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The Basis Number of Mycielski's Graph for Some Cog-Graphs

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Abstract. Let $G = (V, E)$ be a simple connected graph, then the basis number of G is denoted by $b(G)$ and is defined by the least positive integer k such that the graph G has a $k-fold$ basis for it is cycle space. In this paper we studied the basis number of Mycielski's graph for some cog-special graphs, and we compute the basis number of Mycielski's graph for cog-path graph, cog-cycle graph, cog-star graph, and cog-wheel graph.

2020 Mathematics Subject Classifications: 05C10, 05C25, 05C35
Key Words and Phrases: Basis number, $k-fold$, Mycielski's graph

1. Introduction

Let G be a connected graph with edges sets $\{e_1, e_2, \ldots, e_q\}$. For each subset S of edges of the graph G, there is a vector $(a_1, a_2, a_3, \ldots, a_q)$ corresponding to S such that $a_i = 1$ if $e_i \in S$ and $a_i = 0$ if $e_i \notin S$. These vectors form a vector space of dimension q on the field Z_2, called the vector space associated with the graph G and denoted by $(z_2)^q$. The vectors of $(z_2)^q$ that correspond to the cycles of G generate a vector subspace called the cycles space of G and denoted by $C(G)$. Each vector in $C(G)$ represents either a cycle in G or the union of separate cycles with respect to the edges.

A known corollary of graph theory is that a dimension of $C(G)$ is $q - p + 1$ where p represents the number of vertices of graph G and q the number of edges. The method for finding the base for the cycles space of $C(G)$ is as follows:

Let T be a generating tree for the graph G; If the edge e_i belongs to $G - T$ then $T + e_i$ contains only one cycle, let it be C_{e_i}. Clearly, $q - p + 1$ of cycles C_{e_i}, where $e_i \in G - T$ for $i = 1, 2, \ldots, q$ forms the base of the cycles space $C(G)$.

The base B of cycles space $C(G)$ is said to have a $k - fold$ if each edge of G shows no more than k times (iterations) in the cycles that corresponding to the vectors in the base B.

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The basis number of the graph G is defined as the smallest integer k, such that $C(G)$ have a $k-fold$ base; It is denoted by $b(G)$. If B is the base of the cycles space $C(G)$ and e is an edge in G, then the fold of the edge e in B is defined the number of cycles that exist in B and containing the edge e, and is denoted by $f_B(e)$.

In recent years, interest in the basic number has increased, we refer the reader to references [3–6, 9, 10, 13] for more information. In this paper, we will assume that all graphs that we encounter are finite, unguided and simple; For undefined terms, refer to the references [7][8].

There are other types of numbers that are important in graph theory such as: detour number [1] and number of domination [17], and graph theory has an important applications at the present time, see [11, 14, 15].

Mycielski's graph [16]: Let G be the graph, such that the set of its vertices is $V = \{u_1, u_2, u_3, \ldots, u_n\}$, then the Mycielski's graph for G consists of G itself as a sub graph isomorphic with $(n+1)$ additional vertices, the vertex v_i corresponding to u_i in G, for $i = 1, 2, 3, \ldots, n$; and another vertex w which is adjacent to each vertex v_i such that these vertices form a sub graph isomorphic with star $K_{(1,n)}$; In addition, for each edge $u_i\ u_j$ in G, the Mycielski's graph includes two edges $u_i\ v_j$ and $v_i\ u_j$, therefore if G is a graph of n vertices and m edges, then the Mycielski's graph of G has $2n+1$ vertices and $3m+n$ edges and is denoted by $\mu(G)$. Figure (1) represents Mycielski's graph of the cycle C_3.

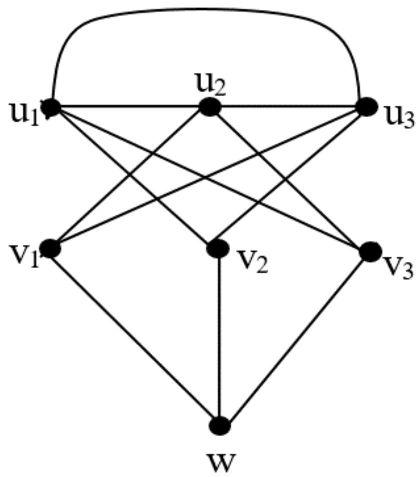

Figure 1: $\mu(C_3)$

2. Main Results

2.1. Cog-Path Graph P_m^c

It is a graph consists of a path $P_m : u_1, u_2, \ldots, u_m$ where $m \geq 3$, with $m-1$ additional vertices $v_1, v_2, \ldots, v_{m-1}$ and additional edges $\{u_iv_i, v_iu_{i+1}, i = 1, 2, \ldots, m-1\}$. The number of vertices of P_m^c is $2m-1$ and the number of its edges is $3m-3$ [2].

2.1.1. The Basis Number for Mycielski's Graph of the Cog-Path $\mu(P_m^c)$

Let the vertices of the Cog path graph P_m^c be $u_1, u_2, u_3, \ldots, u_{2m-1}$ and the vertices opposite to them are $v_1, v_2, v_3, \ldots, v_{2m-1}$ and let the other vertex be w, from the definition of the Mycielski's graph, it becomes clear that the number of vertices of $\mu(P_m^c)$ is $4m-1$ and the number of its edges are $11m-10$. See Figure (2).

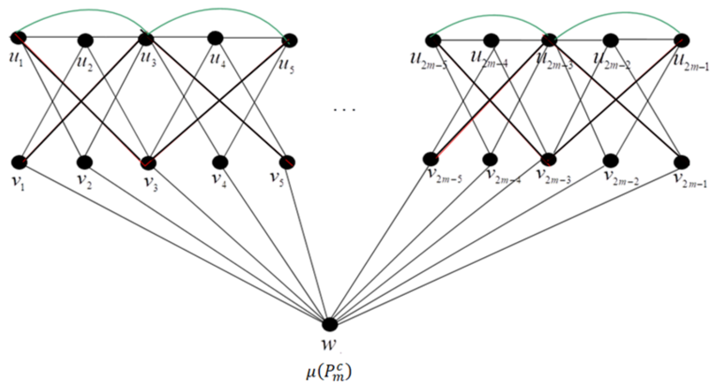

Figure 2: $\mu(P_m^c)$

Theorem 1. *Let P_m be a path of order $m \geq 3$ then $b(\mu(P_m^c)) = 3$*

Proof. We can prove that for each $m \geq 3$, there is a subgraph of $\mu(P_m^c)$ that topologically equivalent $K_{3,3}$, according to Kurtowski's Theorem [8], $\mu(P_m^c)$ is not planar, and according to McLean's Theorem [12] we have

$$b(\mu(P_m^c)) \geq 3 \tag{1}$$

We will prove that there is a base B for the cycles space of a graph $\mu(P_m^c)$ with 3-fold.

Let B be a set of cycles of $\mu(P_m^c)$ which defined by the following formula:

$B = \cup_{j=1}^{5} M_j \cup \{C\}, where$

$M_1 = \{u_{2i-1}u_{2i+1}v_{2i-1}u_{2i}v_{2i+1}u_{2i-1} : i = 1, 2, 3, \ldots, m-1\},$

$M_2 = \{wv_{2i-1}u_{2i+1}v_{2i}w : i = 1, 2, 3, \ldots, m-1\},$

$M_3 = \{u_{2i-1}u_{2i}u_{2i+1}u_{2i-1} : i = 1, 2, 3, \ldots, m-1\},$

$M_4 = \{wv_iu_{i+1}v_{i+2}w : i = 1, 2, 3, \ldots, 2m-3\},$

$M_5 = \{u_iu_{i+1}u_{i+2}v_{i+1}u_i : i = 1, 2, 3, \ldots, 2m-3\},$

$C = \{wv_1u_2u_1v_2w\}.$

In order B to be the base for the cycles space of the graph $\mu(P_m^c)$, it must be $|B| = \dim C(\mu(P_m^c))$ and B must be a linearly independent set. It is known that

$$\dim C(\mu(P_m^c)) = 7m - 8$$

$$|B| = |\cup_{j=1}^{5} M_j| + 1$$

$$= 3(m-1) + 2(2m-3) + 1 = 7m - 8.$$

It remains to show that B is linearly independent cycles.

Clearly that the cycles of each M_1, M_2 and M_3 are independent because they are separate cycles with respect to edges; and the cycles of each M_4 and M_5 are independent because it is represent the boundaries of the faces of a planar subgraph.

Now; the cycle C is independent of M_5 because contains the edges wv_1 and wv_2 but these edges are not available in any linear combination of cycle M_5, hence $M_5 \cup \{C\}$ is linearly independent. Also, any linear combination of $M_5 \cup \{C\}$ contains the edges of type $u_i\ u_{i+1}, i = 1, 2, \ldots, 2m-2$ and these edges are not available in any linear combination of cycle M_4, therefore $M_5 \cup \{C\} \cup M_4$ is linearly independent. Further more $M_4 \cup M_5 \cup \{C\} \cup M_3$ are independent set of cycles since any linear combination of cycles M_3 contains the edges of type $u_{2i-1}\ u_{2i+1}$, $i = 1, 2, \ldots, m-1$ and these edges are not available in $M_4 \cup M_5 \cup \{C\}. Also, M_3 \cup M_4 \cup M_5 \cup \{C\} \cup M_2$ are independent set of cycles because M_2 contains the edges of type $v_{2i-1}\ u_{2i+1}, i = 1, 2, \ldots, m-1$ but these edges are not available in $M_3 \cup M_4 \cup M_5 \cup \{C\}$.

Finally; the cycles of the set $B = \cup_{j=1}^{5} M_j \cup \{C\}$ are independent since any linear combination of cycles M_1 contains the edges of type $u_{2i-1}v_{2i+1}, i = 1, 2, \ldots, m-1$ while these edges are not available in any linear combination of cycles $M_2 \cup M_3 \cup M_4 \cup M_5 \cup \{C\}$.

To find the fold for base B, we divide the edges of the graph $\mu(P_m^c)$ into:

$E_1 = \{u_iu_{i+1} : i = 1, 2, \ldots, 2m-2\}$

$E_2 = \{u_iv_{i+1} : i = 1, 2, \ldots, 2m-2\}$

$E_3 = \{v_{2i-1}u_{2i} : i = 1, 2, \ldots, m-1\}$

$E_4 = \{v_{2i}u_{2i+1} : i = 1, 2, \ldots, m-1\}$

$E_5 = \{wv_i : i = 1, 2, \ldots, 2m-1\}$

$E_6 = \{u_{2i-1}u_{2i+1} : i = 1, 2, \ldots, m-1\}$

$E_7 = \{u_{2i-1}v_{2i+1} : i = 1, 2, \ldots, m-1\}$

$E_8 = \{v_{2i-1}u_{2i+1} : i = 1, 2, \ldots, m-1\}$

Now, we calculate the fold for a set of the edges of the graph $\mu(P_m^c)$,

Case I: $f_{B(\mu(P_m^c))}(e)$ is less than or equal to 1 when $e \in E_7$.

Case II: $f_{B(\mu(P_m^c))}(e)$ is less than or equal to 2 for all $e \in E_i$,$i = 6, 8$.

Case III: $f_{B(\mu(P_m^c))}(e)$ is less than or equal to 3 for all $e \in E_i, i = 1, 2, 3, 4, 5$.

From the above three cases, it can be seen that the fold for each edge in the graph $\mu(P_m^c)$ is not more than 3 in the base $B(\mu(P_m^c))$; That is

$$b\left(\mu(P_m^c)\right) \leq 3 \tag{2}$$

From (1) and (2), we get $b(\mu(P_m^c)) = 3$.

2.2. Cog-Cycle Graph C_m^c

It is a graph conclude from a cycle $C_m : u_1, u_2, \ldots, u_m$ where $m \geq 3$, by adding m vertices and $2m$ edges of the form $v_1, v_2, \ldots, v_m$ and $\{u_i v_i, u_{i+1} v_i : i = 1, 2, \ldots, m\}$, respectively, where $u_{m+1} \equiv u_1$. It is clear that the number of vertices of a graph C_m^c is $2m$ and the number of edges is $3m$ [2].

2.2.1. The Basis Number for Mycielski's Graph of the Cog-Cycle $\mu(C_m^c)$

Let the vertices of the cog-cycle graph C_m^c are $u_1, u_2, \ldots, u_2m$ where $m \geq 3$, and the corresponding vertices are $v_1, v_2, \ldots, v_{2m}$ and let the other vertex be w. From the definition of the Mycielski's graph we have the number of vertices of the graph $\mu(C_m^c)$ is $4m + 1$ and the number of its edges is $11m$.

Theorem 2. *Let C_m be a cycle of order $m \geq 3$ then $b(\mu(C_m^c)) = 3$*

Proof. We can prove that for each $m \geq 3$, there is a subgraph of $\mu(C_m^c)$ that topologically equivalent $K_{3,3}$, according to Kurtowski's Theorem [8] $\mu(C_m^c)$ is not planar, and according to McLean's Theorem [12] we have

$$b\left(\mu(C_m^c)\right) \geq 3 \tag{3}$$

We will prove that there is a base B for the cycles space of the graph $\mu(C_m^c)$ with 3-fold.

Let B be a set of cycles of $\mu(C_m^c)$ which defined by the following formula:

$$B = B(\mu(P_m^c)) \cup M, \text{where } \; M = \{M_1, M_2, \ldots, M_8\}$$

Where $B(\mu(P_m^c))$ is the base for Michelsky's graph of the cog-path P_m^c, which defined in the previous theorem, also M is a set of cycles of the graph $\mu(C_m^c)$ defined as the following formula:

$M_1 = u_1 u_{2m-1} u_{2m} u_1$,

$M_2 = u_1 u_{2m-1} v_{2m} u_1$,

$M_3 = u_1 u_{2m-1} v_{2m-2} w v_{2m} u_1$,

$M_4 = u_1 u_{2m} v_{2m-1} u_1$,

$M_5 = u_1u_{2m}v_1u_3u_1,$

$M_6 = u_{2m-2}u_{2m-1}u_{2m}v_{2m-1}u_{2m-2},$

$M_7 = v_1u_{2m-1}u_{2m}v_1,$

$M_8 = v_{2m}u_{2m-1}u_{2m-3}v_{2m-1}wv_{2m}.$

In order B to be the base for the cycles space of graph $\mu(C_m^c)$ must be $|B| = dimC(\mu(C_m^c))$, and B must be a linearly independent set of cycles. It is known that

$$dim \;\; C(\mu(C_m^c)) = 11m - (2m+1) + 1 = 7m, \text{and}$$

$$|B| = |B(\mu(P_m^c))| + |M|$$
$$= (7m - 8) + 8 = 7m$$

It remains to show that B is linearly independent.

It is known that $B(\mu(P_m^c))$ is linearly independent because it is represent the base of the cycles space of $\mu(P_m^c)$. In addition, the cycles set M is linearly independent because one of them cannot be written as a linear combination of the other cycles.

Finally, the set of cycles $B = B(\mu(P_m^c)) \cup \{M_1, M_2, \ldots, M_8\}$ is independent because any linear combination of M_i's cycles, $i = 1, 2, \ldots, 8$ contains at least one new edge of type $u_1u_{2m-1}, u_1u_{2m}, u_{2m-1}u_{2m}, u_1v_{2m-1}, u_1v_{2m}, v_1u_{2m-1}, v_1u_{2m}, u_{2m-1}v_{2m}, v_{2m-1}u_{2m}, wv_{2m}$ while these edges are not exist in any linear combination for cycles of $B(\mu(P_m^c))$.

To find the fold for the base B we divide the edges of the graph $\mu(C_m^c)$ into:

$$E_1 = E(\mu(P_m^c)) - E_2$$

$$E_2 = \{u_1u_3, v_1u_3, u_{2m-3}u_{2m-1}, u_{2m-2}u_{2m-1}, u_{2m-2}v_{2m-1}, v_{2m-2}u_{2m-1}, wv_{2m-2}, wv_{2m-1}\}$$

$$E_3 = \{u_1u_{2m-1}, u_1u_{2m}, u_{2m-1}u_{2m}, u_1v_{2m-1}, u_1v_{2m}, v_1u_{2m-1}, v_1u_{2m}, u_{2m-1}v_{2m}, v_{2m-1}u_{2m}, wv_{2m}\}$$

Now, we calculate the fold for a set of the edges of the graph $\mu(C_m^c)$, We note that $f_{B(\mu(C_m^c))}(e)$ is less than or equal to 3 for all $e \in E_i, \; i = 1, 2, 3$, thus the fold for each edge in the graph $\mu(C_m^c)$ is not more than 3 in the base $B(\mu(C_m^c))$; That is

$$b\left(\mu(C_m^c)\right) \leq 3 \tag{4}$$

From (3) and (4), we get $b(\mu(C_m^C)) = 3$.

2.3. Cog-Star Graph S_m^c

It is a graph consisted of a star graph $S_m : u_1, u_2, \ldots, u_{m-1}, u_m$, where $m \geq 4$ with $m-1$ of additional vertices $v_1, v_2, \ldots, v_{m-2}, v_{m-1}$ and additional edges $\{u_iv_{i+1}, u_iv_{i+2}, i = 1, 2, \ldots, m-1\}$, where $v_{m+1} \equiv v_2$ [2].

It is clear that the number of vertices of a graph S_m^c is $2m-1$ and the number of edges is $3m-3$.

2.3.1. The Basis Number for Mycielski's Graph of the Cog-Star $\mu(S_m^c)$

Let the vertices of the cog-star graph S_m^c are $u_1, u_2, \ldots, u_{2m-1}$ and the corresponding vertices are $v_1, v_2, \ldots, v_{2m-1}$ and the other vertex is w.

By Mycielski's definition, it turns out that the number of vertices of $\mu(S_m^c)$ is $4m-1$ and the number of its edges is $11m-10$.

Theorem 3. *Let S_m be a star of order $m \geq 4$ then $b(\mu(S_m^c)) = 3$.*

Proof. We can prove that for each $m \geq 4$, there is a subgraph of $\mu(S_m^c)$ that topologically equivalent $K_{3,3}$, according to Kurtowski's Theorem [8], $\mu(S_m^c)$ is not planar and according to McLean's Theorem [12] we have

$$b\left(\mu(S_m^c)\right) \geq 3 \tag{5}$$

Let B be a set of cycles of $\mu(S_m^c)$ which defined by the following formula:

$$B = B(\mu(S_m^c)) = \cup_{i=1}^{5} S_i \cup \{C_1, C_2, C_3, C_4, C_5, C_6\}$$

Where

$S_1 = \{u_i u_{i+1} u_{i+2} v_{i+1} u_i : i = 1, 2, 3, \ldots, 2m-3\}$,

$S_2 = \{w v_i u_{i+1} v_{i+2} w : i = 1, 2, 3, \ldots, 2m-3\}$,

$S_3 = \{u_{2m-1} v_i u_{i+1} v_{i+2} u_{2m-1}, i = 2, 4, 6, \ldots, 2m-4\}$,

$S_4 = \{u_{2m-1} u_i v_{2m-1} u_{i+2} u_{2m-1}, i = 2, 4, 6, \ldots, 2m-6\}$,

$S_5 = \{u_{2m-1} v_i w v_{i+1} u_{i+2} u_{2m-1}, i = 2, 4, 6, \ldots, 2m-6\}$,

$C_1 = u_{2m-2} u_1 u_2 u_{2m-1} u_{2m-2}$,

$C_2 = u_{2m-2} v_1 w v_{2m-1} u_{2m-2}$,

$C_3 = v_{2m-2} u_1 u_{2m-2} u_{2m-1} v_{2m-2}$,

$C_4 = v_1 u_{2m-2} u_1 v_{2m-2} w v_1$,

$C_5 = v_{2m-2} u_1 v_2 w v_{2m-2}$,

$C_6 = u_{2m-1} v_{2m-4} w v_{2m-3} u_{2m-4} u_{2m-1}$

In order B to be the base for the cycles space of the graph $\mu(S_m^c)$, it must be $|B| = dim \ \ C(\mu(S_m^c))$, and B must be a linearly independent set of cycles. It is known that $dim \ \ C(\mu(S_m^c)) = 7m-8$, and $|B| = |B(\mu(S_m^c))| = |\cup_{i=1}^{5} S_i| + |\{C_1, C_2, C_3, C_4, C_5, C_6\}| = (7m-14) + 6 = 7m-8$, since $|S_1| = |S_2| = 2m-3$ and $|S_3| = m-2, |S_4| = |S_5| = m-3$. It remains to show that B is linearly independent.

It is clear that each of S_1, S_2, S_3, S_4 and S_5 is linearly independent because it is represent the boundaries of the faces of a planar subgraph. $S_1 \cup S_2$ is linearly independent because any linear combination of S_2 contains edges of type $wv_i, i = 1, 2, \ldots, 2m-1$, which are not found in any linear combination of S_1. Also, $S_3 \cup S_4$ is linearly independent because any linear combination of S_4 contains edges of type $u_{2m-1}u_i, i = 2, 4, \ldots, 2m-4$, which are not found in any linear combination of S_3. In addition, $S_3 \cup S_4 \cup S_5$ is linearly independent since any linear combination of S_5 contains edges of type $wv_i, i = 2, 4, \ldots, 2m-1$, which are not found in any linear combination of $S_3 \cup S_4$. Also, $(S_1 \cup S_2) \cup (S_3 \cup S_4 \cup S_5)$ is linearly independent because $S_3 \cup S_4 \cup S_5$ contains edges of type $u_{2m-1}u_i, i = 2, 4, \ldots, 2m-4$, which are not found in any linear combination of $S_1 \cup S_2$.

Finally, $(\cup_{i=1}^{5} S_i) \cup (\{C_1, C_2, C_3, C_4, C_5, C_6\})$ is linearly independent because any linear combination of $\{C_1, C_2, C_3, C_4, C_5, C_6\}$ contains edges of type $u_{2m-2}u_1, v_{2m-2}v_1$, which are not found in any linear combination of $\cup_{i=1}^{5} S_i$.

To find the fold for the base B we divide the edges of the graph $\mu(S_m^c)$ into:

$$E_1 = \{u_i u_{i+1}, i = 2, 3, \ldots, 2m-3\},$$

$$E_2 = \{u_i v_{i+1}, i = 2, 3, \ldots, 2m-5\} \cup \{u_{2m-3}v_{2m-2}\}$$

$$E_3 = \{v_i u_{i+1}, i = 1, 2, \ldots, 2m-3\} \cup \{u_{2m-1}v_j, j = 2, 4, \ldots, 2m-6\}$$

$$E_4 = \{wv_i, i = 3, 4, \ldots, 2m-5\}$$

$$E_5 = \{u_{2m-1}u_i, i = 4, 6, \ldots, 2m-6\} \cup \{v_{2m-1}u_j, j = 2, 4, \ldots, 2m-4\}$$

$$E_6 = \{v_{2m-2}u_1, u_{2m-2}u_1, u_{2m-2}v_1\}$$

$$E_7 = \{u_{2m-2}v_{2m-1}, u_1u_2, wv_{2m-1}, u_{2m-1}u_2, u_1v_2\}$$

$$E_8 = \{wv_1, wv_{2m-2}, u_{2m-2}u_{2m-1}\}$$

$$E_9 = \{wv_2, wv_{2m-4}, wv_{2m-3}, u_{2m-4}v_{2m-3}, u_{2m-1}u_{2m-4}, u_{2m-1}v_{2m-2}, u_{2m-1}v_{2m-4}\}$$

Now, we calculate the fold for a set of the edges of the graph $\mu(S_m^c)$,

Case I: $f_{B(\mu(S_m^c))}(e)$ is equal to 2 for all $e \in E_i$, $i = 1, 7$.

Case II: $f_{B(\mu(S_m^c))}(e)$ is less than or equal to 3 for all $e \in E_i$, $i = 2, 3, 4, 5, 6, 8, 9$.

From the above two cases, it can be seen that the fold for each edge in the graph $\mu(S_m^c)$ is not more than 3 in the base $B(\mu(S_m^c))$; That is

$$b\left(\mu(S_m^c)\right) \leq 3 \tag{6}$$

From (5) and (6), we get $b(\mu(S_m^c)) = 3$.

2.4. Cog-Wheel Graph W_m^c

It is a graph consisted of a wheel $W_m : u_1, u_2, \ldots, u_m$ where $m \geq 4$, by adding $m-1$ vertices and $2m-2$ edges of the form $v_1, v_2, \ldots, v_{m-1}$ and $\{v_i u_i, v_i u_{i+1} : i = 1, 2, \ldots, m-1\}$ respectively, where $u_m \equiv u_1$. It is clear that the number of vertices of a graph W_m^c is $2m-1$ and the number of edges is $4m-4$ [2].

2.4.1. The Basis Number for Mycielski's Graph of the Cog-Wheel $\mu(W_m^c)$

Let the vertices of the cog-wheel graph w_m^c are $u_1, u_2, \dots, u_{2m-1}$ and the corresponding vertices are $v_1, v_2, \dots, v_{2m-1}$ and the other vertex is w, since the number of vertices of cog-wheel is $2m-1$ and the number of its edges is $4(m-1)$, then by Mycielski's definition, it turns out that the number of vertices of $\mu(w_m^c)$ is $4m-1$ and the number of its edges is $14m-13$.

Theorem 4. *Let W_m be a wheel of order $m \geq 5$ then $b(\mu(W_m^c)) = 3$.*

Proof. We can prove that for each $m \geq 5$, there is a subgraph of $\mu(W_m^c)$ that topologically equivalent $K_{3,3}$, according to Kurtowski's Theorem [8] $\mu(W_m^c)$ is not planar and according to McLean's Theorem [12] we have

$$b\left(\mu(W_m^c)\right) \geq 3 \tag{7}$$

We will prove that there is a base B for the cycles space of the graph $\mu(W_m^c)$ of 3-fold.

Let B be a set of cycles of $\mu(W_m^c)$ which defined by the following formula:

$$B = B(\mu(P_m^c)) \cup (\cup_{i=1}^{3} S_i) \cup \{C_1, C_2, C_3\}$$

Where $B(\mu(P_m^c))$ is the base for Michelsky's graph of the cog-path, $m \geq 5$ and

$$S_1 = \{u_{2m-1}v_i u_{i+2} u_{2m-1} : i = 1, 3, 5, \dots, 2m-5\},$$

$$S_2 = \{v_{2m-1}u_i u_{i+2} v_{2m-1} : i = 1, 3, 5, \dots, 2m-5\},$$

$$S_3 = \{u_{2m-1}u_i v_{i+2} u_{2m-1} : i = 1, 3, 5, \dots, 2m-5\},$$

$$C_1 = v_{2m-1}u_1 u_{2m-3} v_{2m-1},$$

$$C_2 = u_1 v_{2m-3} u_{2m-5} u_{2m-1} u_1$$

$$C_3 = u_{2m-3} v_1 u_{2m-1} u_1 u_{2m-3},$$

In order B to be the base for the cycles space of the graph $\mu(W_m^c)$, it must be $|B| = dim\ C(\mu(W_m^c))$, and B must be a linearly independent set of cycles.

Clearly, $dim\ C(\mu(W_m^c)) = 10m - 11$, and since

$$|B| = |B(\mu(P_m^c))| + |\cup_{i=1}^{3} S_i| + |\{C_1, C_2, C_3\}|$$

$$= 7m - 8 + 3m - 6 + 3 = 10m - 11$$

Now, it remains to show that B is linearly independent.

It is known that $B(\mu(P_m^c))$ is linearly independent because it is represent the base of the cycles space of $\mu(P_m^c)$. Note that each of S_1, S_2andS_3 is linearly independent because it is represent the boundaries of the faces of a planar subgraph. Now, $S_1 \cup S_2$ is linearly independent because any linear combination of S_2 contains edges of type

$u_iu_{i+2}, i = 1, 3, 5, \ldots, 2m-3$, which are not found in any linear combination of S_1. Now, $S_1 \cup S_2 \cup S_3$ is linearly independent because any linear combination of S_3 contains edges of type $u_iv_{i+2}, i = 1, 3, \ldots, 2m-5$, which are not found in any linear combination of $S_1 \cup S_2$. In addition, $\{C_1, C_2, C_3\}$ is linearly independent because we cannot write any one of them as a linear combination of the others cycles. Now, $(\cup_{i=1}^{3} S_i) \cup (\{C_1, C_2, C_3\})$ is linearly independent because any linear combination of $\{C_1, C_2, C_3\}$ contains at least one of the edges $u_1u_{2m-3}, u_1v_{2m-3}, v_1u_{2m-3}$, which are not found in any linear combination of $\cup_{i=1}^{3} S_i$.

Finally, the set of cycles $B = B(\mu(P_m^c)) \cup (\cup_{i=1}^{3} S_i) \cup (\{C_1, C_2, C_3\})$ is linearly independent because any linear combination of cycles in $(\cup_{i=1}^{3} S_i) \cup (\{C_1, C_2, C_3\})$ contains edges of type $u_{2m-1}u_i, i = 1, 3, \ldots, 2m-5$, which are not found in any linear combination in $B(\mu(P_m^c))$, therefore $B(\mu(W_m^c))$ is linearly independent.

To find the fold for the base B we divide the edges of the graph $\mu(W_m^c)$ into:

$E_1 = \{u_iu_{i+1}, i = 1, 2, \ldots, 2m-2\}$

$E_2 = \{u_iv_{i+1}, v_iu_{i+1}, i = 1, 2, \ldots, 2m-2\}$

$E_3 = \{wv_i, i = 1, 2, \ldots, 2m-1\}$

$E_4 = \{u_iv_{i+2}, i = 1, 3, \ldots, 2m-3\}$

$E_5 = \{v_iu_{i+2}, i = 1, 3, \ldots, 2m-3\}$

$E_6 = \{u_iu_{i+2}, i = 1, 3, \ldots, 2m-3\}$

$E_7 = \{u_{2m-1}u_i, i = 1, 3, \ldots, 2m-5\}$

$E_8 = \{v_{2m-1}u_i, u_{2m-1}v_i, i = 1, 3, \ldots, 2m-5\}$

$E_9 = \{u_1v_{2m-3}, v_1u_{2m-3}, u_1u_{2m-3}\}$

Now, we calculate the fold for a set of the edges of the graph $\mu(W_m^c)$,

Case I: $f_{B(\mu(W_m^c))}(e)$ is less than or equal to 2 for all $e \in E_i$, $i = 8, 9$.

Case II: $f_{B(\mu(W_m^c))}(e)$ is less than or equal to 3 for all $e \in E_i$, $i = 1, 2, \ldots, 7$.

From the above two cases, it can be seen that the fold for each edge in the graph $\mu(W_m^c)$ is not more than 3 in the base B$(\mu(W_m^c))$; That is

$$b(\mu(W_m^c)) \leq 3 \tag{8}$$

From (7) and (8), we get $b(\mu(W_m^c)) = 3$.

3. Conclusion

After studying the basis number of Mycielski 's graph for some cog-graphs, we concluded that $b(\mu(G)) = 3$, where G are cog-path graph, cog-cycle graph, cog-star graph and cog-wheel graph.

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