Bounds on Intersection Number in the Join and Corona of Graphs

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Abstract. In this paper, we provide an upper bound for the intersection number in the join and corona of graphs. Moreover, we give formulas for the intersection number of $K_n \circ G$, $P_n \circ G$, $C_n \circ G$ and $C_r n$.

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1. Introduction

Let $S$ be a set and $F = \{S_1, S_2, \ldots, S_p\}$, for some integer $p$, a nonempty family of distinct nonempty subsets of $S$ whose union is $S$. The intersection graph of $F$ is denoted by $\Omega(F)$ and defined by $V(\Omega(F)) = F$, with $S_i$ and $S_j$ adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$. A graph $G$ is an intersection graph on $S$ if there exists a family $F$ of subsets of $S$ for which $G \cong \Omega(F)$. The intersection number $\omega(G)$ of a given graph $G$ is the minimum number of elements in a set $S$ such that $G$ is an intersection graph on $S$. The intersection number has been studied by [1]. They obtained the best possible upper bound for the intersection number of a graph with a given number of points. In [2], Frank Harary provided an upper bound for the intersection number of a graph $G$. He showed that $\omega(G) \leq |E(G)|$. In [3], the authors provided a lower bound for the intersection number of a graph $G$. They showed that $\log_2(|V(G)| + 1) \leq \omega(G)$. Moreover, the authors provided formulas for the intersection numbers of $P_n$, $C_n$, $W_n$, $F_n$, $K_n$, and $G+K_1$ for any connected graph $G$. They also defined the concept of an extreme intersection graph. A graph $G$ is an extreme intersection graph if for any family $F$ of subsets of $S = \{1, 2, 3, \ldots, \omega(G)\}$ such that $\Omega(F) \cong G$, then $S \in F$.

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Therefore, \( \phi \) is not an extreme intersection graph. Then for any graph \( H \), \( \omega(G + H) \leq \omega(G)\omega(H) \).

**Theorem 1.** Suppose \( G \) is not an extreme intersection graph. Then for any graph \( H \), \( \omega(G + H) \leq \omega(G)\omega(H) \).

**Proof.** Let \( G \) be not an extreme intersection graph. Then there exists a family \( F_1 \) of nonempty subsets of a set \( S \) such that \( S \not\in F_1 \) and \( \Omega(F_1) \not\cong G \). That is, there is an isomorphism \( \phi_1 : V(G) \to F_1 \) such that \( \phi_1(x) \not= S_1 \), for all \( x \in V(G) \). Let \( H \) be any graph and suppose \( \omega(H) = m \). Let \( S_2 = \{1, 2, \ldots, m\} \) and \( F_2 \) be a nonempty subset of a set \( S_2 \) for which \( \Omega(F_2) \cong H \). That is, there is an isomorphism \( \phi_2 : V(H) \to F_2 \). Let \( S = S_1 \times S_2 \), and \( F = (\cup\{A \times S_2 : A \in F_1\}) \cup (\cup\{S_1 \times B : B \in F_2\}) \). Let \( \phi : V(G + H) \to F \) be a mapping defined by

\[
\phi(x) = \begin{cases} 
\phi_1(x) \times S_2, & \text{if } x \in V(G) \\
S_1 \times \phi_2(x), & \text{if } x \in V(H).
\end{cases}
\]

Let \( x_1, x_2 \in V(G + H) \) such that \( \phi(x_1) = \phi(x_2) \). The case \( x_1 \in V(G) \) and \( x_2 \in V(H) \) is not possible. Since \( \phi(x_1) = \phi_1(x_1) \times S_2 \) and \( \phi(x_2) = S_1 \times \phi_2(x_2) \). Consider the following cases:

Case 1. Suppose \( x_1, x_2 \in V(G) \). Then \( \phi(x_1) = \phi_1(x_1) \times S_2 \) and \( \phi(x_2) = \phi_1(x_2) \times S_2 \). Note that \( \phi(x_1) = \phi(x_2) \), so we have \( \phi_1(x_1) = \phi_1(x_2) \). Since \( \phi_1 \) is one to one, \( x_1 = x_2 \).

Case 2. Suppose \( x_1, x_2 \in V(H) \). Then \( \phi(x_2) = S_1 \times \phi_2(x_1) \) and \( \phi(x_2) = S_1 \times \phi_2(x_2) \). Note that \( \phi(x_1) = \phi(x_2) \), so we have \( \phi_2(x_1) = \phi_2(x_2) \). Since \( \phi_2 \) is one to one, \( x_1 = x_2 \).

Therefore, \( \phi \) is one to one.

Let \( u \in F \). If \( u = S_1 \times B, B \in F_2 \). Since \( \phi_2 \) is onto, there exists \( x \in V(H) \subseteq V(G + H) \) such that \( \phi_2(x) = B \). Thus, \( \phi(x) = S_1 \times \phi_2(x) = S_1 \times B = u \). Therefore, \( \phi \) is onto.

If \( u = A \times S_2, A \in F_1 \). Since \( \phi_1 \) is onto, there exists \( x \in V(G) \subseteq V(G + H) \) such that \( \phi_1(x) = A \). Thus, \( \phi(x) = S_1 \times S_2 = A \times S_2 = u \). Therefore, \( \phi \) is onto.

Let \( x_1 \) and \( x_2 \) be adjacent in \( G + H \). Consider the following cases:

Case 1. Suppose \( x_1 \) and \( x_2 \) are adjacent in \( G \). Then \( \phi(x_1) = \phi_1(x_1) \times S_2 \) and \( \phi(x_2) = \phi_1(x_2) \times S_2 \). Now,

\[
\phi(x_1) \cap \phi(x_2) = (\phi_1(x_1) \times S_2) \cap (\phi_1(x_2) \times S_2) = (\phi_1(x_1) \cap \phi_1(x_2)) \times S_2 \\
\neq \emptyset, \text{since } \phi_1 \text{ preserves adjacency.}
\]

Therefore, \( \phi(x_1) \) and \( \phi(x_2) \) are adjacent in \( \Omega(F) \).

Case 2. Suppose \( x_1 \) and \( x_2 \) are adjacent in \( H \). Then \( \phi(x_1) = S_1 \times \phi_2(x_1) \) and \( \phi(x_2) = S_1 \times \phi_2(x_2) \). Now,

\[
\phi(x_1) \cap \phi(x_2) = (S_1 \times \phi_2(x_1)) \cap (S_1 \times \phi_2(x_2)) = S_1 \times (\phi_2(x_1) \cap \phi_2(x_2))
\]
Therefore, $\phi(x_1)$ and $\phi(x_2)$ are adjacent $\Omega(F)$.

Case 3. Suppose $x_1 \in V(G)$ and $x_2 \in V(H)$. Then $\phi(x_1) = \phi_1(x_1) \times S_2$ and $\phi(x_2) = S_1 \times \phi_2(x_2)$. Now,

$$
\phi(x_1) \cap \phi(x_2) = (\phi_1(x_1) \times S_2) \cap (S_1 \times \phi_2(x_2))
= (\phi_1(x_1) \cap S_1) \times (S_2 \cap \phi_2(x_2))
= \phi_1(x_1) \times \phi_2(x_2),\text{ since } \phi_1(x_1) \subseteq S_1 \text{ and } \phi_2(x_2) \subseteq S_2
\neq \emptyset.
$$

Therefore, $\phi(x_1)$ and $\phi(x_2)$ are adjacent $\Omega(F)$.

Let $u, v \in F$. If $u = A \times S_2$ and $v = S_1 \times B$ for some $A \in F_1$ and $B \in F_2$, then $u = \phi_1(x) \times S_2$ and $v = S_1 \times \phi_2(y)$ for some $x \in V(G)$ and $y \in V(H)$. Consequently, $\phi^{-1}(u) = x \in V(G)$ and $\phi^{-1}(v) = y \in V(H)$. It follows that $x$ and $y$ are adjacent in $G + H$. 

If $u = A_1 \times S_2$ and $v = A_2 \times S_2$, for some $A_1, A_2 \in F_1$ then $u = \phi_1(x_1) \times S_2 = \phi(x_1)$ and $v = \phi_1(x_2) \times S_2 = \phi(x_2)$, for some $x_1, x_2 \in V(G)$. Consequently, $\phi^{-1}(u) = x_1 \in V(G)$ and $\phi^{-1}(v) = x_2 \in V(G)$. Thus, $x_1$ and $x_2$ are adjacent in $G$.

If $u = S_1 \times B_1$ and $v = S_1 \times B_2$, for some $B_1, B_2 \in F_2$ then $u = S_1 \times \phi_1(y_1) = \phi(y_1)$ and $v = S_1 \times \phi_1(y_2) = \phi(y_2)$ for some $y_1, y_2 \in V(H)$. Consequently, $\phi^{-1}(u) = y_1 \in V(H)$ and $\phi^{-1}(v) = y_2 \in V(H)$. Thus, $y_1$ and $y_2$ are adjacent in $H$. Therefore, $\phi$ preserves adjacency.

Hence, $\Omega(F) \cong G + H$

Accordingly, $\omega(G + H) \leq |S|$, since $S = S_1 \times S_2$. Then $|S| = |S_1||S_2| = \omega(G)\omega(H)$.

Hence, $\omega(G + H) \leq \omega(G)\omega(H)$. \hfill \Box

Let $G$ be a connected graph. A subset $S$ of $V(G)$ is a **clique** if $\langle S \rangle$ is a complete graph. A clique $M$ is **maximal** if $a \in V(G) - M$, then $M \cup \{a\}$ is no longer a clique in $G$. The **clique graph** of $G$, denoted by $\zeta(G)$, is the intersection graph of the set of all maximal cliques of $G$. The **clique order** of $G$, denoted by $co(G)$, is $|V(\zeta(G))|$. That is, $co(G)$ is the number of maximal cliques in $G$.

**Theorem 2.** Let $K_n$, $P_n$ and $C_n$ be a complete graph, path and cycle, respectively. Then

(i) $co(K_n) = 1$, $n \geq 1$

(ii) $co(P_n) = n - 1$, $n \geq 2$

(iii) $co(C_n) = \begin{cases} 1, & \text{if } n = 3 \\ n, & \text{if } n \geq 4 \end{cases}$

The **corona** $G \circ H$ of two graphs $G$ and $H$, is the graph obtained by making $n$ copies $(n$ is the order of $G$) of $H$ and joining every vertex of the $i$th copy of $H$ with the vertex $v_i$ of $G$. For each $a \in V(G)$, we denote by $H^a$ the copy of $H$ corresponding to the vertex $a$. 

Theorem 3. Let $G$ be a connected graph and $H$ be any graph. Then

$$\omega(G \circ H) \leq co(G) + |V(G)| \cdot \omega(H).$$

Proof. Let $V(G) = \{a_1, a_2, a_3, \ldots, a_n\}$ and $V(\zeta(G)) = \{B_1, B_2, \ldots, B_{co(G)}\}$. For each $i = 1, 2, \ldots, n$, let $F_i$ be a collection of nonempty subsets of $S_i = \{(i, j) : 1 \leq j \leq \omega(H)\}$ such that $\Omega(F_i) \cong H_{a_i}$. For each $i = 1, 2, \ldots, n$, let $\phi_i : V(H_{a_i}) \to F_i$ be an isomorphism. Let $S_o = \{(0, j) : 1 \leq j \leq co(G)\}$ and $S = \bigcup_{i=0}^n S_i$. For each $i = 1, 2, \ldots, n$, let $T_i = \{(0, j) : a_i \in B_j\}$ for some $j$. Let $F = (\bigcup_{i=1}^n F_i) \bigcup \{S_i \cup T_i : 1 \leq i \leq n\}$.

Define a mapping $\phi : V(G \circ H) \to F$ as follows

$$\phi(x) = \begin{cases} \phi_i(x), & \text{if } x \in V(H_{a_i}), \text{ for some } i \\ S_i \cup T_i, & \text{for some } i. \end{cases}$$

Let $x_1, x_2 \in V(G \circ H)$ such that $\phi(x_1) = \phi(x_2)$. Suppose $x_1 \in V(G)$ and $x_2 \in V(H_{a_i})$ for some $i$. Then $x_1 \in B_j$ for some $j$. Thus, $(0, j) \in \phi(x_1)$. Now, $\phi(x_2) = \phi_i(x_2) \subseteq S_i$, so $(0, j) \notin S_j$. This is a contradiction. Therefore, the case $x_1 \in V(G)$ and $x_2 \in V(H_{a_i})$ is not possible. Consider the following cases:

Case 1. Suppose $x_1, x_2 \in V(G)$. Then $x_1 = a_i$ and $x_2 = a_j$. Thus, $\phi(x_1) = S_i \cup T_i$ and $\phi(x_2) = S_j \cup T_j$. Note that $(i, 1) \in S_i \subseteq \phi(x_1) = \phi(x_2)$. It follows that $(i, 1) \in S_j \subseteq \{(j, 1), (j, 2), \ldots, (j, \omega(H))\}$. Consequently, $i = j$. In effect $x_1 = x_2$.

Case 2. Suppose $x_1 \in V(H_{a_i})$ and $x_2 \in V(H_{a_j})$. Suppose $i \neq j$. Then $\phi(x_1) \cap \phi(x_2) = \phi_i(x_1) \cap \phi_j(x_2) \subseteq S_i \cap S_j \neq \emptyset$. This is a contradiction. Hence, $i = j$. Consequently, $\phi_i(x_1) = \phi_i(x_2) = \phi_i(x_2) = \phi_i(x_2)$. Since $\phi_i$ is one to one, $x_1 = x_2$. Therefore, $\phi$ is one to one.

Suppose $B \in F_i$ for some $i$. Since $\phi_i : V(H_{a_i}) \to F_i$ is onto, there exists $x \in V(H_{a_i})$ such that $\phi_i(x) = B$. Consequently, $\phi(x) = \phi_i(x) = B$. Suppose $B = S_i \cup T_i$, for some $i$. Take $x = a_i$. Then $\phi(x) = \phi(a_i) = B$. Hence, $\phi$ is onto.

Let $x_1$ and $x_2$ be adjacent in $G \circ H$. Consider the following cases:

Case 1. Suppose $x_1$ and $x_2$ are adjacent in $G$. Then $x_1 = a_i$ and $x_2 = a_j$, for some $i$ and $j$. In effect, $\phi(x_1) = S_i \cup T_i$ and $\phi(x_2) = S_j \cup T_j$. Since $a_i$ and $a_j$ are adjacent in $G$, there exists $k$ such that $a_i, a_j \in B_k$. This implies that $(0, k) \in T_i$ and $(0, k) \in T_j$. It follows $\phi(x_1) \cap \phi(x_2) \neq \emptyset$. Therefore, $\phi(x_1)$ and $\phi(x_2)$ are adjacent in $\Omega(F)$.

Case 2. Suppose $x_1$ and $x_2$ are adjacent in $H_{a_i}$, for some $i$. Then $x_1, x_2 \in V(H_{a_i})$. It follows $\phi(x_1) = \phi_i(x_1)$ and $\phi(x_2) = \phi_i(x_2)$. Since $\phi_i$ preserves adjacency, $\phi(x_1) \cap \phi(x_2) = \phi_i(x_1) \cap \phi_i(x_2) \neq \emptyset$. Thus, $\phi(x_1)$ and $\phi(x_2)$ are adjacent in $\Omega(F)$.

Case 3. Suppose $x_1 = a_i$ and $x_2 \in V(H_{a_i})$. Then $\phi(x_1) = \phi(a_i) = S_i \cup T_i$ and $\phi(x_2) = \phi_i(x_2)$. Since $\phi_i(x_2) \subseteq S_i$, $\phi(x_1) \cap \phi(x_2) \neq \emptyset$. Thus, $\phi(x_1)$ and $\phi(x_2)$ are adjacent in $\Omega(F)$.

Suppose $A$ and $B$ are adjacent in $\Omega(F)$. That is, $A \cap B \neq \emptyset$. The case $A \in F_i$ and $B \in F_j$, where $i, j \neq 0$ and $i \neq j$, is not possible, since $S_i \cap S_j = \emptyset$ in this case. Consider the following cases:

Case 1. Suppose $A, B \in F_i$, for some $i$. Since $\phi_i$ is onto, there exists $x_1, x_2 \in V(H_{a_i})$ such that $\phi_i(x_1) = A$ and $\phi_i(x_2) = B$. Since $\phi_i$ preserves adjacency, $x_1$
and $x_2$ are adjacent in $H_{n_i}$. It follows that $x_1$ and $x_2$ are adjacent in $G \circ H$.

Case 2. Suppose $A = S_i \cup T_i$ and $B = S_j \cup T_j$ for some $i, j = 1, 2, 3, \ldots, n$, $i \neq j$. Since $A \cap B \neq \emptyset$, $(S_i \cap S_j) \cup (S_i \cap T_j) \cup (T_i \cap S_j) \cup (T_i \cap T_j) \neq \emptyset$. Note that $S_i \cap S_j = \emptyset$, $S_i \cap T_j = \emptyset$, $T_i \cap S_j = \emptyset$. Consequently, $(T_i \cap T_j) \neq \emptyset$. Moreover, $\phi(a_i) = A$ and $\phi(a_j) = B$. Let $t \in T_i \cap T_j$. Then $t \in T_i$ and $t \in T_j$. This implies that $t = (0, r)$ where $a_i \in B_r$ and $t = (0, s)$ where $a_j \in B_s$. Obviously, $r = s$ and $a_i, a_j \in B_r$. It follows that $a_i$ and $a_j$ are adjacent in $G$. Accordingly, $a_i$ and $a_j$ are adjacent in $G \circ H$.

Case 3. Suppose $A \in F_i$ and $B = S_j \cup T_j$ for some $i$ and $j$. Suppose $i \neq j$. Then $\phi(a) = \phi_i(a) = A$ for some $a \in V(H_{n_i})$ and $\phi(a_j) = B$. Since $A \cap B \neq \emptyset$, $(A \cap S_j) \cup (A \cap T_j) \neq \emptyset$. Since $A \subseteq S_i$, $A \cap T_j \subseteq S_i \cap T_j = \emptyset$ and $A \cap S_j \subseteq S_i \cap S_j = \emptyset$. This is a contradiction. Thus, $i = j$. Consequently, $a \in V(H_{n_j})$. It follows that $a$ and $a_j$ are adjacent in $G \circ H$. Hence $\phi$ preserves adjacency.

Therefore, $\Omega(F) \cong G \circ H$.

Accordingly,

$$\omega(G \circ H) \leq |S| = \sum_{i=0}^{n} |S_i| = |S_a| + \sum_{i=1}^{n} |S_i| = \text{co}(G) + \sum_{i=1}^{n} \omega(H) = \text{co}(G) + n \cdot \omega(H) = \text{co}(G) + |V(G)| \cdot \omega(H).$$

Therefore, $\omega(G \circ H) \leq \text{co}(G) + |V(G)| \cdot \omega(H)$.

**Corollary 1.** Let $G$ be a connected graph and $n \geq 2$. Then $\omega(K_n \circ G) = 1 + n \cdot \omega(G)$.

**Proof.** By Theorem 3, $\omega(K_n \circ G) \leq \text{co}(K_n) + |V(K_n)| \cdot \omega(G)$. By Theorem 2, $\text{co}(K_n) = 1$. Thus,

$$\omega(K_n \circ G) \leq \text{co}(K_n) + |V(K_n)| \cdot \omega(G) = 1 + n \cdot \omega(G).$$

Suppose $\omega(K_n \circ G) < 1 + n \cdot \omega(G)$. Let $V(K_n) = \{a_1, a_2, \ldots, a_n\}$ and for each $i$, $1 \leq i \leq n$, let $G_i$ be the $i$th copy of $G$ corresponding to the vertex $a_i$. Let $F$ be a collection of subsets of $S = \{1, 2, 3, \ldots, \omega(K_n \circ G)\}$ such that $\Omega(F) \cong K_n \circ G$. Let $\phi : V(K_n \circ G) \to F$ be an isomorphism. For each $i$, $1 \leq i \leq n$, $\{\phi(x) : x \in V(G_i)\}$ is a set representation for $G_i$. Thus, $|\bigcup_{x \in V(G_i)} \phi(x)| \geq \omega(G_i) = \omega(G)$. Note that for each $i, j$, $i \neq j$, and each $a \in G_i$
and \( b \in G_j \), \( ab \notin E(K_n \circ G) \). Consequently, \( E_i = \bigcup_{x \in V(G_i)} \phi(x) \) and \( E_j = \bigcup_{x \in V(G_j)} \phi(x) \) are disjoint whenever \( i \neq j \). Now,

\[
| \bigcup_{i=1}^n E_i | = \sum_{i=1}^n |E_i| \\
\geq \sum_{i=1}^n \omega(G) \\
= n \cdot \omega(G).
\]

It follows that the elements of \( S - (\bigcup_{i=1}^n E_i) \) are used for the set representation of \( G \). Note that

\[
|S - (\bigcup_{i=1}^n E_i)| = |S| - |(\bigcup_{i=1}^n E_i)| \\
\leq \omega(K_n \circ G) - n \cdot \omega(G), \text{ since we suppose } \omega(K_n \circ G) < 1 + n \cdot \omega(G). \\
< 1.
\]

That is, \( |S - (\bigcup_{i=1}^n E_i)| = 0 \). This implies, \( S = \bigcup_{i=1}^n E_i \). Since \( a_1 \) and \( a_2 \) are adjacent, \( \phi(a_1) \cap \phi(a_2) \neq \emptyset \). Let \( t \in \phi(a_1) \cap \phi(a_2) \). Then \( t \in \phi(a_1) \) and \( t \in \phi(a_2) \). Since \( S = \bigcup_{i=1}^n E_i \), \( t \in E_r \) for some \( r \). Thus \( t \in \phi(x) \) for \( x \in V(G_r) \). Therefore, \( \{ \{x, a_1, a_2\} \} \) is complete. This is a contradiction.

Therefore, \( \omega(K_n \circ G) = 1 + n \cdot \omega(G) \).

\( \square \)

**Corollary 2.** Let \( G \) be a connected graph and \( n \geq 2 \). Then \( \omega(P_n \circ G) = (n - 1) + n \cdot \omega(G) \).

**Proof.** By Theorem 3, \( \omega(P_n \circ G) \leq \text{co}(P_n) + |V(P_n)| \cdot \omega(G) \). By Theorem 2, \( \text{co}(P_n) = n - 1 \). Thus,

\[
\omega(P_n \circ G) \leq \text{co}(P_n) + |V(P_n)| \cdot \omega(G) \\
= (n - 1) + n \cdot \omega(G).
\]

Suppose \( \omega(P_n \circ G) < (n - 1) + n \cdot \omega(G) \). Let \( V(P_n) = \{a_1, a_2, ..., a_n\} \), \( E(P_n) = \{a_ia_{i+1} : 1 \leq i \leq n - 1\} \) and for each \( i, 1 \leq i \leq n \), let \( G_i \) be the \( i \)th copy of \( G \) corresponding to the vertex \( a_i \). Let \( F \) be a collection of subsets of \( S = \{1, 2, 3, ..., \omega(P_n \circ G)\} \) such that \( \Omega(F) \cong P_n \circ G \). Let \( \phi : V(P_n \circ G) \to F \) be an isomorphism. For each \( i, 1 \leq i \leq n \), \( \{\phi(x) : x \in V(G_i)\} \) is a set representation for \( G_i \). Thus, \( |\bigcup_{x \in V(G_i)} \phi(x)| \geq \omega(G_i) = \omega(G) \).

Note that for each \( i, j, i \neq j \), and each \( a \in G_i \) and \( b \in G_j \), \( ab \notin E(P_n \circ G) \). Consequently, \( E_i = \bigcup_{x \in V(G_i)} \phi(x) \) and \( E_j = \bigcup_{x \in V(G_j)} \phi(x) \) are disjoint whenever \( i \neq j \). Now,

\[
| \bigcup_{i=1}^n E_i | = \sum_{i=1}^n |E_i| \\
\geq \sum_{i=1}^n \omega(G)
\]
It follows that the elements of $S - (\bigcup_{i=1}^{n} E_i)$ are used for the set representation of $G$. Note that

$$|S - (\bigcup_{i=1}^{n} E_i)| = |S| - |(\bigcup_{i=1}^{n} E_i)|$$

$$\leq \omega(P_n \circ G) - n \cdot \omega(G),$$

since we suppose $\omega(P_n \circ G) < (n - 1) + n \cdot \omega(G)$.

Since $a_i$ and $a_{i+1}$ are adjacent, $\phi(a_i) \cap \phi(a_{i+1}) \neq \emptyset$, for every $i$, $1 \leq i \leq n - 1$. Let $A_i = \phi(a_i) \cap \phi(a_{i+1})$, $1 \leq i \leq n - 1$. Since $|S - (\bigcup_{i=1}^{n} E_i)| < n - 1$, there exist $i, j$ with $i < j$, such that $A_i \cap A_j \neq \emptyset$. Let $t \in A_i \cap A_j$. Then $t \in A_i$ and $t \in A_j$. It follows that $t \in \phi(a_i)$ and $t \in \phi(a_{j+1})$. Note that $j \geq i + 1$, it follows $a_i$ and $a_{j+1}$ are adjacent. This is a contradiction.

Hence, $\omega(P_n \circ G) = (n - 1) + n \cdot \omega(G)$.

**Corollary 3.** Let $G$ be a connected graph. Then

$$\omega(C_n \circ G) = \begin{cases} 1 + 3\omega(G), & \text{if } n = 3 \\ n + n \cdot \omega(G), & \text{if } n \geq 4 \end{cases}$$

**Proof.** By Theorem 3, $\omega(C_n \circ G) \leq co(C_n) + |V(C_n)| \cdot \omega(G)$. By Theorem 2,

$$co(C_n) = \begin{cases} 1, & \text{if } n = 3 \\ n, & \text{if } n \geq 4 \end{cases}$$

The case $n = 3$, follows from Corollary 1 and for $n \geq 4$,

$$\omega(C_n \circ G) \leq co(C_n) + |V(C_n)| \cdot \omega(G)$$

$$= n + n \cdot \omega(G).$$

Suppose $\omega(C_n \circ G) < n + n \cdot \omega(G)$. Let $V(C_n) = \{a_1, a_2, ..., a_n\}$, $E(C_n) = \{a_i a_{i+1} : 1 \leq i \leq n - 1\} \cup \{a_1 a_n\}$ and for each $i$, $1 \leq i \leq n$, let $G_i$ be the $i$th copy of $G$ corresponding to the vertex $a_i$. Let $F$ be a collection of subsets of $S = \{1, 2, 3, ..., \omega(C_n \circ G)\}$ such that $\Omega(F) \cong C_n \circ G$. Let $\phi : V(C_n \circ G) \rightarrow F$ be an isomorphism. For each $i$, $1 \leq i \leq n$, $\{\phi(x) : x \in V(G_i)\}$ is a set representation for $G_i$. Thus, $|\cup_{x \in V(G_i)} \phi(x)| \geq \omega(G_i) = \omega(G)$. Note that for each $i, j$, $i \neq j$, and each $a \in G_i$ and $b \in G_j$, $ab \notin E(C_n \circ G)$. Consequently, $E_i = \cup_{x \in V(G_i)} \phi(x)$ and $E_j = \cup_{x \in V(G_j)} \phi(x)$ are disjoint whenever $i \neq j$. Now,

$$|\cup_{i=1}^{n} E_i| = \sum_{i=1}^{n} |E_i|$$
\[
\geq \sum_{i=1}^{n} \omega(G) = n \cdot \omega(G).
\]

It follows that the elements of \( S - (\bigcup_{i=1}^{n} E_i) \) are used for the set representation of \( G \). Note that

\[
|S - (\bigcup_{i=1}^{n} E_i)| = |S| - |(\bigcup_{i=1}^{n} E_i)| \leq \omega(C_n \circ G) - n \cdot \omega(G), \text{since we suppose } \omega(C_n \circ G) < n + n \cdot \omega(G).
\]

Since \( a_i \) and \( a_{i+1} \) are adjacent, \( \phi(a_i) \cap \phi(a_{i+1}) \neq \emptyset \), for every \( i, \ 1 \leq i \leq n \). Let \( A_i = \phi(a_i) \cap \phi(a_{i+1}), \ 1 \leq i \leq n \). Since \( |S - (\bigcup_{i=1}^{n} E_i)| < n \), there exist \( i, j \) with \( i < j \), such that \( A_i \cap A_j \neq \emptyset \). Let \( t \in A_i \cap A_j \). Then \( t \in A_i \) and \( t \in A_j \). It follows that \( t \in \phi(a_i) \) and \( t \in \phi(a_{j+1}) \). Note that \( j \geq i + 1 \), it follows \( a_i \) and \( a_{j+1} \) are adjacent. This is a contradiction.

Hence, \( \omega(C_n \circ G) = n + n \cdot \omega(G) \). \( \square \)

**Corollary 4.** Let \( n \geq 3 \). Then

\[
\omega(Cr_n) = \begin{cases} 
4, & \text{if } n = 3 \\
2n, & \text{if } n \geq 4.
\end{cases}
\]

**Proof.** The proof follows from Corollary 3. \( \square \)

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**References**

