EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 1302-1317
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Degree of convergence of a function in generalized Zygmund norm using Karamata-Matrix ($K^\lambda A$) product operator

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Abstract. In the present paper, we obtain the results on the degree of convergence of a function of Fourier series in generalized Zygmund space using Karamata-Matrix ($K^\lambda A$) product operator. We also study an application of our main result.

2020 Mathematics Subject Classifications: 40C05, 40C10, 42A10

Key Words and Phrases: Degree of convergence, generalized Zygmund space ($Z_r^{(\eta)}; r \geq 1$), Karamata-Matrix ($K^\lambda A$) product operator, Fourier series, modulus of continuity.

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1. Introduction

Karamata ([3]) introduced K^λ-summability method for the first time. This method was again introduced by Lotosky ([8]) for $\lambda = 1$. A deep study on K^λ and their similar cases is studied after the publication of the work of Agnew ([1]). The degree of approximation of a function in function spaces viz, Lipschitz, Hölder and generalized Hölder using different transforms of Fourier series, has been studied by the researchers [4–6, 9–12] etc. Therefore, in the present paper we study the degree of convergence of a function in generalized Zygmund space ($Z_r^{(\eta)}; r \geq 1$) using Karamata-Matrix ($K^\lambda A$) product operator of Fourier series.

1.1. Fourier series

Let g be a Lebesgue integrable function with period 2π on the interval $[-\pi, \pi]$. The Fourier series of a function g is given by

$$g(t) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty}(a_\nu \cos \nu t + b_\nu \sin \nu t), \tag{1}$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4756

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where a_0, a_ν and b_ν are Fourier co-efficients.
The ν^{th} partial sum of (1) is given ([15]) by

$$s_\nu(g;t) = s_\nu(t) - g(t) = \frac{1}{2\pi}\int_0^\pi \phi(t,w)D_\nu(w)dw, \tag{2}$$

where

$$\phi(t,w) = g(t+w) + g(t-w) - 2g(t),$$

and $D_\nu(w)$ (Dirichlet Kernal) is defined by

$$D_\nu(w) = \frac{\sin\left(\nu+\frac{1}{2}\right)w}{\sin\left(\frac{w}{2}\right)}. \tag{3}$$

1.2. Summability operator

Let

$$u_0 + u_1 + u_3 + \cdots = \sum_{\nu=0}^{\infty} u_\nu \tag{4}$$

be an infinite series with the sequence of its ν^{th} partial sum s_ν.

1.2.1. Karamata (K^λ) operator

Let us define, for $\nu \in \mathbb{N}\cup\{0\}$, the numbers $\begin{bmatrix}\nu\\k\end{bmatrix}$, for $0 \le k \le \nu$, by

$$\prod_{p=0}^{\nu-1}(t+p) = t(t+1)\cdots(t+\nu-1) = \sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}t^k = \frac{\Gamma(t+\nu)}{\Gamma t}.$$

The numbers $\begin{bmatrix}\nu\\k\end{bmatrix}$ are said to be the absolute value of stirling number of first kind.
Let $\{s_\nu\}$ be the seuqence of the partial sums of the series (4) and we write [3, 8]

$$s_\nu^\lambda = \frac{\Gamma\lambda}{\Gamma(\lambda+\nu)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k s_k \tag{5}$$

to denote the ν^{th} K^λ-operator of order $\lambda > 0$. If $s_\nu^\lambda \to s$ as $\nu \to \infty$, where s is a definite number, then the series (4) is said to be summable by Karamata metnhod (K^λ) of order $\lambda > 0$ to the sum s.
Thus,

$$s_\nu^\lambda \to s(K^\lambda) \qquad as\ \nu \to \infty. \tag{6}$$

1.2.2. Matrix (A) operator

Let $A = (a_{\nu,k});\ \nu, k = 0, 1, 2, \cdots$ be an infinite lower triangular matrix satisfying the Silverman-Toeplitz [14] conditions of regularity i.e.

$$\sum_{k=0}^{\nu} a_{\nu,k} = 1 \quad \text{as} \quad \nu \to \infty,$$

$$a_{\nu,k} = 0 \quad \text{for} \quad k > \nu,$$

$$\sum_{k=0}^{\nu} |a_{\nu,k}| \leq M, \quad \text{a finite constant.}$$

The sequence to sequence transformation

$$d_{\nu}^{A} := \sum_{k=0}^{\nu} a_{\nu,k} s_k \tag{7}$$

defines the sequence d_{ν}^{A} of matrix operator of the sequence $\{s_{\nu}\}$ obtained by the sequence of co-efficient $(a_{\nu,k})$. If $d_{\nu}^{A} \to s$ as $n \to \infty$, then (4) is said to be summable by matrix (A) method to a definite number s.

1.2.3. Karamata-Matrix $(K^{\lambda}A)$ product operator

Superimposing A operator on K^{λ}, a Karamata-Matrix $(K^{\lambda}A)$ product operator is obtained and is given by

$$\begin{aligned} d_{\nu}^{K^{\lambda}A} &= \frac{\Gamma\lambda}{\Gamma\nu + \lambda} \sum_{k=0}^{\nu} \begin{bmatrix} \nu \\ k \end{bmatrix} \lambda^{k} (d_{k}^{A}) \\ &= \frac{\Gamma\lambda}{\Gamma\nu + \lambda} \sum_{k=0}^{\nu} \begin{bmatrix} \nu \\ k \end{bmatrix} \lambda^{k} \sum_{j=0}^{k} a_{\nu,j} s_j. \end{aligned} \tag{8}$$

If $d_{\nu}^{K^{\lambda}A} \to s$ as $\nu \to \infty$, then the series (4) is said to be summable to s by $(K^{\lambda}A)$ product operator.
Regularity of the K^{λ} and A methods implies the regularity of the $K^{\lambda}A$ method.

1.3. Generalized Zygmund space

Let $C_{2\pi}$ denotes the Banach space of all continuous and 2π-periodic functions defined on the interval $[0, 2\pi]$ with the supremum norm.
The function space for $0 < \alpha < 1$,

$$Z_{\alpha} := \{g \in C_{2\pi} : |g(t + w) + g(t - w) - 2g(t)| = O(|w|^{\alpha})\} \tag{9}$$

is a Banach space with the norm $\|\cdot\|_{(\alpha)}$ defined by

$$\|g\|_{(\alpha)} := \sup_{0\leq t\leq 2\pi} |g(t)| + \sup_{\substack{t,w\\ w\neq 0}} \frac{|g(t+w)+g(t-w)-2g(t)|}{|w|^{\alpha}}$$

The space of all Lebesgue integrable and periodic functions with period 2π be

$$L^r := \left\{ g:[0,2\pi]\to\mathbb{R}; \int_0^{2\pi} |g(t)|^r dt < \infty, r\geq 1 \right\}. \tag{10}$$

The norm of (10) is defined by

$$\|g\|_r = \begin{cases} \left\{ \frac{1}{2\pi}\int_0^{2\pi} |g(t)|^r dt \right\}^{\frac{1}{r}} & \text{for } \ 1\leq r<\infty, \\ ess\sup_{t\in[0,2\pi]} |g(t)| & \text{for } \ r=\infty. \end{cases}$$

We define

$$Z_{(\alpha),r} := \left\{ g\in L^r[0,2\pi] : \left(\int_0^{2\pi} |g(t+w)+g(t-w)-2g(t)|^r dt \right)^{\frac{1}{r}} = O(|w|^{\alpha}) \right\}. \tag{11}$$

The space $Z_{(\alpha),r}, r\geq 1, 0<\alpha\leq 1$ is a Banach space with the norm $\|\cdot\|_{\alpha,r}$:

$$\|g\|_{\alpha,r} := \|g\|_r + \sup_{w\neq 0} \frac{\|g(t+w)+g(t-w)-2g(t)\|_r}{|w|^{\alpha}}.$$

$$\|g\|_{\alpha,r} := \|g\|_r.$$

The function space $Z^{(\eta_1)}$ is a defined as

$$Z^{(\eta_1)} := \{g\in C_{2\pi} : |g(t+w)+g(t-w)-2g(t)| = O(\eta_1(w))\}$$

where η_1 is a integral modulus of continuity, that is, η_1 is a non-decreasing continuous function together with the property $\eta_1(0)=0, \eta_1(w_1+w_2)\leq \eta_1(w_1)+\eta_1(w_2)$.

Let $\eta_1 : [0,2\pi]\to\mathbb{R}$ be a real valued arbitrary function with $\eta_1(w)>0$ for $0<w\leq 2\pi$ and $\lim_{n\to 0+}\eta_1(w)=\eta_1(0)=0$.
Now, we define ([15])

$$Z_r^{(\eta_1)} = \left\{ g\in L^r[0,2\pi] : \sup_{w\neq 0} \frac{\|g(\cdot+w)+g(\cdot-w)-2g(\cdot)\|_r}{\eta_1(w)} < \infty, r\geq 1 \right\}, \tag{12}$$

with its norm given by

$$\|g\|_r^{(\eta_1)} = \|g\|_r + \sup_{w\neq 0} \frac{\|g(\cdot+w)+g(\cdot-w)-2g(\cdot)\|_r}{\eta_1(w)}, r\geq 1. \tag{13}$$

Hence, the generalized Zygmund space (12) with (13) is a Banach space.
The space $\|\cdot\|_r^{(\eta_1)}$ is complete in view of $L^r(r \geq 1)$ space.

Note 1: $\eta_1(w)$ and $\eta_2(w)$ denote the moduli of continuity of order two ([15]).
If $\frac{\eta_1(w)}{\eta_2(w)}$ be non-decreasing and positive, then

$$\|g\|_r^{(\eta_2)} \leq \max\left(1, \frac{\eta_1(2\pi)}{\eta_2(2\pi)}\right) \|g\|_r^{(\eta_1)} < \infty.$$

Note 2: We observe that

$$Z_r^{(\eta_1)} \subset Z_r^{(\eta_2)} \subset L^r, r \geq 1.$$

Remark 1:

(i) If $r \to \infty$ in $Z_r^{(\eta_1)}$ then $Z_r^{(\eta_1)}$ reduces to $Z^{(\eta_1)}$.

(ii) If $\eta_1(w) = w^\alpha$ in $Z^{(\eta_1)}$ then $Z^{(\eta_1)}$ reduces to Z_α.

(iii) If $\eta_1(w) = w^\alpha$ in $Z_r^{(\eta_1)}$ then $Z_r^{(\eta_1)}$ reduces to $Z_{\alpha,r}$.

(iv) If $r \to \infty$ in $Z_{\alpha,r}$ then $Z_{\alpha,r}$ reduces to Z_α.

(v) If $\eta_1(w) = w^{\alpha_1}, \eta_2(w) = w^{\alpha_2}, r \to \infty$ and $\alpha_2 = 0$ in $Z_{\alpha,r}$ then $Z_r^{(\eta_1)}$ reduces to $Lip(\alpha)$.

(vi) Let $0 \leq \delta_2 < \delta_1 < 1$, if $\eta_1(w) = w^{\delta_1}$ and $\eta_2(w) = w^{\delta_2}$ then $\frac{\eta_1(w)}{\eta_2(w)}$ is non-decreasing, while $\frac{\eta_1(w)}{w\eta_2(w)}$ is non-increasing.

1.4. Degree of convergence

The degree of convergence of a summation method to a given function g is a measure that how fast w_ν converges to g, which is given by ([7])

$$\|g - w_\nu\| = O\left(\frac{1}{\gamma_\nu}\right),$$

where $\gamma_\nu \to \infty$ as $\nu \to \infty$.
We write

$$\Phi(w) = \phi(t, w) = g(t + w) + g(t - w) - 2g(t);$$

$$\Phi(t) = \int_0^t |\phi(u)| du;$$

$$M_\nu(w) = \frac{1}{2\pi} \sum_{k=0}^{\nu} \begin{bmatrix} \nu \\ k \end{bmatrix} \lambda^k \sum_{j=0}^{k} a_{k,j} \frac{\sin\left(j + \frac{1}{2}\right) w}{\sin(\frac{w}{2})}.$$

The organization of the paper is as follows: In section 2, we give a motivation and propose our main results. In section 3, we establish two lemmas, which are used in the proofs of our main results. In section 4, we establish our main results. In section 5, we give applications of our main results and in section 6, we give a conclusion of the main results.

2. Main Results

In this section, we state our main results:

Theorem 1. *Let g be a Lebesgue integrable function with period 2π then the degree of convergence of g of Fourier series in the generalized Zygmund space $(Z_r^{(\eta_1)}, r \geq 1)$ using $(K^\lambda A)$ operator, is given by*

$$\|d_\nu^{K^\lambda A}(g;\cdot) - g(\cdot)\|_r^{(\eta_2)} = O\left[\left(\frac{(1+\Gamma\lambda)\{2\pi(\nu+1)-1\}}{(\nu+1)\Gamma\lambda\{\pi(\nu+1)-1\}}\right)\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right], \tag{14}$$

where $\eta_1(w)$ and $\eta_2(w)$ are as defined in Note 1 and $\frac{\eta_1(w)}{\eta_2(w)}$ is positive and non-decreasing.

Theorem 2. *Following the conditions of Theorem 1, if $\frac{\eta_1(w)}{w\eta_2(w)}$ is non-increasing, then the degree of convergence of g of Fourier series in the generalized Zygmund space $(Z_r^{(\eta_1)}, r \geq 1)$ using $(K^\lambda A)$ operator, is given by*

$$\|d_\nu^{K^\lambda A}(g;\cdot) - g(\cdot)\|_r^{(\eta_2)} = O\left[\left(\frac{(1+\Gamma\lambda)\{2\pi(\nu+1)-1\}}{\Gamma\lambda\{\pi(\nu+1)-1\}}\right)\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\log\{(\nu+1)\pi\}\right]. \tag{15}$$

3. Lemmas

In this section, we prove the following lemmas:

Lemma 1. *([6]) Let $f \in Z_r^{(\eta_1)}$, then for $0 < w \leq \pi$.*
If $\eta_1(w)$ and $\eta_2(w)$ are as defined in Note 1, then

$$\|\phi(\cdot+z,w)+\phi(\cdot-z,w)-2\phi(\cdot,w)\|_r = O\left(\eta_2(|z|)\frac{\eta_1(w)}{\eta_2(w)}\right).$$

Lemma 2. $|M_\nu(w)| = O\left(\frac{\nu+1}{\Gamma\lambda}\right)$ *for* $0 < w \leq \frac{1}{\nu+1}$.

Proof. For $0 < w \leq \frac{1}{\nu+1}, \sin(\frac{w}{2}) \geq \frac{w}{\pi}, |\sin(\nu w)| \leq \nu w$.

$$\begin{aligned}
|M_\nu(w)| &= \frac{1}{2\pi}\left|\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{j=0}^{k}a_{k,j}\frac{\sin\left(j+\frac{1}{2}\right)w}{\Gamma(n+\lambda)\sin(\frac{w}{2})}\right| \\
&\leq \frac{1}{2\pi}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{q=0}^{j}a_{k,j}\frac{|\sin\left(j+\frac{1}{2}\right)w|}{|\sin(\frac{w}{2})|} \\
&\leq \frac{1}{2\pi}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{j=0}^{k}a_{k,j}\frac{\left(j+\frac{1}{2}\right)w}{\frac{w}{\pi}}
\end{aligned}$$

$$\leq \frac{1}{4}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{j=0}^{k}a_{k,j}(2j+1)$$
$$\leq \frac{1}{4}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\left\{2\sum_{j=0}^{k}ja_{k,j}+\sum_{j=0}^{k}a_{k,j}\right\}$$
$$\leq \frac{1}{4}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\left\{2(a_{k,1}+2a_{k,2}+\cdots+ka_{k,k})+1\right\}$$
$$\leq \frac{1}{4}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\left\{2(ka_{k,1}+ka_{k,2}+\cdots+ka_{k,k})+1\right\}$$
$$\leq \frac{1}{4}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\left\{2k(a_{k,0}+a_{k,1}+a_{k,2}+\cdots+a_{k,k})-2ka_{k,0}+1\right\}$$
$$\leq \frac{1}{4}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k 2k\left\{(a_{k,0}+a_{k,1}+a_{k,2}+\cdots+a_{k,k})-a_{k,0}\right\}+1$$
$$\leq \frac{1}{4}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k 2k\left\{1-a_{k,0}\right\}+1$$
$$\leq \frac{1}{4}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k(2k+1)$$
$$\leq \frac{(2n+1)}{4}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k$$
$$\leq \frac{(2n+1)}{4}\frac{1}{\Gamma(\nu+\lambda)}\frac{\Gamma(\nu+\lambda)}{\Gamma\lambda}$$
$$= O\left(\frac{\nu+1}{\Gamma\lambda}\right).$$

Lemma 3. $|M_\nu(w)| = O\left(\frac{1}{w^2(\nu+1)\Gamma\lambda}\right)$ *for* $\frac{1}{\nu+1} < w \leq \pi$.

Proof. For $\frac{1}{\nu+1} < w \leq \pi, \sin(\frac{w}{2}) \geq \frac{w}{\pi}$.

$$|M_\nu(w)| = \frac{1}{2\pi}\left|\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{j=0}^{k}a_k b_j\frac{\sin\left(j+\frac{1}{2}\right)w}{\Gamma(\nu+\lambda)\sin(\frac{w}{2})}\right|$$
$$\leq \frac{1}{2\pi}\frac{1}{\Gamma(\nu+\lambda)}\left|\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{j=0}^{k}a_{k,j}\frac{\sin\left(j+\frac{1}{2}\right)w}{|\sin(\frac{w}{2})|}\right|$$
$$\leq \frac{1}{2\pi}\frac{1}{\Gamma(\nu+\lambda)}\left|\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{j=0}^{k}a_{k,j}\frac{\sin\left(j+\frac{1}{2}\right)w}{\frac{w}{\pi}}\right|$$

$$\leq \frac{1}{2w}\frac{1}{\Gamma(\nu+\lambda)}\left|\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{j=0}^{k}a_{k,j}\sin\left(j+\frac{1}{2}\right)w\right|$$

By Abel's lemma, we get

$$\begin{aligned}
&|M_\nu(w)|\\
&\leq \frac{1}{2w}\frac{1}{\Gamma(\nu+\lambda)}\left[\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\left|\sum_{j=0}^{k-1}(a_{k,j}-a_{k-1,j+1})\sum_{p=0}^{j}\sin\left(p+\frac{1}{2}\right)w\right|\right.\\
&\left.+\,a_{k,k}\sum_{j=0}^{k}\sin\left(j+\frac{1}{2}\right)w\right]\\
&\leq \frac{1}{2w}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\left|\sum_{j=0}^{k-1}\Delta a_{k,j}\sum_{p=0}^{j}\sin\left(p+\frac{1}{2}\right)w\right|+a_{k,k}\left|\sum_{j=0}^{k}\sin\left(j+\frac{1}{2}\right)w\right|\\
&\leq \frac{1}{2w}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\left[\sum_{j=0}^{k-1}|\Delta a_{k,j}|+a_{k,k}\right]\max_{0\leq p\leq m}\left|\sum_{p=0}^{m}\sin\left(p+\frac{1}{2}\right)l\right|\\
&\leq \frac{1}{2w}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\left[O\left(\frac{1}{k+1}\right)+O\left(\frac{1}{k+1}\right)\right]\cdot\frac{1}{w}\\
&\leq \frac{1}{w^2}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\left(\frac{1}{k+1}\right).\\
&\leq \frac{1}{w^2(\nu+1)}\frac{1}{\Gamma(\nu+\lambda)}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\\
&\leq \frac{1}{w^2(\nu+1)}\frac{1}{\Gamma(\nu+\lambda)}\frac{\Gamma(\nu+\lambda)}{\Gamma\lambda}\\
&=O\left(\frac{1}{w^2(\nu+1)\Gamma\lambda}\right).
\end{aligned}$$

4. Proof of Main Results

Proof. [**Proof of Theorem 1**] By using the integral representation ([13]) of $s_\nu(g;t)$, we have

$$s_\nu(g;t)-g(t)=\frac{1}{2\pi}\int_0^\pi\phi(t,w)\frac{\sin(n+\frac{1}{2})w}{\sin(\frac{w}{2})}dw. \tag{16}$$

Denoting $K^\lambda A$ operator of $s_\nu(g;t)$ by $d_\nu^{K^\lambda A}$, we get

$$d_\nu^{K^\lambda A}(g;t)-g(t)=\frac{\Gamma\lambda}{\Gamma\nu+\lambda}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{j=0}^{k}a_{k,j}\left\{s_j(f;t)-f(t)\right\}$$

$$= \frac{\Gamma\lambda}{\Gamma\nu+\lambda}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{j=0}^{k}a_{k,j}\left\{\frac{1}{2\pi}\int_0^{\pi}\phi(t,w)\frac{\sin(j+\frac{1}{2})w}{\sin(\frac{w}{2})}dw\right\}$$
$$= \Gamma\lambda\int_0^{\pi}\phi(t,w)\frac{1}{2\pi}\sum_{k=0}^{\nu}\begin{bmatrix}\nu\\k\end{bmatrix}\lambda^k\sum_{j=0}^{k}a_{k,j}\frac{\sin(j+\frac{1}{2})w}{\Gamma\nu+\lambda\cdot\sin(\frac{w}{2})}dw$$
$$= \Gamma\lambda\int_0^{\pi}\phi(t,w)M_\nu(w)dw.$$

Let

$$\rho_\nu(t) := d_\nu^{K^\lambda A}(g;t) - g(t)$$
$$= \Gamma\lambda\int_0^{\pi}\phi(t,w)M_\nu(w)dw. \tag{17}$$

Now,

$$\rho_\nu(t+z)+\rho_\nu(t-z)-2\rho_\nu(t) = \Gamma\lambda\int_0^{\pi}\{\phi(t+z,w)+\phi(t-z,w)-2\phi(t,w)\}M_\nu(w)dw.$$

Using generalized Minkowski inequality ([2]), we can write

$$\|\rho_\nu(\cdot+z)+\rho_\nu(\cdot-z)-2\rho_\nu(\cdot)\|_r$$
$$\leq \Gamma\lambda\int_0^{\frac{1}{\nu+1}}\|\phi(\cdot+z,w)+\phi(\cdot-z,w)-2\phi(\cdot,w)\|_r|M_\nu(w)|dw$$
$$+\Gamma\lambda\int_{\frac{1}{\nu+1}}^{\pi}\|\phi(\cdot+z,w)+\phi(\cdot-z,w)-2\phi(\cdot,w)\|_r|M_\nu(w)|dw$$
$$= I_1+I_2. \tag{18}$$

Now, using Lemmas 1 and 2, we have

$$I_1 = \left[\Gamma\lambda\int_0^{\frac{1}{\nu+1}}\eta_2(|z|)\frac{\eta_1(w)}{\eta_2(w)}\frac{(\nu+1)}{\Gamma\lambda}dw\right]$$
$$= O\left[(\nu+1)\eta_2(|z|)\int_0^{\frac{1}{\nu+1}}\frac{\eta_1(w)}{\eta_2(w)}dw\right]$$
$$= O\left[(\nu+1)\eta_2(|z|)\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\int_0^{\frac{1}{\nu+1}}dw\right]$$
$$= O\left[\eta_2(|z|)\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\right]. \tag{19}$$

Now, using Lemmas 1 and 3, we have

$$I_2 = O\left[\Gamma\lambda \int_{\frac{1}{\nu+1}}^{\pi} \eta_2(|z|)\frac{\eta_1(w)}{\eta_2(w)}\left\{\frac{1}{w^2(\nu+1)\Gamma\lambda}\right\}dw\right]$$
$$= O\left[\frac{\eta_2(|z|)}{(\nu+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right]. \tag{20}$$

Combining (18)-(20), we have

$$\|\rho_\nu(\cdot+z)+\rho_\nu(\cdot-z)-2\rho_\nu(\cdot)\|_r = O\left[\eta_2(|z|)\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\right]$$
$$+ O\left[\frac{\eta_2(|z|)}{(v+1)}\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right]. \tag{21}$$

Now,

$$\sup_{z\neq 0}\frac{\|\rho_\nu(\cdot+z)+\rho_\nu(\cdot-z)-2\rho_\nu(\cdot)\|_r}{\eta_2(|z|)}$$
$$= O\left[\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\right] + O\left[\frac{1}{(n+1)}\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right]. \tag{22}$$

Now,

$$\|\rho_\nu(\cdot)\|_r \leq \int_0^{\pi}\|\phi(\cdot,w)\|_r|M_\nu(w)|dw$$
$$= O\left[\int_0^{\frac{1}{\nu+1}}\|\phi(\cdot,w)\|_r|M_\nu(w)|dw\right] + \left[\int_{\frac{1}{\nu+1}}^{\pi}\|\phi(\cdot,w)\|_r|M_\nu(w)|dw\right]$$
$$= J_1 + J_2. \tag{23}$$

Using Lemma 2, we get

$$J_1 = O\left[\int_0^{\frac{1}{\nu+1}}\|\phi(\cdot,w)\|_r|M_\nu(w)|dw\right]$$
$$= O\left[\frac{(\nu+1)}{\Gamma\lambda}\int_0^{\frac{1}{\nu+1}}\eta_1(w)dw\right]$$
$$= O\left[\frac{(\nu+1)}{\Gamma\lambda}\eta_1\left(\frac{1}{\nu+1}\right)\int_0^{\frac{1}{\nu+1}}dw\right]$$
$$= O\left[\frac{1}{\Gamma\lambda}\eta_1\left(\frac{1}{\nu+1}\right)\right]. \tag{24}$$

Using Lemma 3, we get

$$\begin{aligned} J_2 &= O\left[\int_{\frac{1}{\nu+1}}^{\pi} \|\phi(\cdot,w)\|_r |M_\nu(w)| dw\right] \\ &= O\left[\int_{\frac{1}{\nu+1}}^{\pi} \left\{\frac{1}{w^2(\nu+1)\Gamma\lambda}\right\} \eta_1(w) dw\right] \\ &= O\left[\frac{1}{(\nu+1)\Gamma\lambda}\int_{\frac{1}{\nu+1}}^{\pi} \frac{\eta_1(w)}{w^2} dw\right]. \end{aligned} \tag{25}$$

Combining (23)-(25), we have

$$\|\rho_\nu(\cdot)\|_r = O\left[\frac{1}{\Gamma\lambda}\eta_1\left(\frac{1}{\nu+1}\right)\right] + O\left[\frac{1}{(\nu+1)\Gamma\lambda}\int_{\frac{1}{\nu+1}}^{\pi} \frac{\eta_1(w)}{w^2} dw\right]. \tag{26}$$

Now, we have

$$\|\rho_\nu(\cdot)\|_r^{(\eta_2)} = \|\rho_\nu(\cdot)\|_r + \sup_{z\neq 0} \frac{\|\rho_\nu(\cdot+z)+\rho_\nu(\cdot-z)-2\rho_\nu(\cdot)\|_r}{\eta_2(|z|)}.$$

From (22) and (26), we get

$$\begin{aligned} \|\rho_v(\cdot)\|_r^{(\eta_2)} &= O\left[\frac{1}{\Gamma\lambda}\eta_1\left(\frac{1}{\nu+1}\right)\right] + O\left[\frac{1}{(\nu+1)\Gamma\lambda}\int_{\frac{1}{\nu+1}}^{\pi} \frac{\eta_1(w)}{w^2} dw\right] \\ &= O\left[\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\right] + O\left[\frac{1}{(\nu+1)}\int_{\frac{1}{\nu+1}}^{\pi} \frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2} dw\right]. \end{aligned}$$

In view of monotonicity of $\eta_2(w)$, we have
$\eta_1(w) = \frac{\eta_1(w)}{\eta_2(w)}\eta_2(w) \leq \eta_2(\pi)\frac{\eta_1(w)}{\eta_2(w)} = O\left(\frac{\eta_1(w)}{\eta_2(w)}\right)$ for $0 < w \leq \pi$. Hence,

$$\begin{aligned} \|\rho_\nu(\cdot)\|_r^{(\eta_2)} &= O\left[\frac{1}{\Gamma\lambda}\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\right] + O\left[\frac{1}{(\nu+1)\Gamma\lambda}\int_{\frac{1}{\nu+1}}^{\pi} \frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2} dw\right] \\ &+ O\left[\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\right] + O\left[\frac{1}{(\nu+1)}\int_{\frac{1}{\nu+1}}^{\pi} \frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2} dw\right]. \end{aligned} \tag{27}$$

Since η_1 and η_2 are as defined in Note 1 and $\frac{\eta_1(w)}{\eta_2(w)}$ is positive, non-decreasing, therefore,

$$\int_{\frac{1}{\nu+1}}^{\pi} \frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2} dw \geq \frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\int_{\frac{1}{\nu+1}}^{\pi} \frac{1}{w^2} dw$$

$$\geq \frac{\eta_1\left(\frac{1}{n+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\left[-\frac{1}{\pi}+(\nu+1)\right]$$

$$\geq \frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\left[\frac{\pi(\nu+1)-1}{\pi}\right].$$

Then,

$$\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}=O\left[\frac{\pi}{\{\pi(\nu+1)-1\}}\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right]. \tag{28}$$

From (27) and (28), we get

$$\begin{aligned}
&\|\rho_\nu(\cdot)\|_r^{(\eta_2)}\\
&=O\left[\frac{1}{\Gamma\lambda}\frac{\pi}{\{\pi(\nu+1)-1\}}\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right]\\
&+O\left[\frac{1}{(\nu+1)\Gamma\lambda}\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right]\\
&+O\left[\frac{\pi}{\{\pi(\nu+1)-1\}}\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right]+O\left[\frac{1}{(\nu+1)}\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right]\\
&=O\left[\left(\frac{\pi}{\Gamma\lambda\{\pi(\nu+1)-1\}}+\frac{1}{\Gamma\lambda(\nu+1)}+\frac{\pi}{\{\pi(\nu+1)-1\}}+\frac{1}{(\nu+1)}\right)\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right]\\
&=O\left[\left(\frac{(1+\Gamma\lambda)\{2\pi(\nu+1)-1\}}{(\nu+1)\Gamma\lambda\{\pi(\nu+1)-1\}}\right)\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right].
\end{aligned}$$

Proof. [**Proof of Theorem 2**] Following the proof of Theorem 1, we have

$$E_\nu(g)=O\left[\left(\frac{(1+\Gamma\lambda)\{2\pi(\nu+1)-1\}}{(\nu+1)\Gamma\lambda\{\pi(\nu+1)-1\}}\right)\int_{\frac{1}{\nu+1}}^{\pi}\frac{\eta_1(w)}{\eta_2(w)}\frac{1}{w^2}dw\right].$$

Since $\frac{\eta_1(w)}{w\eta_2(w)}$ is non-increasing and positive, thus using second mean value theorem of the integral calculus, we have

$$E_\nu(g)=O\left[\left(\frac{(1+\Gamma\lambda)\{2\pi(\nu+1)-1\}}{(\nu+1)\Gamma\lambda\{\pi(\nu+1)-1\}}\right)(\nu+1)\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\int_{\frac{1}{\nu+1}}^{\pi}\frac{1}{w}dw\right]$$

$$= O\left[\left(\frac{(1+\Gamma\lambda)\{2\pi(\nu+1)-1\}}{\Gamma\lambda\{\pi(\nu+1)-1\}}\right)\frac{\eta_1\left(\frac{1}{\nu+1}\right)}{\eta_2\left(\frac{1}{\nu+1}\right)}\log\{(\nu+1)\pi\}\right].$$

.

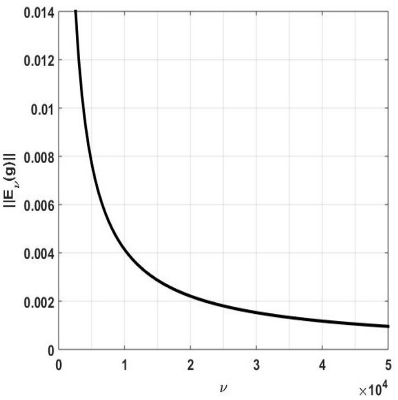
5. Application

In this section, we study an application of our main result.
We take $\eta_1(\frac{1}{\nu+1}) = \left(\frac{1}{\nu+1}\right)^{\delta_1}$, $\eta_2(\frac{1}{\nu+1}) = \left(\frac{1}{\nu+1}\right)^{\delta_2}$, $\delta_1 = 1, \delta_2 = 0$, and $\lambda = 2$ then from Theorem 2, we have

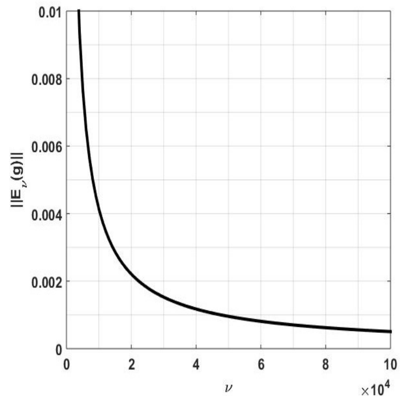
$$E_\nu(g) = O\left[\left(\frac{(1+\Gamma\lambda)\{2\pi(\nu+1)-1\}}{\Gamma\lambda\{\pi(\nu+1)-1\}}\right)\frac{1}{(\nu+1)}\log\{(\nu+1)\pi\}\right].$$

Table 1: Degree of convergence of g for different ν.

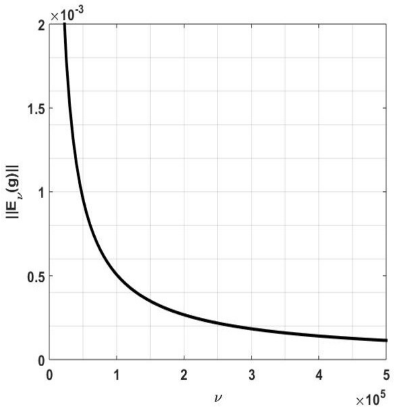
ν	**Degree of convergence of g**
1000	0.0322
10000	0.0041
50000	0.0009572
100000	0.0005063
500000	0.00011414
1000000	0.000005984
.	.
.	.
∞	0



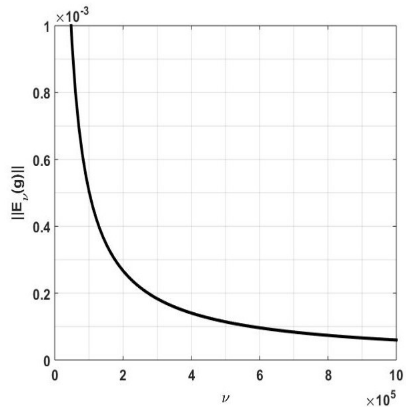
(a) For $\nu = 50000$



(b) For $\nu = 100000$



(c) For $\nu = 500000$



(d) For $\nu = 1000000$

Figure 1: Degree of convergence of function g.

6. Conclusion

From the Table 1 and figures $1(a)$ to $1(d)$, we observed that the error estimation tends to zero rapidely as ν tends to infinity. Thus, the results obtained in Theorems 1 and 2 provide the best approximation of the function g in generalized Zygmund space $(Z_r^{(\eta)}; r \geq 1)$ using Karamata-Matrix $(K^\lambda A)$ product operator.

Acknowledgements

The first author expresses his gratitude towards his mother for her blessings. The first author also expresses his gratitude towards his father in heaven whose soul is always guiding and encouraging him. The second author also expresses his gratitude towards his parents for their blessings and encouragement. Both the authors wish to thank the referees for their insightful comments and valuable suggestions for improvement of the paper.

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