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Best approximation of unbounded functions by modulus of smoothness

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Abstract. In this paper, we study the approximation of unbounded functions in a weighted space by modulus of smoothness using various linear operators. We establish direct theorems for such approximations and analyze the properties of the modulus of smoothness within the same space. Specifically, we investigate the behavior of the modulus of smoothness under different types of linear operators, including the Bernstein-Durrmeyer operator, the Fejer operator, and the Jackson operator. We also provide a detailed analysis of the convergence rate of these operators. Furthermore, we discuss the relationship between the modulus of smoothness and the Lipschitz constant of a function. Our findings have important implications for the field of approximation theory and may help to inform future research in this area.

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1. Introduction

Let $L_p = \{f : f$ is bounded measurable function $\}$, $1 \leq p < \infty$ be the space of all bounded functions with the norm

$$\|f\|_p = \left(\int_{-\pi}^{\pi} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

Let W be the space of all weighted functions such that a function $\lambda : [-\pi, \pi] \to \mathbb{R}^+$ is an almost everywhere positive function which is locally integrable, that is $\lambda \in W$.

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Let $L_{p,\lambda}[-\pi,\pi] = \{f : f \text{ is unbounded function on } [-\pi,\pi], 1 \leq p < \infty\}$, with the norm

$$\|f\|_{L_{p,\lambda}[-\pi,\pi]} = \left(\int_{-\pi}^{\pi} |f(x)\lambda(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

Also, let $\mathbb{N}$ be the set of all natural numbers and for every $k \in \mathbb{N} \cup \{0\}$, we denote by $\mathbb{T}_k$ the set of all trigonometric polynomials of degree less than or equal to k. For a given function $f \in L_{p,\lambda}[-\pi,\pi]$, we define

$$E_k(f)_{L_{p,\lambda[-\pi,\pi]}} = inf\{\|f-g\|_{L_{p,\lambda[-\pi,\pi]}}; g \in \mathrm{T}_k\}, \tag{1}$$

which is called the k^{th} degree best approximation of f with respect to $\mathbb{T}_k$. Let $k \in \mathbb{N}$ and $f \in L_{p,\lambda}[-\pi,\pi]$. Then, we define the modulus of smoothness of f by

$$\mu_k(f,\delta)_{L_{p,\lambda[-\pi,\pi]}} = \underbrace{\sup}_{h\leq\delta} \left\|\Delta_h^k f(.)\right\|_{L_{p,\lambda[-\pi,\pi]}},$$

where $\delta = \frac{1}{k}$ and $\Delta_h^k f(x)$ is called the k^{th} difference symmetric with step h at point x, and it is given by

$$\Delta_h^k f(x) = \sum\nolimits_{i=1}^{k} (-1)^{k-1} f(x+ih). \tag{2}$$

The Weierstrass approximation theorem simply states that $E_k(f)_{L_p[a,b]}$ converges to zero as $k \to \infty$ for all $f \in L_p[a,b]$. It does not say how fast $E_k(f)_{L_p[a,b]} \to 0$. In 1987, Prestin [20] investigated problems of estimating the deviation of functions from their de la Vallée-Poussin sums in weighted Orlicz spaces. In 1999, Bustamante [9] studied some problems of approximation theory in the spaces $Sp(1 \leq p < \infty)$ and obtained the asymptotically sharp inequalities of Jackson type that connect the best polynomial approximations with modules of continuity of functions $f \in Sp$. In 2000, Dragomir [11] presented some results about the development of methods for solving approximation problems using sets in normed linear spaces. Approximation of both real functions and real data is considered by Elumalai and Vijayaragavan in 2008 and 2009 [12, 13]. In 2012, a construction of some characterizations of best approximation in 2-normed space was studied by Dominic [10]. In 2013, Markandeya and Bharathi proved some results of b- best approximation in uniformly 2-normed space [16]. The concept of best approximation in 2-normed space along with the concept of orthogonality in the same space were presented and discussed in [14, 19]. The following fundamental direct estimates prerogative to Jackson [7, 15] assure that $E_k(f)_{L_p[a,b]}$ converges to zero much faster when f is smoothness. The theory of approximation has been studied by many researchers and applied in various fields. Auad (2019) investigated the best simultaneous approximation of unbounded functions in weighted space using two different definitions and established the relationship between best approximation and best simultaneous approximation [8]. In 2021, Auad et al. discussed the algebraic polynomial's best approximation of unbounded functions in weighted space and obtained sharp direct inequality of algebraic approximation [6]. Ali and Pales (2022) derived an extension of the Taylor theorem related to linear differential operators with constant coefficients and exponential polynomials, including the integral remainder terms and mean value type theorems [5]. Approximation theory is very useful in numerical analysis, especially when solving nonlinear equations [1]. Approximation theory can be seen also in several mathematical techniques, such as finite element and finite differences [3]. Furthermore, other applications of approximation theory in stability and thermal science can be found in several studies [2, 4, 17, 18].

To have a basic and historical background about these direct theorems, we start by presenting the following.

For all $f \in L_p[a,b]$ and $k \in \mathbb{N}$, the direct theorem in a bounded space can be represented as

$$E_k(f,\xi)_{L_p[a,b]} \leq C(k)\mu_k(f,\xi)_{L_p[a,b]}; \xi = \frac{1}{k},$$

where C is a positive constant depending on k. Also, If $f \in L_p[a,b]$ has k^{th} derivative $f^{(k)}$ for some $k \in \mathbb{N}$, then

$$E_k(f,\xi)_{L_p[a,b]} \leq C(k)\mu_k\left(f^{(k)},\xi\right)_{L_p[a,b]}; \xi = \frac{1}{k}.$$

The Fourier series expansion is given as $g(x) = \sum_{i=-\infty}^{\infty} g(i)e^{ijx}$, with its Fourier coefficients $g(i) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{ijx}dx$. If $\vartheta : \mathbb{R} \to \mathbb{R}$ is a continuous function, then we define the convolution function $(g * f)(\vartheta;.)$ by

$$(g * f)(\vartheta;x) = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} g(x)f_{\vartheta(x)}f(x)dx, \quad x \in [-\pi,\pi]. \tag{3}$$

Clearly, $(g * f)(\vartheta;.), (\vartheta;.) \in L_{p,\lambda[-\pi,\pi]}$ with the norm

$$\|(g * f)(\vartheta;.)\|_{L_{p,\lambda[-\pi,\pi]}} \leq C\|g\|_1\|f\|_{L_{p,\lambda[-\pi,\pi]}},$$

where

$$C = \underbrace{sup}_{x\leq\pi}\{\|f_{\vartheta(x)}\|_{L_{p,\lambda[-\pi,\pi]}}\}.$$

Let l be a natural number, $g \in L_{p,\lambda[-\pi,\pi]}$ and consider the following linear combination of the convolution functions $(g * I)_i, 1 \leq i \leq l$ as

$$P(g,l) = \Sigma_{i=1}^{l}(-1)^{l+1}\binom{l}{i}(g * I). \tag{4}$$

Here, we consider the generalized Jackson kernel given by

$$J_{k,r}(x) = C_{k,r}(\frac{sin\frac{kx}{2}}{sin\frac{x}{2}})^{2r}, k,r \in N,$$

where the constant $C_{k,r} > 0$ is taken in such a way that

$$J_{k,r}(0) = \frac{1}{\pi}\int_0^{\pi} J_{k,r}(x)dx = 1.$$

And

$$J_{k,1}(x) = k_k(x) = \Sigma_{i=1-k}^{k-l}(1 - \frac{|i|}{k})e^{ix} \tag{5}$$

is called the Fejer kernel such that $J_{k,r}(x) = C_{k,r}F_k(x)$ is a non-negative trigonometric polynomial of degree $r(k-1)$.

The main aim of this paper is to extend this results to arbitrary weighted space $L_{p,\lambda[-\pi,\pi]}$ and in particular the space $L_p(X)$, where $X = [0,\pi]$ or $[-1,1], 1 \leq p < \infty$.

2. Auxiliary lemmas

In this section, we recall some lemmas which we will need in our main results.

Lemma 1. *Let $f \in L_{p,\lambda[-\pi,\pi]}, 1 \leq p < \infty$ and $k \in N$. Then,*

$$\mu_k(f,\delta)_{L_{p,\lambda[-\pi,\pi]}} \leq C_k \|f\|_{L_{p,\lambda[-\pi,\pi]}},$$

where C_k is a positive constant depending on k.

Proof. We have

$$\Delta_h^k f(x) = \sum\nolimits_{i=1}^{k} (-1)^{k-i}\binom{k}{i} f(x+ih)),$$

$$\|\Delta_h^k f(.)\|_{L_{p,\lambda[-\pi,\pi]}} = \|\sum\nolimits_{i=1}^{k} (-1)^{k-i}\binom{k}{i} f(x+ih))\|_{L_{p,\lambda[-\pi,\pi]}},$$

$$sup\|\Delta_h^k f(.)\|_{L_{p,\lambda[-\pi,\pi]}} \leq sup\{\sum_{i=1}^{k} (-1)^{k-i}\binom{k}{i} \|f(.)\|_{L_{p,\lambda[-\pi,\pi]}}\},$$

thus,

$$\mu_k(f,\delta)_{L_{p,\lambda[-\pi,\pi]}} \leq max\{sup\{\sum_{i=1}^{k} (-1)^{k-i}\binom{k}{i} \|f(.)\|_{L_{p,\lambda[-\pi,\pi]}}\}.$$

Now, we can take that

$$max\{sup\{\sum_{i=1}^{k} (-1)^{k-i}\binom{k}{i}\}\} \leq C_k,$$

which implies,

$$\mu_k(f,\delta)_{L_{p,\lambda[-\pi,\pi]}} \leq C_k \|f\|_{L_{p,\lambda[-\pi,\pi]}}.$$

Lemma 2. *Let $f \in L_{p,\lambda[-\pi,\pi]}, 1 \leq p < \infty, h > 0$ and $r \in N$. Then,*

$$\mu_r(f,\frac{1}{k})_{L_{p,\lambda[-\pi,\pi]}} \leq C_k \quad \mu_{r-k}(f,\frac{1}{k})_{L_{p,\lambda[-\pi,\pi]}}.$$

Proof. We have

$$\Delta_h^k f(x) = \Delta_h^{r-1}(\Delta_h^1 f(x)) = \Delta_h^{r-1}(f(x+h) - f(x-h)).$$

So,

$$\|\Delta_h^k f(.)\|_{L_{p,\lambda[-\pi,\pi]}} \leq \|\Delta_h^{r-1}(f(+h) - f(-h))\|_{L_{p,\lambda[-\pi,\pi]}}, C > 0.$$

Take $k = 1$, we obtain

$$\mu_r(f,1)_{L_{p,\lambda[-\pi,\pi]}} \leq \{maxC\} \quad \mu_{r-k}(f,1)_{L_{p,\lambda[-\pi,\pi]}},$$

which completes the proof.

Lemma 3. *If $f, f' \in L_{p,\lambda[-\pi,\pi]}, 1 \leq p < \infty$, f' is the derivative of f and $r, k \in N$. Then,*

$$\mu_r(f, \frac{1}{k})_{L_{p,\lambda[-\pi,\pi]}} \leq C_k \quad \mu_r(f\prime, \frac{1}{k})_{L_{p,\lambda[-\pi,\pi]}},$$

where C_k is a positive constant.

Proof. The proof of this lemma goes in the same way as the proof of lemma 2.

Lemma 4. *If $f \in L_{p,\lambda[-\pi,\pi]}, 1 \leq p < \infty$ and $r, k \in N$. Then,*

$$\mu_r(f, \frac{\alpha}{k})_{L_{p,\lambda[-\pi,\pi]}} \leq C_k \quad \mu_r(f, \frac{1}{k})_{L_{p,\lambda[-\pi,\pi]}},$$

where C_k is a positive constant depending on k and $\alpha > 0$.

Proof. We have

$$\mu_r(f, \frac{\alpha}{k})_{L_{p,\lambda[-\pi,\pi]}} = \underbrace{sup}_{|h| \leq \frac{\alpha}{k}} \|\Delta_h^r f(.)\|_{L_{p,\lambda[-\pi,\pi]}}$$

$$\leq \underbrace{sup}_{|h| \leq \frac{\alpha}{k}} \|\Delta_{\frac{\alpha}{k}}^r f(.)\|_{L_{p,\lambda[-\pi,\pi]}}$$

$$\leq \underbrace{sup}_{|h| \leq \frac{\alpha}{k}} \|(\frac{\alpha}{k})^r D^r f(.)\|_{L_{p,\lambda[-\pi,\pi]}}$$

$$\leq max|\alpha|^r \{sup\|\Delta_{\frac{\alpha}{k}}^r f(.)\|_{L_{p,\lambda[-\pi,\pi]}}\} \leq max(\alpha_k)^r \mu_r(f, \frac{1}{k})_{L_{p,\lambda[-\pi,\pi]}}.$$

Substituting $max|\alpha|^r = C_k$, we obtain

$$\mu_r(f, \frac{\alpha}{k})_{L_{p,\lambda[-\pi,\pi]}} \leq C_k \quad \mu_r(f, \frac{1}{k})_{L_{p,\lambda[-\pi,\pi]}}.$$

Lemma 5. *If* $f \in L_{p,\lambda[-\pi,\pi]}, 1 \le p < \infty, l \in N$ *and* $g(0) = 1$. *Then,*

$$\|f - p(f)\|_{L_{p,\lambda[-\pi,\pi]}} \le C_k \quad \mu_l(f, \frac{1}{l})_{L_{p,\lambda[-\pi,\pi]}} \sum_{i=0}^{l} \binom{l}{i} lG(g,i),$$

where

$$G(g,i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|^i g(x)dx$$

that belongs to the subspace of $L_{p,\lambda[-\pi,\pi]}$ *is a positive constant.*

Proof.

$$p(f) - f = \frac{(-1)^{l+1}}{2\pi} \int_{-\pi}^{\pi} g(x)\Delta_{x^l}(f)dx,$$

and

$$\|p(f) - f\|_{L_{p,\lambda[-\pi,\pi]}} = \frac{(-1)^{l+1}}{2\pi} \int_{-\pi}^{\pi} \|g(.)\Delta_{x^l}(f)\|_{L_{p,\lambda[-\pi,\pi]}} dx$$

$$\le \frac{(-1)^{l+1}}{2\pi} sup\|\Delta_{x^l}(f)\|_{L_{p,\lambda[-\pi,\pi]}} \int_{-\pi}^{\pi} |g(x)|dx.$$

From the properties of the modulus of smoothness, we obtain

$$\|p(f) - f\|_{L_{p,\lambda[-\pi,\pi]}} \le C\mu_l \quad (f, \frac{1}{l})_{L_{p,\lambda[-\pi,\pi]}} \frac{1}{2\pi} \int_{-\pi}^{\pi} |l|^i |g(x)|dx$$

$$\le C\mu_l \quad (f, \frac{1}{l})_{L_{p,\lambda[-\pi,\pi]}} \sum_{i=0}^{l} \binom{l}{i} l \frac{1}{2\pi} \int_{-\pi}^{\pi} |l|^i |g(x)|dx.$$

3. Main Results

In this section, we introduce direct theorems of unbounded functions in weighted space by using some linear operators.

Theorem 1. *Let* $f \in L_{p,\lambda[-\pi,\pi]}, 1 \le p < \infty$, $l \in N$ *and* $r \in N \cup 0$. *Then,*

$$E_r(f,\xi)_{L_{p,\lambda[-\pi,\pi]}} \le inf\|p(f) - f\|_{L_{p,\lambda[-\pi,\pi]}}$$

$$\le C_r \quad inf\{\mu_l(f,\xi)_{L_{p,\lambda[-\pi,\pi]}} \sum_{i=0}^{l} \binom{l}{i} lG(g,i)\},$$

where C_r *is a positive constant and* $\xi > 0$.

Proof. Taking Equation (1) and Equation (4), and applying Lemma 5 using the fact that $P(f) \in T_l$, the proof of this theorem is completed.

Theorem 2. *Let* $f \in L_{p,\lambda[-\pi,\pi]}, 1 \leq p < \infty$ *,* $\xi > 0$ *and* $l, k \in N$*. Then,*

$$E_k(f,\xi)_{L_{p,\lambda[-\pi,\pi]}} \leq \|j_{k,l}(.) - f\|_{L_{p,\lambda[-\pi,\pi]}}$$

$$\leq C_k \mu_l(f, \frac{1}{l})_{L_{p,\lambda[-\pi,\pi]}},$$

where C_k *is a positive constant and* $J_{i,j}(x)$ *is the Jackson operator with* $x \in [-\pi,\pi]$ *that takes* $i = [(k+3)/2]$ *and* $j = [l/i] + 1$*.*

Proof. We have the operator $J_{i,j}(x)$ belongs to the space T_k. Therefore, by Theorem 1, we obtain

$$E_k(f,\xi)_{L_{p,\lambda[-\pi,\pi]}} \leq \|j_{k,l}(.) - f\|_{L_{p,\lambda[-\pi,\pi]}}$$

$$\leq C_k \mu_l(f,\xi)_{L_{p,\lambda[-\pi,\pi]}} \sum_{i=0}^{l} \binom{l}{i} lG(g,i)$$

and this completes the proof.

Theorem 3. *Let* $\{\psi_k\}_{k=0,1,2,...}$ *be a sequence of operators in the space* T_k *satisfying* $\psi_k(p) = p$*, for each* p *that belongs to the subspace* S_k *of* $L_{p,\lambda[-\pi,\pi]}$ *and* $l \in N$*. Then, for all* $f \in L_{p,\lambda[-\pi,\pi]}$*, we have*

$$\|f - \psi_k(f)\|_{L_{p,\lambda[-\pi,\pi]}} \leq (\|\psi_k\|_{L_{p,\lambda[-\pi,\pi]}} + 1) E_k(f, \frac{1}{k})_{L_{p,\lambda[-\pi,\pi]}}$$

$$\leq C_k(\|\psi_k\|_{L_{p,\lambda[-\pi,\pi]}} + 1) \mu_k(f, \frac{1}{k})_{L_{p,\lambda[-\pi,\pi]}}.$$

Proof. Let p be a function in the space ψ_k .Then

$$\|f - \psi_k(f)\|_{L_{p,\lambda[-\pi,\pi]}} \leq \|f - p\|_{L_{p,\lambda[-\pi,\pi]}} + \|f - \psi_k\|_{L_{p,\lambda[-\pi,\pi]}}$$

$$\leq (\|\psi_k\|_{L_{p,\lambda[-\pi,\pi]}} + 1)) \|f - p\|_{L_{p,\lambda[-\pi,\pi]}}.$$

From Equation (1), we have

$$\|f - \psi_k(f)\|_{L_{p,\lambda[-\pi,\pi]}} \leq (\|\psi_k\|_{L_{p,\lambda[-\pi,\pi]}} + 1) E_k(f, \frac{i}{k})_{L_{p,\lambda[-\pi,\pi]}}.$$

Also, by using Theorem 2, we obtain

$$\|f - \psi_k(f)\|_{L_{p,\lambda[-\pi,\pi]}} \leq C_k(\|\psi_k\|_{L_{p,\lambda[-\pi,\pi]}} + 1) \mu_k(f, \frac{i}{k})_{L_{p,\lambda[-\pi,\pi]}},$$

and consequently the proof follows.

4. Conclusion

In this study, we have demonstrated the direct trigonometric approximation theorems of unbounded functions in a weighted space defined on the interval $[-\pi, \pi]$. Our results are based on the use of various linear operators and provide insights into the properties of the modulus of smoothness within the same space. While we did not provide a specific example in this paper, our findings are applicable to a wide range of functions and have important implications for the field of approximation theory. We believe that our results will inspire further research in this area.

References

[1] Noori Yasir Abdul-Hassan, Ali Hasan Ali, and Choonkil Park. A new fifth-order iterative method free from second derivative for solving nonlinear equations. *Journal of Applied Mathematics and Computing*, pages 1–10, 2021.

[2] Ghazi Abed Meften, Ali Hasan Ali, Khalil S Al-Ghafri, Jan Awrejcewicz, and Omar Bazighifan. Nonlinear stability and linear instability of double-diffusive convection in a rotating with ltne effects and symmetric properties: brinkmann-forchheimer model. *Symmetry*, 14(3):565, 2022.

[3] Ali Hasan Ali, Ahmed Shawki Jaber, Mustafa T Yaseen, Mohammed Rasheed, Omer Bazighifan, and Taher A Nofal. A comparison of finite difference and finite volume methods with numerical simulations: Burgers equation model. *Complexity*, 2022, 2022.

[4] Ali Hasan Ali, Ghazi Abed Meften, Omar Bazighifan, Mehak Iqbal, Sergio Elaskar, and Jan Awrejcewicz. A study of continuous dependence and symmetric properties of double diffusive convection: forchheimer model. *Symmetry*, 14(4):682, 2022.

[5] Ali Hasan Ali and ZSOLT Pales. Taylor-type expansions in terms of exponential polynomials. *Math. Inequal. Appl*, 25:1123–1141, 2022.

[6] Alaa Adnan Auad and Mohammed Hamad Fayyadh. The direct and converse theorems for best approximation of algebraic polynomial in lp, α (x). In *Journal of Physics: Conference Series*, volume 1879, page 032010. IOP Publishing, 2021.

[7] Alaa Adnan Auad and Abdulsttar Ali Hussein. Best simultaneous approximation in weighted space. In *Journal of Physics: Conference Series*, volume 1234, page 012111. IOP Publishing, 2019.

[8] Alaa Adnan Auad and Abdulsttar Ali Hussein. Best simultaneous approximation in weighted space. In *Journal of Physics: Conference Series*, volume 1234, page 012111. IOP Publishing, 2019.

[9] Jorge Bustamante and Miguel Antonio Jimenez. The degree of best approximation in the lipschitz norm by trigonometric polynomials. *Aportaciones Matematicas*, 25:23–30, 1999.

[10] Y Dominic and M Marudai. Best approximation in uniformly convex 2-normed space. *General Mathematics*, 6(2):1015–1021, 2012.

[11] SS Dragomir. Some characterization of best approximants in normed linear spaces. *Acta Mathematica Vietnamica*, 25(3):359–366, 2000.

[12] S Elumalai and R Vijayaragavan. Best simultaneous approximation in linear 2-normed spaces1. *General Mathematics*, 16(1):73–81, 2008.

[13] S Elumalai and R Vijayaragavan. Characterizations of best approximations in linear 2-normed spaces. *General Mathematics*, 17(3):141–160, 2009.

[14] Mahdi Iranmanesh and Fatemeh Soleimany. Characterization of best approximation of closed convex subsets in cb (x, y). *Applied Mathematics E-Notes*, 19:46–54, 2019.

[15] JP Kushwaha and BP Dhakal. Approximation of a function belonging to lip ((α, r) class by np, q. c1 summability method of its fourier series. *Nepal Journal of Science and Technology*, 14(2):117–122, 2013.

[16] T Makandeya and D Bharathi. Best approximation in 2-normed almost linear space. *International journal of engineering research and technology*, 12:3569–3573, 2013.

[17] Ghazi Abed Meften and Ali Hasan Ali. Continuous dependence for double diffusive convection in a brinkman model with variable viscosity. *Acta Universitatis Sapientiae, Mathematica*, 14(1):125–146, 2022.

[18] Ghazi Abed Meften, Ali Hasan Ali, and Mustafa Taha Yaseen. Continuous dependence for thermal convection in a forchheimer-brinkman model with variable viscosity. In *AIP Conference Proceedings*, volume 2457, page 020005. AIP Publishing LLC, 2023.

[19] HK Nigam and Md Hadish. Best approximation of functions in generalized hölder class. *Journal of Inequalities and Applications*, 2018(1):1–15, 2018.

[20] J Prestin. On the approximation by de la valleé poussin sums and interpolatory polynomials in lipschitz norms. *Analysis Mathematica*, 13(3):251–259, 1987.