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# Upper and lower weak $s\beta(\star)$-continuity

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**Abstract.** This paper is concerned with the concepts of upper and lower weakly $s\beta(\star)$-continuous multifunctions. Moreover, some characterizations of upper and lower weakly $s\beta(\star)$-continuous multifunctions are investigated.

**2020 Mathematics Subject Classifications**: 54C08, 54C60
**Key Words and Phrases**: Upper weakly $s\beta(\star)$-continuous multifunction, lower weakly $s\beta(\star)$-continuous multifunction

## 1. Introduction

In topology, there has been recently significant interest in characterizing and investigating the characterizations of some weak forms of continuity for functions and multifunctions. As weak forms of continuity in topological spaces, weak continuity [12], quasi-continuity [14], semi-continuity [13] and almost continuity in the sense of Husain [9] are well-known. It is shown in [15] that quasicontinuity is equivalent to semi-continuity. It will be shown that weak continuity, semi-continuity and almost continuity are respectively independent. Popa and Stan [23] introduced weak quasi-continuity which is implied by both weak continuity and quasicontinuity. Janković [10] introduced almost weak continuity as a generalization of both weak continuity and almost continuity. Noiri [16] obtained some characterizations of almost weak continuity and some relations between almost weak continuity and weak continuity. Popa [20] and Smithson [24] independently introduced the notion of weakly continuous multifunctions. The present authors introduced and studied other weak forms of continuous multifunctions: weakly quasicontinuous multifunctions [17], almost weakly continuous multifunctions [18], weakly $\alpha$-continuous multifunctions [22], weakly $\beta$-continuous multifunctions [21]. These multifunctions have similar characterizations. The analogy in their definitions and results suggests the need of formulating a unified theory. Noiri and Popa [19] introduced and studied the notions of upper and lower weakly $m$-continuous multifunctions as a multifunction from a set satisfying certain minimal condition into a topological space. In [2], the present author introduced and studied

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the concepts of upper and lower $\star$-continuous multifunctions in ideal topological spaces. Moreover, several characterizations of upper and lower $\star$-continuous multifunctions were investigated in [3]. The purpose of the present paper is to introduce the notions of upper and lower weakly $s\beta(\star)$-continuous multifunctions. Furthermore, some characterizations of upper and lower weakly $s\beta(\star)$-continuous multifunctions are discussed.

## 2. Preliminaries

Throughout the present paper, spaces $(X,\tau)$ and $(Y,\sigma)$ (or simply $X$ and $Y$) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of a topological space $(X,\tau)$. The closure of $A$ and the interior of $A$ are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathscr{I}$ on a topological space $(X,\tau)$ is a nonempty collection of subsets of $X$ satisfying the following properties: (1) $A\in\mathscr{I}$ and $B\subseteq A$ imply $B\in\mathscr{I}$; (2) $A\in\mathscr{I}$ and $B\in\mathscr{I}$ imply $A\cup B\in\mathscr{I}$. A topological space $(X,\tau)$ with an ideal $\mathscr{I}$ on $X$ is called an ideal topological space and is denoted by $(X,\tau,\mathscr{I})$. For an ideal topological space $(X,\tau,\mathscr{I})$ and a subset $A$ of $X$, $A^\star(\mathscr{I})$ is defined as follows: $A^\star(\mathscr{I})=\{x\in X : U\cap A\notin\mathscr{I}$ for every open neighbourhood $U$ of $x\}$. In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [11], $A^\star$ is called the local function of $A$ with respect to $\mathscr{I}$ and $\tau$ and $\mathrm{Cl}^\star(A)=A^\star\cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than $\tau$. A subset $A$ is said to be $\star$-closed [10] if $A^\star\subseteq A$. The interior of a subset $A$ in $(X,\tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$.

**Lemma 1.** *For a subset $A$ of an ideal topological space $(X,\tau,\mathscr{I})$, the following properties hold:*

*(1) If $V\in\tau$, then $V\cap Cl^\star(A)\subseteq Cl^\star(V\cap A)$* [8].

*(2) If $F$ is closed in $X$, then $Int^\star(A\cup F)\subseteq Int^\star(A)\cup F$.*

A subset $A$ of an ideal topological space $(X,\tau,\mathscr{I})$ is called *semi-$\mathscr{I}$-open* [7] (resp. *$pre^\star_{\mathscr{I}}$-open* [5], *strong $\beta$-$\mathscr{I}$-open* [7]) if $A\subseteq\mathrm{Cl}^\star(\mathrm{Int}(A))$ (resp. $A\subseteq\mathrm{Int}^\star(\mathrm{Cl}(A))$, $A\subseteq\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$). The complement of a semi-$\mathscr{I}$-open (resp. $\mathrm{pre}^\star_{\mathscr{I}}$-open, strong $\beta$-$\mathscr{I}$-open) set is called *semi-$\mathscr{I}$-closed* [7] (resp. *$pre^\star_{\mathscr{I}}$-closed* [5], *strong $\beta$-$\mathscr{I}$-closed* [7]).

**Lemma 2.** *For a subset $A$ of an ideal topological space $(X,\tau,\mathscr{I})$, the following properties hold:*

*(1) $sCl_{\mathscr{I}}(A)=A\cup Int^\star(Cl(A))$* [6].

*(2) $s\beta Cl_{\mathscr{I}}(A)=A\cup Int^\star(Cl(Int^\star(A)))$* [6].

*(3) $s\beta Int_{\mathscr{I}}(A)=A\cap Cl^\star(Int(Cl^\star(A)))$.*

**Lemma 3.** [4] *Let $(X,\tau,\mathscr{I})$ be an ideal topological space. If $V$ is $\star$-open, then*

$$sCl_{\mathscr{I}}(V)=Int^\star(Cl(V)).$$

**Lemma 4.** [4] *For a subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$, $x \in s\beta Cl_{\mathscr{I}}(A)$ if and only if $U \cap A \neq \emptyset$ for every strong $\beta$-$\mathscr{I}$-open set $U$ containing $x$.*

**Lemma 5.** [4] *For a subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are hold:*

*(1) $X - s\beta Cl_{\mathscr{I}}(A) = s\beta Int_{\mathscr{I}}(X - A)$.*

*(2) $X - s\beta Int_{\mathscr{I}}(A) = s\beta Cl_{\mathscr{I}}(X - A)$.*

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from $X$ into $Y$, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set $B$ of $Y$ by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then $F$ is said to be surjection if $F(X) = Y$, or equivalent, if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$ and $F$ is called injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

## 3. Upper and lower weakly $s\beta(\star)$-continuous multifunctions

In this section, we introduce the notions of upper and lower weakly $s\beta(\star)$-continuous multifunctions. Moreover, several characterizations of upper and lower weakly $s\beta(\star)$-continuous multifunctions are discussed.

**Definition 1.** *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

*(1) upper weakly $s\beta(\star)$-continuous at a point $x \in X$ if for each $\star$-open set $V$ of $Y$ containing $F(x)$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq Cl^\star(V)$;*

*(2) lower weakly $s\beta(\star)$-continuous at a point $x \in X$ if for each $\star$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(z) \cap Cl^\star(V) \neq \emptyset$ for every $z \in U$;*

*(3) upper (resp. lower) weakly $s\beta(\star)$-continuous if $F$ has this property at each point of $X$.*

**Theorem 1.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $F$ is upper weakly $s\beta(\star)$-continuous at a point $x \in X$;*

*(2) $x \in Cl^\star(Int(Cl^\star(F^+(Cl^\star(V)))))$ for every $\star$-open set $V$ of $Y$ containing $F(x)$;*

*(3)* $x \in s\beta Int_{\mathscr{I}}(F^{+}(Cl^{\star}(V)))$ *for every* $\star$*-open set* $V$ *of* $Y$ *containing* $F(x)$*.*

*Proof.* (1) $\Rightarrow$ (2): Let $V$ be any $\star$-open set of $Y$ containing $F(x)$. By (1), there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \mathrm{Cl}^{\star}(V)$. Then, $x \in U \subseteq F^{+}(\mathrm{Cl}^{\star}(V))$. Since $U$ is strong $\beta$-$\mathscr{I}$-open, we have

$$x \in U \subseteq \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(U))) \subseteq \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(F^{+}(\mathrm{Cl}^{\star}(V))))).$$

(2) $\Rightarrow$ (3): Let $V$ be any $\star$-open set of $Y$ containing $F(x)$. Thus, by (2), we have $x \in \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(F^{+}(\mathrm{Cl}^{\star}(V)))))$. Since $x \in F^{+}(\mathrm{Cl}^{\star}(V))$ and by Lemma 2, we obtain $x \in F^{+}(\mathrm{Cl}^{\star}(V)) \cap \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(F^{+}(\mathrm{Cl}^{\star}(V))))) = s\beta\mathrm{Int}_{\mathscr{I}}(F^{+}(\mathrm{Cl}^{\star}(V)))$.

(3) $\Rightarrow$ (1): Let $V$ be any $\star$-open set of $Y$ containing $F(x)$. By (3), we have

$$x \in s\beta\mathrm{Int}_{\mathscr{I}}(F^{+}(\mathrm{Cl}^{\star}(V)))$$

and so there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $U \subseteq F^{+}(\mathrm{Cl}^{\star}(V))$; hence $F(U) \subseteq \mathrm{Cl}^{\star}(V)$. This shows that $F$ is upper weakly $s\beta(\star)$-continuous at $x$.

**Theorem 2.** *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, the following properties are equivalent:*

*(1)* $F$ *is lower weakly* $s\beta(\star)$*-continuous at a point* $x \in X$*;*

*(2)* $x \in Cl^{\star}(Int(Cl^{\star}(F^{-}(Cl^{\star}(V)))))$ *for every* $\star$*-open set* $V$ *of* $Y$ *such that* $F(x) \cap V \neq \emptyset$*;*

*(3)* $x \in s\beta Int_{\mathscr{I}}(F^{-}(Cl^{\star}(V)))$ *for every* $\star$*-open set* $V$ *of* $Y$ *such that* $F(x) \cap V \neq \emptyset$*.*

*Proof.* The proof is similar to that of Theorem 1.

**Definition 2.** *A function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ *is said to be weakly* $s\beta(\star)$*-continuous at a point* $x \in X$ *if for each* $\star$*-open set* $V$ *of* $Y$ *containing* $f(x)$*, there exists a strong* $\beta$*-*$\mathscr{I}$*-open set* $U$ *of* $X$ *containing* $x$ *such that* $f(U) \subseteq Cl^{\star}(V)$*. A function*

$$f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$$

*is said to be weakly* $s\beta(\star)$*-continuous if* $f$ *has this property at each point of* $X$*.*

**Corollary 1.** *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, the following properties are equivalent:*

*(1)* $f$ *is weakly* $s\beta(\star)$*-continuous at a point* $x \in X$*;*

*(2)* $x \in Cl^{\star}(Int(Cl^{\star}(f^{-1}(Cl^{\star}(V)))))$ *for every* $\star$*-open set* $V$ *of* $Y$ *containing* $f(x)$*;*

*(3)* $x \in s\beta Int_{\mathscr{I}}(f^{-1}(Cl^{\star}(V)))$ *for every* $\star$*-open set* $V$ *of* $Y$ *containing* $f(x)$*.*

**Theorem 3.** *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, the following properties are equivalent:*

*(1) $F$ is upper weakly $s\beta(\star)$-continuous;*

*(2) $F^+(V) \subseteq Cl^\star(Int(Cl^\star(F^+(Cl^\star(V)))))$ for every $\star$-open set $V$ of $Y$;*

*(3) $Int^\star(Cl(Int^\star(F^-(V)))) \subseteq F^-(Cl^\star(V))$ for every $\star$-open set $V$ of $Y$;*

*(4) $Int^\star(Cl(Int^\star(F^-(Int^\star(K))))) \subseteq F^-(K)$ for every $\star$-closed set $K$ of $Y$;*

*(5) $s\beta Cl_{\mathscr{I}}(F^-(Int^\star(K))) \subseteq F^-(K)$ for every $\star$-closed set $K$ of $Y$;*

*(6) $s\beta Cl_{\mathscr{I}}(F^-(Int^\star(Cl^\star(B)))) \subseteq F^-(Cl^\star(B))$ for every subset $B$ of $Y$;*

*(7) $F^+(Int^\star(B)) \subseteq s\beta Int_{\mathscr{I}}(F^+(Cl^\star(Int^\star(B))))$ for every subset $B$ of $Y$;*

*(8) $F^+(V) \subseteq s\beta Int_{\mathscr{I}}(F^+(Cl^\star(V)))$ for every $\star$-open set $V$ of $Y$;*

*(9) $s\beta Cl_{\mathscr{I}}(F^-(V)) \subseteq F^-(Cl^\star(V))$ for every $\star$-open set $V$ of $Y$.*

*Proof.* (1) $\Rightarrow$ (2): Let $V$ be any $\star$-open set of $Y$ and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by Theorem 1, $x \in s\beta\mathrm{Int}_{\mathscr{I}}(F^+(\mathrm{Cl}^\star(V)))$ and hence $F^+(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(\mathrm{Cl}^\star(V)))))$ by Lemma 2.

(2) $\Rightarrow$ (3): Let $V$ be any $\star$-open set of $Y$. Thus, by (2), we have

$$\begin{aligned} X - F^-(\mathrm{Cl}^\star(V)) &= F^+(Y - \mathrm{Cl}^\star(V)) \\ &\subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(\mathrm{Cl}^\star(Y - \mathrm{Cl}^\star(V)))))) \\ &= \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(Y - \mathrm{Int}^\star(\mathrm{Cl}^\star(V)))))) \\ &\subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(Y - V)))) \\ &= \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(X - F^-(V)))) \\ &= X - \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^-(V)))) \end{aligned}$$

and hence $\mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^-(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$.

(3) $\Rightarrow$ (4): Let $K$ be any $\star$-closed set of $Y$. Then, $\mathrm{Int}^\star(K)$ is $\star$-open in $Y$ and so

$$\mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^-(\mathrm{Int}^\star(K))))) \subseteq F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(K))) \subseteq F^-(\mathrm{Cl}^\star(K)) = F^-(K).$$

(4) $\Rightarrow$ (5): Let $K$ be any $\star$-closed set of $Y$. Then, we have

$$\mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^-(\mathrm{Int}^\star(K))))) \subseteq F^-(K)$$

and $F^-(\mathrm{Int}^\star(K)) \subseteq F^-(K)$. Thus, by Lemma 2, $s\beta_{\mathscr{I}}\mathrm{Cl}(F^-(\mathrm{Int}^\star(K))) \subseteq F^-(K)$.

(5) $\Rightarrow$ (6): Let $B$ be any subset of $Y$. Then, $\mathrm{Cl}^\star(B)$ is $\star$-closed in $Y$ and by (5), $s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(B)))) \subseteq F^-(\mathrm{Cl}^\star(B))$.

(6) $\Rightarrow$ (7): Let $B$ be any subset of $Y$. By (6),

$$\begin{aligned} F^+(\mathrm{Int}^\star(B)) &= X - F^-(\mathrm{Cl}^\star(Y - B)) \\ &\subseteq X - s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(Y - B)))) \end{aligned}$$

$$= s\beta\mathrm{Int}_{\mathscr{I}}(F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(B)))).$$

(7) $\Rightarrow$ (8): The proof is obvious.

(8) $\Rightarrow$ (9): Let $V$ be any $\star$-open set of $Y$. Thus, by (8), we have

$$\begin{aligned} s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(V)) &\subseteq s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \\ &= s\beta\mathrm{Cl}_{\mathscr{I}}(X - F^+(Y - \mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \\ &= X - s\beta\mathrm{Int}_{\mathscr{I}}(F^+(Y - \mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \\ &= X - s\beta\mathrm{Int}_{\mathscr{I}}(F^+(\mathrm{Cl}^\star(Y - \mathrm{Cl}^\star(V)))) \\ &\subseteq X - F^+(Y - \mathrm{Cl}^\star(V)) \\ &= F^-(\mathrm{Cl}^\star(V)). \end{aligned}$$

(9) $\Rightarrow$ (1): Let $x \in X$ and $V$ be any $\star$-open set of $Y$ containing $F(x)$. By (9),

$$\begin{aligned} x \in F^+(V) &\subseteq F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V))) \\ &= X - F^-(\mathrm{Cl}^\star(Y - \mathrm{Cl}^\star(V))) \\ &\subseteq X - s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(Y - \mathrm{Cl}^\star(V))) \\ &= s\beta\mathrm{Int}_{\mathscr{I}}(F^+(\mathrm{Cl}^\star(V))) \end{aligned}$$

and hence $F$ is upper weakly $s\beta(\star)$-continuous by Theorem 1.

**Theorem 4.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $F$ is lower weakly $s\beta(\star)$-continuous;*

*(2) $F^-(V) \subseteq Cl^\star(Int(Cl^\star(F^-(Cl^\star(V)))))$ for every $\star$-open set $V$ of $Y$;*

*(3) $Int^\star(Cl(Int^\star(F^+(V)))) \subseteq F^+(Cl^\star(V))$ for every $\star$-open set $V$ of $Y$;*

*(4) $Int^\star(Cl(Int^\star(F^+(Int^\star(K))))) \subseteq F^+(K)$ for every $\star$-closed set $K$ of $Y$;*

*(5) $s\beta Cl_{\mathscr{I}}(F^+(Int^\star(K))) \subseteq F^+(K)$ for every $\star$-closed set $K$ of $Y$;*

*(6) $s\beta Cl_{\mathscr{I}}(F^+(Int^\star(Cl^\star(B)))) \subseteq F^+(Cl^\star(B))$ for every subset $B$ of $Y$;*

*(7) $F^-(Int^\star(B)) \subseteq s\beta Int_{\mathscr{I}}(F^-(Cl^\star(Int^\star(B))))$ for every subset $B$ of $Y$;*

*(8) $F^-(V) \subseteq s\beta Int_{\mathscr{I}}(F^-(Cl^\star(V)))$ for every $\star$-open set $V$ of $Y$;*

*(9) $s\beta Cl_{\mathscr{I}}(F^+(V)) \subseteq F^+(Cl^\star(V))$ for every $\star$-open set $V$ of $Y$.*

*Proof.* The proof is similar to that of Theorem 3.

**Corollary 2.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1)* $f$ *is weakly* $s\beta(\star)$*-continuous;*

*(2)* $f^{-1}(V) \subseteq Cl^\star(Int(Cl^\star(f^{-1}(Cl^\star(V)))))$ *for every* $\star$*-open set* $V$ *of* $Y$*;*

*(3)* $Int^\star(Cl(Int^\star(f^{-1}(V)))) \subseteq f^{-1}(Cl^\star(V))$ *for every* $\star$*-open set* $V$ *of* $Y$*;*

*(4)* $Int^\star(Cl(Int^\star(f^{-1}(Int^\star(K))))) \subseteq f^{-1}(K)$ *for every* $\star$*-closed set* $K$ *of* $Y$*;*

*(5)* $s\beta Cl_{\mathscr{I}}(f^{-1}(Int^\star(K))) \subseteq f^{-1}(K)$ *for every* $\star$*-closed set* $K$ *of* $Y$*;*

*(6)* $s\beta Cl_{\mathscr{I}}(f^{-1}(Int^\star(Cl^\star(B)))) \subseteq f^{-1}(Cl^\star(B))$ *for every subset* $B$ *of* $Y$*;*

*(7)* $f^{-1}(Int^\star(B)) \subseteq s\beta Int_{\mathscr{I}}(f^{-1}(Cl^\star(Int^\star(B))))$ *for every subset* $B$ *of* $Y$*;*

*(8)* $f^{-1}(V) \subseteq s\beta Int_{\mathscr{I}}(f^{-1}(Cl^\star(V)))$ *for every* $\star$*-open set* $V$ *of* $Y$*;*

*(9)* $s\beta Cl_{\mathscr{I}}(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ *for every* $\star$*-open set* $V$ *of* $Y$*.*

Recall that a subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$ is called *R-*$\mathscr{I}^\star$*-open* [2] (resp. $\mathscr{I}^\star$*-preopen* [2], $\mathscr{I}^\star$*-semi-open* [3]) if $A = \mathrm{Int}^\star(\mathrm{Cl}^\star(A))$ (resp. $A \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(A))$, $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(A))$). The complement of a $R$-$\mathscr{I}^\star$-open (resp. $\mathscr{I}^\star$-preopen, $\mathscr{I}^\star$-semi-open) set is called *R-*$\mathscr{I}^\star$*-closed* [2] (resp. $\mathscr{I}^\star$*-preclosed* [2], $\mathscr{I}^\star$*-semi-closed* [3]). Let $A$ be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. A point $x$ in an ideal topological space $(X, \tau, \mathscr{I})$ is called a $\star_\theta$*-cluster point* of $A$ [3] if $\mathrm{Cl}^\star(U) \cap A \neq \emptyset$ for every $\star$-open set $U$ of $X$ containing $x$. The set of all $\star_\theta$-cluster points of $A$ is called the $\star_\theta$*-closure* [3] of $A$ and is denoted by $\star_\theta\mathrm{Cl}(A)$. A subset $B$ of an ideal topological space $(X, \tau, \mathscr{I})$ is called $\star_\theta$*-closed* [3] if $\star_\theta\mathrm{Cl}(B) = B$. The complement of a $\star_\theta$-closed set is called $\star_\theta$*-open* [3].

**Lemma 6.** [3] *For a subset* $A$ *of an ideal topological space* $(X, \tau, \mathscr{I})$*, the following properties hold:*

*(1) If* $A$ *is* $\star$*-open in* $X$*, then* $Cl^\star(A) = \star_\theta Cl(A)$*.*

*(2)* $\star_\theta Cl(A)$ *is* $\star$*-closed in* $X$*.*

**Theorem 5.** *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, the following properties are equivalent:*

*(1)* $F$ *is upper weakly* $s\beta(\star)$*-continuous;*

*(2)* $s\beta Cl_{\mathscr{I}}(F^-(Int^\star(\star_\theta Cl(B)))) \subseteq F^-(\star_\theta Cl(B))$ *for every subset* $B$ *of* $Y$*;*

*(3)* $s\beta Cl_{\mathscr{I}}(F^-(Int^\star(Cl^\star(B)))) \subseteq F^-(\star_\theta Cl(B))$ *for every subset* $B$ *of* $Y$*;*

*(4)* $s\beta Cl_{\mathscr{I}}(F^-(Int^\star(Cl^\star(V)))) \subseteq F^-(Cl^\star(V))$ *for every* $\star$*-open set* $V$ *of* $Y$*;*

*(5)* $s\beta Cl_{\mathscr{I}}(F^-(Int^\star(Cl^\star(V)))) \subseteq F^-(Cl^\star(V))$ *for every* $\mathscr{J}^\star$*-preopen set* $V$ *of* $Y$*;*

*(6)* $s\beta Cl_{\mathscr{I}}(F^-(Int^\star(K))) \subseteq F^-(K)$ *for every R-*$\mathscr{J}^\star$*-closed set* $K$ *of* $Y$*;*

*(7)* $s\beta Cl_{\mathscr{I}}(F^-(Int^\star(Cl^\star(V)))) \subseteq F^-(Cl^\star(V))$ *for every strong* $\beta$-$\mathscr{J}$*-open set* $V$ *of* $Y$*;*

*(8)* $s\beta Cl_{\mathscr{I}}(F^-(Int^\star(Cl^\star(V)))) \subseteq F^-(Cl^\star(V))$ *for every* $\mathscr{J}^\star$*-semi-open set* $V$ *of* $Y$*.*

*Proof.* (1) $\Rightarrow$ (2): Let $B$ be any subset of $Y$. Thus, by Lemma 6, $\star_\theta\mathrm{Cl}(B)$ is $\star$-closed in $Y$ and by Theorem 3, $s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Int}^\star(\star_\theta\mathrm{Cl}(B)))) \subseteq F^-(\star_\theta\mathrm{Cl}(B))$.

(2) $\Rightarrow$ (3): This is obvious since $\mathrm{Cl}^\star(B) \subseteq \star_\theta\mathrm{Cl}(B)$ for every subset $B$ of $Y$.

(3) $\Rightarrow$ (4): This is obvious since $\mathrm{Cl}^\star(V) = \star_\theta\mathrm{Cl}(V)$ for every $\star$-open set $V$ of $Y$.

(4) $\Rightarrow$ (5): Let $V$ be any $\mathscr{J}^\star$-preopen set of $Y$. Then, we have $V \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(V))$ and so $\mathrm{Cl}^\star(V) = \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))$. Now, put $G = \mathrm{Int}^\star(\mathrm{Cl}^\star(V))$, then $G$ is $\star$-open in $Y$ and $\mathrm{Cl}^\star(G) = \mathrm{Cl}^\star(V)$. Thus, by (4), we have $s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$.

(5) $\Rightarrow$ (6): Let $K$ be any $R$-$\mathscr{J}^\star$-closed set of $Y$. Then, $\mathrm{Int}^\star(K)$ is $\mathscr{J}^\star$-preopen in $Y$, by (5),

$$\begin{aligned} s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Int}^\star(K))) &= s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(K))))) \\ &\subseteq F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(K))) \\ &= F^-(K). \end{aligned}$$

(6) $\Rightarrow$ (7): Let $V$ be any strong $\beta$-$\mathscr{J}$-open set of $Y$. Then, $V \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(V)))$. Since $\mathrm{Cl}^\star(V)$ is $R$-$\mathscr{J}^\star$-closed in $Y$. Thus, by (6), $s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$.

(7) $\Rightarrow$ (8): This is obvious since every $\mathscr{J}^\star$-semi-open set is strong $\beta$-$\mathscr{J}$-open.

(8) $\Rightarrow$ (1): Let $V$ be any $\star$-open set of $Y$. Then, since $V$ is $\mathscr{J}^\star$-semi-open set in $Y$, by (8), we have $s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(V)) \subseteq s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$. By Theorem 3, $F$ is upper weakly $s\beta(\star)$-continuous.

**Theorem 6.** *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, the following properties are equivalent:*

*(1)* $F$ *is lower weakly* $s\beta(\star)$*-continuous;*

*(2)* $s\beta Cl_{\mathscr{I}}(F^+(Int^\star(\star_\theta Cl(B)))) \subseteq F^+(\star_\theta Cl(B))$ *for every subset* $B$ *of* $Y$*;*

*(3)* $s\beta Cl_{\mathscr{I}}(F^+(Int^\star(Cl^\star(B)))) \subseteq F^+(\star_\theta Cl(B))$ *for every subset* $B$ *of* $Y$*;*

*(4)* $s\beta Cl_{\mathscr{I}}(F^+(Int^\star(Cl^\star(V)))) \subseteq F^+(Cl^\star(V))$ *for every* $\star$*-open set* $V$ *of* $Y$*;*

*(5)* $s\beta Cl_{\mathscr{I}}(F^+(Int^\star(Cl^\star(V)))) \subseteq F^+(Cl^\star(V))$ *for every* $\mathscr{J}^\star$*-preopen set* $V$ *of* $Y$*;*

*(6)* $s\beta Cl_{\mathscr{I}}(F^+(Int^\star(K))) \subseteq F^+(K)$ *for every* $R$-$\mathscr{J}^\star$*-closed set* $K$ *of* $Y$*;*

*(7)* $s\beta Cl_{\mathscr{I}}(F^+(Int^\star(Cl^\star(V)))) \subseteq F^+(Cl^\star(V))$ *for every strongly* $\beta$-$\mathscr{J}$*-open set* $V$ *of* $Y$*;*

*(8)* $s\beta Cl_{\mathscr{I}}(F^+(Int^\star(Cl^\star(V)))) \subseteq F^+(Cl^\star(V))$ *for every* $\mathscr{J}^\star$*-semi-open set* $V$ *of* $Y$*.*

*Proof.* The proof is similar to that of Theorem 5.

**Corollary 3.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $f$ is weakly $s\beta(\star)$-continuous;*

*(2) $s\beta Cl_{\mathscr{I}}(f^{-1}(Int^{\star}(\star_{\theta}Cl(B)))) \subseteq f^{-1}(\star_{\theta}Cl(B))$ for every subset $B$ of $Y$;*

*(3) $s\beta Cl_{\mathscr{I}}(f^{-1}(Int^{\star}(Cl^{\star}(B)))) \subseteq f^{-1}(\star_{\theta}Cl(B))$ for every subset $B$ of $Y$;*

*(4) $s\beta Cl_{\mathscr{I}}(f^{-1}(Int^{\star}(Cl^{\star}(V)))) \subseteq f^{-1}(Cl^{\star}(V))$ for every $\star$-open set $V$ of $Y$;*

*(5) $s\beta Cl_{\mathscr{I}}(f^{-1}(Int^{\star}(Cl^{\star}(V)))) \subseteq f^{-1}(Cl^{\star}(V))$ for every $\mathscr{J}^{\star}$-preopen set $V$ of $Y$;*

*(6) $s\beta Cl_{\mathscr{I}}(f^{-1}(Int^{\star}(K))) \subseteq f^{-1}(K)$ for every $R$-$\mathscr{J}^{\star}$-closed set $K$ of $Y$;*

*(7) $s\beta Cl_{\mathscr{I}}(f^{-1}(Int^{\star}(Cl^{\star}(V)))) \subseteq f^{-1}(Cl^{\star}(V))$ for every strongly $\beta$-$\mathscr{J}$-open set $V$ of $Y$;*

*(8) $s\beta Cl_{\mathscr{I}}(f^{-1}(Int^{\star}(Cl^{\star}(V)))) \subseteq f^{-1}(Cl^{\star}(V))$ for every $\mathscr{J}^{\star}$-semi-open set $V$ of $Y$.*

**Theorem 7.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $F$ is upper weakly $s\beta(\star)$-continuous;*

*(2) $s\beta Cl_{\mathscr{I}}(F^{-}(V)) \subseteq F^{-}(Cl^{\star}(V))$ for every $\mathscr{J}^{\star}$-preopen set $V$ of $Y$;*

*(3) $F^{+}(V) \subseteq s\beta Int_{\mathscr{I}}(F^{+}(Cl^{\star}(V)))$ for every $\mathscr{J}^{\star}$-preopen set $V$ of $Y$.*

*Proof.* (1) $\Rightarrow$ (2): Let $V$ be any $\mathscr{J}^{\star}$-preopen set of $Y$. Since $F$ is upper weakly $s\beta(\star)$-continuous, by Theorem 3, $s\beta\mathrm{Cl}_{\mathscr{I}}(F^{-}(V)) \subseteq s\beta\mathrm{Cl}_{\mathscr{I}}(F^{-}(\mathrm{Int}^{\star}(\mathrm{Cl}^{\star}(V)))) \subseteq F^{-}(\mathrm{Cl}^{\star}(V))$.

(2) $\Rightarrow$ (3): Let $V$ be any $\mathscr{J}^{\star}$-preopen set of $Y$. Then, we have $V \subseteq \mathrm{Int}^{\star}(\mathrm{Cl}^{\star}(V))$ and $Y - V \supseteq \mathrm{Cl}^{\star}(\mathrm{Int}^{\star}(Y - V))$. Thus, by (3),

$$\begin{aligned} X - F^{+}(V) &= F^{-}(Y - V) \\ &\supseteq F^{-}(\mathrm{Cl}^{\star}(\mathrm{Int}^{\star}(Y - V))) \\ &\supseteq s\beta\mathrm{Cl}_{\mathscr{I}}(F^{-}(\mathrm{Int}^{\star}(Y - V))) \\ &= s\beta\mathrm{Cl}_{\mathscr{I}}(F^{-}(Y - \mathrm{Cl}^{\star}(V))) \\ &= s\beta\mathrm{Cl}_{\mathscr{I}}(X - F^{+}(\mathrm{Cl}^{\star}(V))) \\ &= X - s\beta\mathrm{Int}_{\mathscr{I}}(F^{+}(\mathrm{Cl}^{\star}(V))) \end{aligned}$$

and hence $F^{+}(V) \subseteq s\beta\mathrm{Int}_{\mathscr{I}}(F^{+}(\mathrm{Cl}^{\star}(V)))$.

(3) $\Rightarrow$ (1): Let $V$ be any $\star$-open set of $Y$. Then, $V$ is $\mathscr{J}^{\star}$-preopen in $Y$, by (4), $F^{+}(V) \subseteq s\beta\mathrm{Int}_{\mathscr{I}}(F^{+}(\mathrm{Cl}^{\star}(V)))$. Thus, $F$ is upper weakly $s\beta(\star)$-continuous by Theorem 3.

**Theorem 8.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $F$ is lower weakly $s\beta(\star)$-continuous;*

*(2) $s\beta Cl_{\mathscr{I}}(F^+(V)) \subseteq F^+(Cl^\star(V))$ for every $\mathscr{J}^\star$-preopen set $V$ of $Y$;*

*(3) $F^-(V) \subseteq s\beta Int_{\mathscr{I}}(F^-(Cl^\star(V)))$ for every $\mathscr{J}^\star$-preopen set $V$ of $Y$.*

*Proof.* The proof is similar to that of Theorem 7.

**Corollary 4.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $f$ is weakly $s\beta(\star)$-continuous;*

*(2) $s\beta Cl_{\mathscr{I}}(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ for every $\mathscr{J}^\star$-preopen set $V$ of $Y$;*

*(3) $f^{-1}(V) \subseteq s\beta Int_{\mathscr{I}}(f^{-1}(Cl^\star(V)))$ for every $\mathscr{J}^\star$-preopen set $V$ of $Y$.*

**Definition 3.** [4] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

*(1) upper almost $s\beta(\star)$-continuous at a point $x \in X$ if for each $\star$-open set $V$ of $Y$ containing $F(x)$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq Int^\star(Cl(V))$;*

*(2) lower almost $s\beta(\star)$-continuous at a point $x \in X$ if for each $\star$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(z) \cap Int^\star(Cl(V)) \neq \emptyset$ for every $z \in U$;*

*(3) upper (resp. lower) almost $\beta(\star)$-continuous if $F$ has this property at each point of $X$.*

**Remark 1.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following implication holds:*

*upper almost $s\beta(\star)$-continuity $\Rightarrow$ upper weak $s\beta(\star)$-continuity.*

The converse of the implication is not true in general. We give an example for the implication as follows.

**Example 1.** *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{1\}\}$. Let $Y = \{a, b, c\}$ with a topology $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$ and an ideal $\mathscr{J} = \{\emptyset, \{c\}\}$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $F(1) = \{c\}$ and $F(2) = F(3) = \{a, b\}$. Then, $F$ is upper weakly $s\beta(\star)$-continuous but $F$ is not upper almost $s\beta(\star)$-continuous, since $\{a, b\}$ is $\star$-open in $Y$ but $F^+(\{a, b\})$ is not strong $\beta$-$\mathscr{I}$-open in $X$.*

**Lemma 7.** [2] *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

*(1) $(X, \tau, \mathscr{I})$ is $\star$-$\mathscr{I}$-normal.*

*(2) For each $\star$-closed set $F$ and each $\star$-open set $V$ containing $F$, there exists a $\star$-open set $U$ such that $F \subseteq U \subseteq Cl^{\star}(U) \subseteq V$.*

**Theorem 9.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ such that $F(x)$ is $\star$-closed in $Y$ for each $x \in X$ and $(Y, \sigma, \mathscr{J})$ is a $\star$-$\mathscr{J}$-normal space, the following properties are equivalent:*

*(1) $F$ is upper $s\beta(\star)$-continuous;*

*(2) $F$ is upper almost $s\beta(\star)$-continuous;*

*(3) $F$ is upper weakly $s\beta(\star)$-continuous.*

*Proof.* We show only the implication (3) $\Rightarrow$ (1) since the others are obvious. Suppose that $F$ is upper weakly $s\beta(\star)$-continuous. Let $x \in X$ and $V$ be any $\star$-open set of $Y$ such that $F(x) \subseteq V$. Since $F(x)$ is $\star$-closed in $Y$ and $Y$ is $\star$-$\mathscr{J}$-normal, there exists a $\star$-open set $G$ of $Y$ such that $F(x) \subseteq G \subseteq \mathrm{Cl}^{\star}(G) \subseteq V$. Since $F$ is upper weakly $s\beta(\star)$-continuous, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \mathrm{Cl}^{\star}(G)$; hence $F(U) \subseteq V$. This shows that $F$ is upper $s\beta(\star)$-continuous.

**Theorem 10.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ such that $F(x)$ is $\star$-open in $Y$ for each $x \in X$, the following properties are equivalent:*

*(1) $F$ is lower $s\beta(\star)$-continuous;*

*(2) $F$ is lower almost $s\beta(\star)$-continuous;*

*(3) $F$ is lower weakly $s\beta(\star)$-continuous.*

*Proof.* (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3): The proofs of these implications are obvious.

(3) $\Rightarrow$ (1): Suppose that $F$ is lower weakly $s\beta(\star)$-continuous. Let $x \in X$ and $V$ be any $\star$-open set such that $F(x) \cap V \neq \emptyset$. Then, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(z) \cap \mathrm{Cl}^{\star}(V) \neq \emptyset$ for each $z \in U$. Since $F(z)$ is $\star$-open, we have $F(z) \cap V \neq \emptyset$ for each $z \in U$ and so $F$ is lower $s\beta(\star)$-continuous.

**Definition 4.** [4] *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called almost $s\beta(\star)$-continuous at a point $x \in X$ if for each $\star$-open set $V$ of $Y$ containing $f(x)$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq Int^{\star}(Cl(V))$. A function*

$$f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$$

*is called almost $\beta(\star)$-continuous if $f$ has this property at each point of $X$.*

**Corollary 5.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ such that $f(x)$ is $\star$-open in $Y$ for each $x \in X$, the following properties are equivalent:*

*(1) $f$ is $s\beta(\star)$-continuous;*

*(2) $f$ is almost $s\beta(\star)$-continuous;*

*(3) $f$ is weakly $s\beta(\star)$-continuous.*

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