x^2. \quad (42)$$

To solve 40 and 41 by means of the standard HAM, we choose the initial approximation

$$u_{1,0}(x) = f_1(x), \quad u_{2,0}(x) = f_2(x), \quad u_{3,0}(x) = f_3(x), \quad (43)$$

and the linear operator

$$L[\phi_1(x, q)] = \phi_1(x, q), \quad L[\phi_2(x, q)] = \phi_2(x, q), \quad L[\phi_3(x, q)] = \phi_3(x, q). \quad (44)$$

Furthermore, the system 40 suggests that we define the non-linear operator as

$$\begin{aligned} N_1[\phi_1(x, q), \phi_2(x, q), \phi_3(x, q)] &= \phi_1(x, q) - f_1(x) - 4 \int_0^x \phi_1(t, q) \phi_2(t, q) dt, \\ N_2[\phi_1(x, q), \phi_2(x, q), \phi_3(x, q)] &= \phi_2(x, q) - f_2(x) - \int_0^x t \phi_1(t, q) \phi_3^2(t, q) dt, \\ N_3[\phi_1(x, q), \phi_2(x, q), \phi_3(x, q)] &= \phi_3(x, q) - f_3(x) - \frac{1}{3} \int_0^x t \phi_2(t, q) \phi_3(t, q) dt. \end{aligned} \quad (45)$$

Using the above definition, we construct the zeroth-order deformation equation as in 6 and 7 and the m th-order deformation equation for $m \geq 1$ is

$$\begin{aligned} L[u_{1,m}(x) - \chi_m u_{1,m-1}(x)] &= h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1}), R_{3,m}(\vec{u}_{3,m-1})], \\ L[u_{2,m}(x) - \chi_m u_{2,m-1}(x)] &= h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1}), R_{3,m}(\vec{u}_{3,m-1})], \\ L[u_{3,m}(x) - \chi_m u_{3,m-1}(x)] &= h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1}), R_{3,m}(\vec{u}_{3,m-1})]. \end{aligned} \tag{46}$$

where

$$\begin{aligned} R_{1,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}, \vec{u}_{3,m-1}) &= u_{1,m-1}(x) - f_1(x) - 4 \int_0^x u_{1,m-1}(t) u_{2,m-1}(t) dt, \\ R_{2,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}, \vec{u}_{3,m-1}) &= u_{2,m-1}(x) - f_2(x) - \int_0^x t u_{1,m-1}(t) u_{3,m-1}^2(t) dt, \\ R_{3,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}, \vec{u}_{3,m-1}) &= u_{3,m-1}(x) - f_3(x) + \frac{1}{3} \int_0^x t u_{2,m-1}(t) u_{3,m-1}(t) dt. \end{aligned} \tag{47}$$

Now, for $m \geq 1$, the solutions of the m th-order deformation Equation 47 are

$$\begin{aligned} u_{1,m}(x) &= \chi_m u_{1,m-1}(x) + h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1}), R_{3,m}(\vec{u}_{2,m-1})], \\ u_{2,m}(x) &= \chi_m u_{2,m-1}(x) + h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1}), R_{3,m}(\vec{u}_{2,m-1})], \\ u_{3,m}(x) &= \chi_m u_{3,m-1}(x) + h [R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1}), R_{3,m}(\vec{u}_{2,m-1})], \end{aligned} \tag{48}$$

we now successively obtain

$$\begin{aligned} u_{1,1}(x) &= h \left(2 \ln(x) x^2 + x^2 - \frac{4}{27} \ln^2(x) x^9 + \frac{32}{243} \ln(x) x^9 \right. \\ &\quad \left. - \frac{59}{2187} x^9 + 2 \ln(x) x^4 - \frac{3}{2} x^4 \right). \end{aligned}$$

$$\begin{aligned} u_{2,1}(x) &= h \left(\frac{1}{6} \ln(x) x^6 + \frac{1}{36} x^6 + \frac{1}{1575} \ln(x) x^{14} - \frac{4}{11025} x^{14} - \frac{1}{2700} \ln(x) x^{12} \right. \\ &\quad \left. + \frac{1}{32400} x^{12} + \frac{4}{165} \ln(x) x^{11} - \frac{26}{1815} x^{11} - \frac{2}{135} \ln(x) x^9 + \frac{2}{1215} x^9 + \frac{1}{4} \ln(x) x^8 - \frac{5}{32} x^8 \right), \end{aligned}$$

$$u_{3,1}(x) = h \left(\frac{1}{15} x^5 - \frac{1}{3510} \ln(x) x^{13} + \frac{19}{273780} x^{13} - \frac{1}{180} \ln(x) x^{10} + \frac{1}{675} x^{10} + \frac{1}{360} x^8 \right),$$

⋮

Thus, the approximate solution in a series form is given by

$$\begin{aligned} u_1(x) &= u_{1,0}(x) + \sum_{i=1}^6 u_{1,i}(x), \\ u_2(x) &= u_{2,0}(x) + \sum_{i=1}^6 u_{2,i}(x), \end{aligned}$$

$$u_3(x) = u_{3,0}(x) + \sum_{i=1}^6 u_{3,i}(x).$$

This series has the closed form $m! - 1$

$$u_1(x) = \ln(x), \quad u_2(x) = x, \quad u_3(x) = x^2.$$

which are the exact solutions of the Problem 3.

Table 11 shows a comparison of the numerical results with the errors applying the standard HAM ($m=6$) and the numerical results applying the HAM developed by genetic algorithm (HAM-GA) with the exact solutions 42 within the interval $0 < x < 0.6$. Table 12 the MAE is tabulated for different frequencies as well as the MRE. Table 13 gives the errors on the interval h -curves $[1.2, 0.8]$.

Table 11: Numerical results for (Problem 3)

x	i	$u_{Exacti}(x)$	HAM $h = 1$	AE	HAM-GA $h=1.01064$	AE
0.1	1	-2.30258509	-2.302585092	3.094E-13	-2.302585092	3.734E-13
	2	0.10000000	0.1000000000	5.644E-18	0.0999999999	1.234E-17
	3	0.01000000	0.0099999999	2.932E-21	0.0099999999	9.274E-17
0.3	1	-1.20397280	-1.203972712	9.189E-08	-1.203972808	4.117E-09
	2	0.30000000	0.3000000000	1.477E-10	0.3000000000	5.030E-12
	3	0.09000000	0.0899999999	7.052E-13	0.0899999999	9.033E-14
0.5	1	-0.69314718	-0.693128726	1.845E-05	-0.693144508	2.672E-06
	2	0.50000000	0.500000343	3.436E-07	0.500000159	1.596E-07
	3	0.25000000	0.249999995	4.934E-09	0.249999997	2.706E-09

Table 12: MAE and MRE of HAM and HAM-GA for (Problem 3)

m	i	HAM $h = 1$		HAM-GA $h = 1.01064$	
		MAE	MRE	MAE	MRE
2	1	3.266E-01	1.102E-01	3.223E-01	1.087E-01
	2	4.773E-03	1.241E-03	4.768E-03	1.251E-03
	3	7.305E-05	2.150E-05	1.289E-04	4.453E-05
3	1	8.783E-02	2.910E-02	8.276E-02	2.744E-02
	2	1.582E-03	4.708E-04	1.514E-03	4.544E-04
	3	1.783E-05	4.884E-06	1.606E-05	4.295E-06
4	1	1.626E-02	5.420E-03	1.404E-02	4.690E-03
	2	3.342E-04	1.057E-04	2.950E-04	9.430E-05
	3	4.897E-06	1.510E-06	4.505E-06	1.409E-06
5	1	2.020E-03	6.799E-04	1.452E-03	4.914E-04
	2	5.048E-05	1.679E-05	3.905E-05	1.319E-05
	3	8.725E-07	2.839E-07	7.070E-07	2.332E-07
6	1	8.969E-05	3.016E-05	2.672E-06	9.493E-07
	2	5.256E-06	1.880E-06	3.111E-06	1.169E-06
	3	1.138E-07	3.870E-08	7.714E-08	2.675E-08

Table 13: Errors on the interval h -curves for (Problem 3)

x	i	$h=-1.2$	$h=-1.01064$	$h=-1$	$h=-0.8$
-0.5	1	1.758E-04	1.337E-04	1.806E-04	9.697E-03
	2	2.235E-06	8.834E-07	1.363E-06	8.807E-05
	3	5.742E-07	9.568E-09	1.562E-08	1.219E-06
-0.3	1	4.996E-05	4.619E-08	2.375E-07	8.175E-04
	2	6.683E-08	4.114E-11	3.324E-10	8.066E-07
	3	5.017E-08	1.777E-13	1.546E-12	5.382E-08
-0.1	1	1.921E-05	8.053E-13	4.255E-13	3.286E-05
	2	1.200E-10	1.517E-17	8.126E-18	2.866E-10
	3	2.130E-10	8.919E-17	4.310E-21	2.135E-10
0.1	1	1.255E-05	3.734E-13	3.094E-13	2.206E-05
	2	6.870E-11	1.234E-17	5.644E-18	1.815E-10
	3	2.136E-10	9.274E-17	2.932E-21	2.131E-10
0.3	1	2.885E-05	4.117E-09	9.189E-08	4.264E-04
	2	3.622E-08	5.030E-12	1.477E-10	3.910E-07
	3	5.355E-08	9.033E-14	7.052E-13	4.991E-08
0.5	1	8.557E-05	2.672E-06	1.845E-05	4.074E-03
	2	1.003E-06	1.596E-07	3.436E-07	3.433E-05
	3	7.582E-07	2.706E-09	4.934E-09	3.348E-07

In Figures 14-16 an illustration of how well the exact solution matches with the approximate solution using the genetic algorithm $h=-1.01064$. In Figures 17-19 present the h -curves for (Problem 3). We present the error residual $ER(u_i(x))$, $i = 1, 2, 3$ of HAM-GA when $h = -1.01064$ in Figures 20-22. The CPU time = 10.890 Seconds.

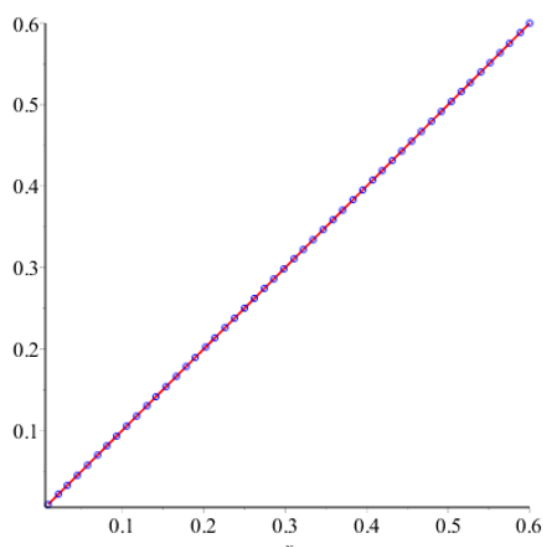
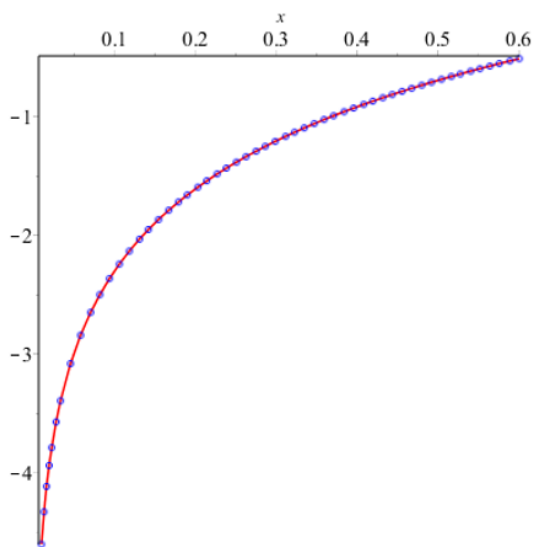


Figure 14: Line : $u_{Exact1}(x)$, o : HAM GA₁(x) Figure 15: Line : $u_{Exact2}(x)$, o : HAM GA₂(x)

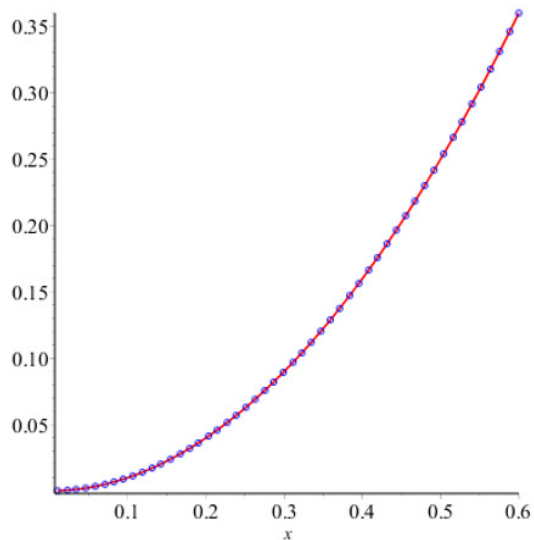


Figure 16: Line : $u_{Exact3}(x)$, o : HAM $GA_3(x)$

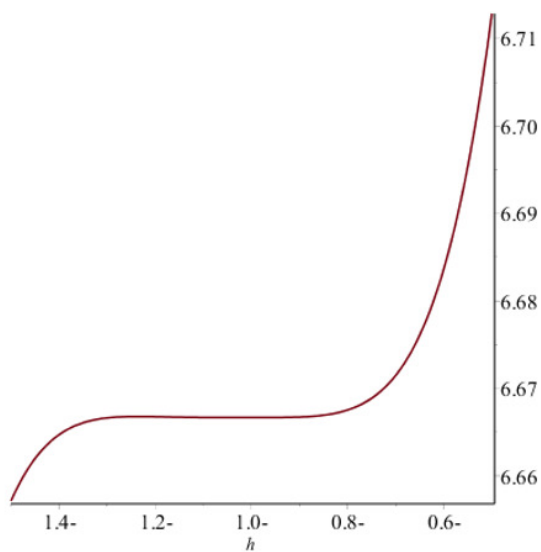


Figure 17: h -curve for $u_1'(0.15, h)$

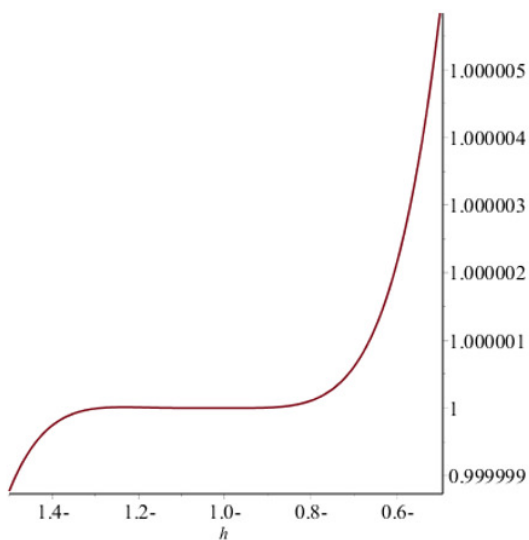


Figure 18: h curve for $u_2'(0.15, h)$

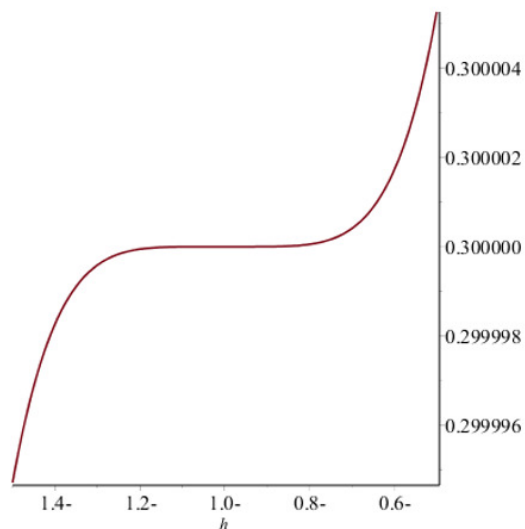


Figure 19: h curve for $u_3'(0.15, h)$

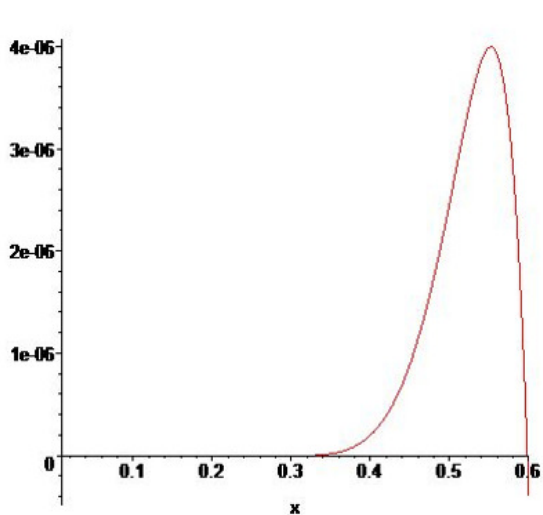


Figure 20: $ER(u_1(x))$ of HAM-GA

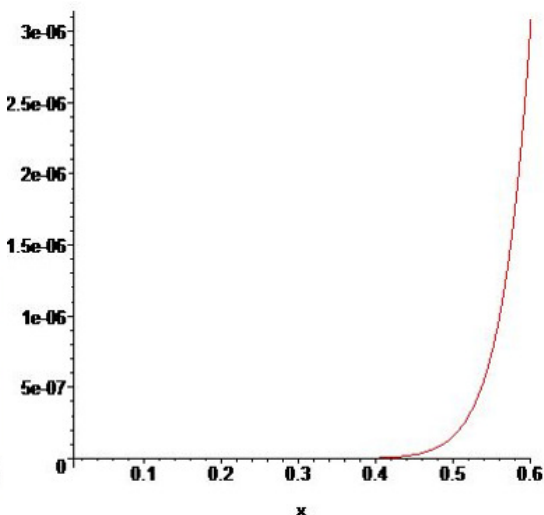
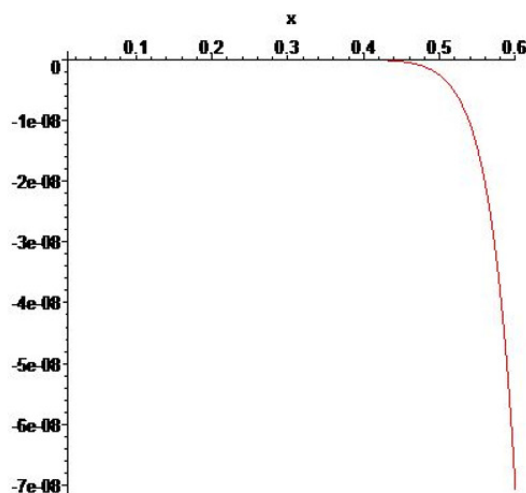


Figure 21: $ER(u_2(x))$ of HAM-GA

Figure 22: $ER(u_3(x))$ of HAM-GA

6. Conclusion

The present study suggests a new algorithm (HAM-GA) merging both the HAM and the GA for solving the nonlinear system of Volterra integral equations. Based on four consecutive cases, the algorithm determines the solution. The algorithm deems the residual error function as a fitness function and based on which the optimum value of λ_1 , λ_2 and h are selected. In the first case, the results were calculated according to the standard HAM, in the second case, the best value for h were chosen according to the genetic algorithm and based on which the results were calculated using HAM-GA. As for the third case, the best λ_1 and λ_2 for the Volterra system of integral equations were chosen, then the results were calculated using HAM-GA. Finally, the results were calculated using HAM-GA depending on optimal λ_1 , λ_2 and h . The results obtained using the best h were better than those obtained using the standard HAM. As for the results for the fourth case, they were optimal for the exact solution. The findings are in good agreement with the h -curves indicating that the proposed algorithm is successful in finding the solution.

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