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Locally Compact Spaces with Defects

Mohmmad Zailai

Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O.Box 80200 Jeddah 21589, Saudi Arabia

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Abstract. We call a topological space X a locally compact space with defects if all points in X possess compact neighborhoods except for some points. We investigate this weaker version of local compactness. We show that for $x \in X^\bullet$ if the partition of singletons of $X \backslash (X^\bullet \cup (\overline{U} \backslash U))$ is locally finite, where $U \neq X$ is an open neighborhood of x, then X is a Tychonoff space. Let X be a T_{1c} locally compact space with defects such that each $x \in X^\bullet$ has an open neighborhood U such that $\overline{U}$ is a union of pairwise disjoint compact subsets $\bigcup_{s\in S} F_s$. Then, we show that if the family $\{F_s\}_{s\in S}$ is locally finite except for a finite number of points, then X is a Tychonoff space.

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1. Introduction

A T_1 space X is said to be locally compact if every point $x \in X$ possesses a compact neighborhood, i.e, an open neighborhood such that its closure is a compact subspace. In this paper we introduce a weaker version of local compactness, which we call local compactness with defects. A T_1 space X is locally compact with defects if each point of the space has a compact neighborhood except for some points. We denote by $X^\bullet$ the set of points of X which do not have compact neighborhoods. Points of $X^\bullet$ are called defects. A space X is said to be scattered if it contains no non-empty subset which is dense-in-itself. It is proved in [4] that for a Tychonoff space X the set $X^\bullet$ is closed. We extend this result and show that for any space, the set of defects is a closed subset. We use that result to show that for any T_1 topological space, if the set of defects is not empty then the space is not scattered. All compact subspaces in this paper are assumed to be closed and T_2. We denote by T_{1c} a T_1 space such that each compact subspace is closed. This is a space which lies between T_1 and T_2 spaces. $\aleph_0$ stands for a cardinality of a countable set. $\mathbf{N}$ stands for the set of all natural numbers. By a $T_{3\frac{1}{2}}$ space we mean a Tychonoff space. $\mathcal{K}$ stands for the sorgenfrey line, i.e., the space generated by the base $\mathcal{B} = \{[x, y)\}$, where x, y are real numbers such that $x < y$, and y is a rational number. For more details about locally compact spaces, see [1]. More details about points that do not have compact neighborhoods, which we call defects, can be found in [3], [4] and [5].

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Email address: mzailai@kau.edu.sa (M. Zailai)

2. Local Compactness with Defects

Definition 2.1. Let X be a topological space and let $\mathcal{F}$ be any property. If $Y \subset X$ is the set of points which do not satisfy the property $\mathcal{F}$, i.e., only $X\backslash Y$ has the property $\mathcal{F}$ then we call the topological space X a space with defects of type $\mathcal{F}$.

Remark 1. *This paper concerns about defect of type local compactness. Throughout this paper we write l.c.w.d. for a space with defects of type local compactness.*

Definition 2.2. A T_1 space X is a space with defects of type local compactness, *l.c.w.d.*, if all points have compact neighborhoods except for some points. We denote by $X^\bullet$ the set of points which do not possess compact neighborhoods.

Remark 2. *It is clear that if* $X^\bullet = \phi$ *then* X *is a locally compact space. We always assume that* $X^\bullet \neq \phi$ *unless stated otherwise.*

Example 1. *[6, 118 page 137]*
Denote T *the graph of the function* $h(t) = sin(1/t)$ *where* $0 < t \leq 1$, *as a subset of the Euclidean space* $\mathbb{R}^2$ *with the relative topology. The set* $T^\star = \{(0,0)\} \cup T$ *is not locally compact since the point* $(0,0)$ *has no compact neighborhood. Therefore, the topological space* $T^\star$ *is a locally compact space with defect of type local compactness.*

Proposition 2.3. *Suppose that the topological spaces* X *and* Y *are l.c.w.d. If* X *is homeomorphic to* Y *then* $X^\bullet$ *and* $Y^\bullet$ *have same cardinality, i.e., the number of defect points is a topological invariant.*

Proof. It is enough to check that the homeomorphic image of any point in $X^\bullet$ lies in $Y^\bullet$. Suppose that $f : X \to Y$ is a homeomorphism. Take any point $x \in X^\bullet$. If $f(x) \notin Y^\bullet$ then there is a neighborhood U of $y = f(x)$ such that the closure $\overline{U}$ is a compact subspace. Now $x \in f^{-1}(U)$ and also have $\overline{f^{-1}(U)} = f^{-1}(\overline{U})$ which is compact. Therefore, $f^{-1}(U)$ is a compact neighborhood of the point x which is a contradiction as x is a defect point.

Lemma 2.4. *Let* X *be l.c.w.d then* $X^\bullet$ *is a closed subset of* X.

Proof. It is sufficient to that the complement $X\backslash X^\bullet$ is open. If $X^\bullet = \phi$, then $X\backslash X^\bullet = X$ is closed. If $X^\bullet = X$, then $X\backslash X^\bullet = \phi$ is closed. Now suppose $X^\bullet \neq X$ and $X^\bullet \neq \phi$, take any arbitrary point $x \in X\backslash X^\bullet$. Assume that for any neighborhood of U_x of x we have that $U_x \cap X^\bullet = \phi$. Since $x \in X\backslash X^\bullet$, then there is a compact neighborhood U_x of x. Let $y \in X^\bullet$, then y does not belong to U_x. Suppose otherwise, i.e., let $y \in U_x$. then $y \in \overline{U_x}$. However, that means U_x is a compact neighborhood of y. This is a contradiction as y is a defect.

Proposition 2.5. *If* X *is locally compact with defects, then* $X^\bullet$ *is dense-in-itself.*

Proof. If $X^\bullet = \phi$ then it is clearly dense-in-itself. Assume that $X^\bullet \neq \phi$. Take any point $x \in X\backslash X^\bullet$, then x is not an accumulation point of $X^\bullet$. Now, let $x \in X^\bullet$ then the closure $\overline{X^\bullet\backslash\{x\}}$ is the set $X^\bullet$. Therefore, the set $X^\bullet$ contains all of its accumulation points. Hence, $X^\bullet$ is dense-in-itself.

Corollary 2.6. *Let X be l.c.w.d such that $X^\bullet \neq \phi$ then X is not a scattered space.*

Lemma 2.7. *[2, page 17] Suppose that the family $\{W_s\}_{s\in S}$ is locally finite, then the following*

$$\overline{\bigcup_{s\in S} W_s} = \bigcup_{s\in S} \overline{W_s}$$

is always true.

Lemma 2.8. *Let X be a T_1 space and suppose space that the family $\{W_s\}_{s\in S}$ is locally finite except at a finite number of points, say $a_1, a_2, ..., a_n$, then we have*

$$\bigcup_{s\in S} \overline{W_s} \bigcup_{i=1}^{n} \{a_i\} = \overline{\bigcup_{s\in S} W_s \bigcup_{i=1}^{n} \{a_i\}}.$$

Proof. It is clearly that

$$\bigcup_{s\in S} \overline{W_s} \bigcup_{i=1}^{n} \{a_i\} \subset \overline{\bigcup_{s\in S} W_s \bigcup_{i=1}^{n} \{a_i\}}.$$

Now suppose that $x \in \overline{\bigcup_{s\in S} W_s \cup \{a_1\} ... \cup \{a_n\}}$. Let x has an open neighborhood U which intersects finitely many members of the family $\{F_s, \{a_1\}, \{a_2\}, ..., \{a_n\}\}_{s\in S}$. Let $S_1 = \{s \in S : U \cap W_s\}$ be the finite indexing set. It is clear that $x \notin \bigcup_{s\in S\setminus S_1} W_s$. Note that

$$x \in \overline{\bigcup_{s\in S_1} W_s \bigcup_{i=1}^{n} \{a_i\}}\, \overline{\bigcup_{s\notin S\setminus S_1} W_s}.$$

Then, we get $x \in \overline{\bigcup_{s\in S_1} W_s \bigcup_{i=1}^{n} \{a_i\}} = \bigcup_{s\in S_1} \overline{W_s} \bigcup_{i=1}^{n} \overline{\{a_i\}}$. Therefore,

$$x \in \bigcup_{s\in S} \overline{W_s} \cup \{a_1\} \cup \{a_2\} \cup ... \cup \{a_n\}.$$

Assume that X does not have a locally finite neighborhood, then $x = a_i$ for some i. Therefore, it is easy to see that

$$x = a_i \in \bigcup_{s\in S} \overline{W_s} \cup \{a_1\} \cup \{a_2\} \cup ... \cup \{a_n\}.$$

Hence,

$$\bigcup_{s\in S} \overline{W_s} \bigcup_{i=1}^{n} \{a_i\} = \overline{\bigcup_{s\in S} W_s \bigcup_{i=1}^{n} \{a_i\}}.$$

Remark 3. *L.c.w.d. spaces do not have to be normal. Consider product of sorgenfrey line,$X = \mathcal{K} \times \mathcal{K}$, which is locally compact i.e. $X^\bullet = \phi$ but it is not normal.*

Definition 2.9. [2, page 71] Let $\{B_i\}_{i\in I}$ be a cover of the space X. Consider any family of continuous maps $\{g_i\}_{i\in I}$, where $g_i : B_i \to Y$. The maps g_i are said to be compatible if for every i_1, i_2 of I we have

$$g_{i_1}|_{B_{i_1}\bigcap B_{i_2}} = g_{i_2}|_{B_{i_1}\bigcap B_{i_2}}.$$

The combination is defined as $g = \bigcup_{i\in I} g_i : X \to Y$.

Remark 4. *The following two Lemmas and Theorem are used in the proofs of Theorem 2.13 and Theorem 2.15.*

Lemma 2.10. *[2, page 17] If $\{S_i\}_{i\in I}$ is a locally finite closed cover of X and $\{g_i\}_{i\in I}$ is a family of compatible maps, where $g_i : S_i \to Y$. Then the combination is continuous.*

Lemma 2.11. *Let $\mathcal{W} = \{B_s\}_{s\in S} \cup \{\{a_1\}, \{a_2\}, ...\{a_n\}\}$ be a closed cover of X such that the family $\mathcal{W}$ is locally finite except for $a_1, a_2, ..., a_n$. Let $\{g_s\}_{s\in S}$ be a family of compatible maps, where $g_s : B_s \to Y$ such that all members of the family are constant of the form $g_s(B_s) = k$ except for a finite number of members. Then, the combination*

$$g = \bigcup_{s\in S} g_s \bigcup_{i=1}^{n} f_i : X \to Y$$

is continuous, where $f_i : \{a_i\} \to Y$ is defined as $f_i(a_i) = k$ for $i = 1, 2, ..., n$.

Proof. Let F be a closed subset of Y. We Want to show that the the inverse image of F is closed in X under that map g. Assume that $k \notin F$, then $g^{-1}(F) = \bigcup_{s\in S_1} g_s(F)$ for a finite subset $S_1 \subset S$. Therefore, $g^{-1}(F)$ is a finite union of closed subsets, i.e., g^{-1} is closed. Now, let us assume that $k \in F$. We have

$$\begin{aligned} g^{-1}(F) &= \bigcup_{s\in S\setminus S_1} g_s^{-1}(F) \bigcup_{i=1}^{n} f_i^{-1}(F) \\ &= \bigcup_{s\in S\setminus S_1} g_s^{-1}(F) \bigcup_{i=1}^{n} \{a_i\} \end{aligned}$$

Therefore, by using Lemma 2.8 we have that $g^{-1}(F)$ is a closed subset of X. Hence, g is continuous.

Theorem 2.12. *[2, page 148]*
Let X be a T_1 space such that any $x \in X$ possesses a compact neighborhood. Then for any closed subset $F \subset X$ such that $x \notin F$ there exists a continuous $f : X \to I$ where $f(x) = 0$ and $f(F) \subset \{1\}$.

Theorem 2.13. *Let $X^\bullet$ be a compact subset of X. Suppose that each point $x \in X^\bullet$ has an open neighborhood $U \neq X$ such that the partition of singletons of the complement of $X^\bullet \cup (\overline{U}\setminus U)$ is locally finite, then X is $T_{3\frac{1}{2}}$.*

Proof. Let $x \in X^\bullet$ and take any closed subset F of X such that $x \notin F$. Let $U \neq X$ be an open neighborhood of x which satisfies assumption above. Define

$$F_0 = ((\overline{U} \backslash U) \cup (\overline{U} \cap F)) \cap X^\bullet,$$

which is a closed subset of the closed subspace $X^\bullet$ such that $x \notin F_0$. Therefore, there exists a map $f : X^\bullet \to I$ such that $f(x) = 0$ and $f(F_0) \subset \{1\}$. Let $g : \overline{U} \backslash U \to I$ be a constant map which is defined as $g(y) = 1$ for any $y \in \overline{U} \backslash U$. Let us define also the following maps

$$f_{s \in S} : \{a_s\} \to I; a_s \mapsto 1.$$

Now, the combination

$$h = f \cup g \bigcup_{s \in S} f_s : X \to I$$

is continuous such that $h(x) = 0$ and $h(F) \subset \{1\}$.

Proposition 2.14. *Let X be a second countable space such that $X^\bullet$ is a discrete subspace. If $X^\bullet$ is compact such that each of its points satisfies the assumption in theorem 2.13, then $X^\bullet$ is of cardinality $\aleph_0$.*

Proof. First from Theorem 2.13 we have that X is a $T_{3\frac{1}{2}}$ space which tells us that X is a regular space. Since every second countable regular space is metrizable, then X is a metrizable space. Separability and second countability are equivalent in metrizable spaces. Hence, X is a separable space. However, we know that every closed discrete subspace of a separable normal space has cardinality $\leq \aleph_0$.

Theorem 2.15. *Let X be a T_{1c} l.c.w.d space such that for each point $x \in X^\bullet$ there exists an open neighborhood U of x such that the closure $\overline{U} = \bigcup_{s \in S} F_s$ is a union of compact subsets. If the family $\{F_s\}_{s \in S}$ is pairwise disjoint and locally finite except for a finite number of points, then X is $T_{3\frac{1}{2}}$.*

Proof. Let x be a defect, i.e., $x \in X^\bullet$. Let F be closed such that $x \notin F$. Take an open neighborhood U of x such that the closure $\overline{U} = \bigcup_{s \in S} F_s$ is a union pairwise disjoint compact subsets, where $\mathcal{W} = \{F_s\}_{s \in S}$ is locally finite except at $a_1, a_2, ..., a_n$. Note that x belongs to only one member of the family $\mathcal{W}$, say F_{s_k} for some $s_k \in S$. Define

$$F_0 = ((\overline{U} \backslash U) \cup (\overline{U} \cap F)) \cap F_{s_k}$$

which is a closed subset of the subspace F_{s_k} and we have that $x \notin F_0$. Therefore, there is a map $f_{s_k} : F_{s_k} \to I$ such that $f_{s_k}(x) = 0$ and $f_{s_k}(F_0) \subset \{1\}$. Define also following constant maps

$$f_s : F_s \to I, \ y \mapsto 1; \quad for \ s \neq s_k$$

$$g : X \backslash U \to I, \ y \mapsto 1.$$

Now, suppose that one of the $a_i's$ is x, say $a_m = x$. Then, we define the following maps

$$g_i : \{a_i\} \to I,\ a_i \mapsto 1 \quad for\ i = 1, ..., n \ \ and\ i \neq m$$

$$g_m : \{a_m\} \to I,\ \ a_m \mapsto 0.$$

If all $a_i's$ are distinct from x, then we define:

$$g_i : \{a_i\} \to I,\ a_i \mapsto 1 \quad for\ i = 1, ..., n$$

First, we need to check that the map

$$h = \bigcup_{s\in S} f_s \bigcup_{i=1}^{n} g_i \cup g : X \to I$$

is continuous. Let $C \subset I$ be closed. If $1 \notin C$, then $h^{-1}(C) = f_{s_k}^{-1}(C)$ is closed or $h^{-1}(C) = f_{s_k}^{-1}(C) \cup g_m^{-1}(C)$ which is also closed. Assume that $1 \in C$, then $h^{-1}(C) = \bigcup_{s\in S} f_s^{-1}(C) \bigcup_{i=1}^{n} g_i^{-1}(C) \bigcup g^{-1}(C)$ which is clearly closed by using Lemma 2.8. Hence, h is continuous. It clear that $h(x) = 0$. Now, take $y \in F$. If $y \in F_0$, then we have $h(y) = 1$. Assume that $y \notin F_0$, then we have two cases:

- Case 1: $y \notin F_{s_k}$, then it is easy to see that $h(y) = 1$,
- Case 2: $y \in F_{s_k}$, and $y \notin \overline{U}$ which cannot happen as $F_{s_k} \subset \overline{U}$. Then, we conclude that if $y \in F$ and $y \notin F_{s_k}$. Therefore, $h(y) = 1$.

Hence, X is a $T_{3\frac{1}{2}}$ space.

Proposition 2.16. *Let $\{X_s\}_{s\in S}$ be a collection of pairwise disjoint l.c.w.d topological spaces. If each point $x_s \in X_s$ has an open neighborhood U such that its closure is a union of pairwise disjoint compact subsets, then so does each point $x \in X = \oplus_{s\in S} X_s$.*

Example 2. *(Modified Arens-Fort Space):*
Here we modify the Arens-fort space. Let (A, τ) be the set of all ordered pairs of $\mathbf{N}\times\mathbf{N}$. We declare that all the singletons of this set are open sets except the points $(0, 0), (1, 0), ..., (n, 0)$ for some positive integer n. Let us define open neighborhoods of each point of $\{(0, 0), (1, 0), ..., (n, 0)\}$ as any set U such that $\{(0, 0), (1, 0), ..., (n, 0)\} \subset U$, and all but a finite number of points of each but a finite number of the sets $T_d = \{(l, d) : l\ is\ fixed\ and\ d \in \mathbf{N}\}$. Note that this space is not locally compact as the points $(0, 0), (1, 0), ..., (n, 0)$ do not possess compact neighborhoods. Let us check that the point $(0, 0)$ does not have a compact neighborhood and all other points can be verified analogously. Let U be an open neighborhood of $(0, 0)$. Consider the following $\mathcal{U} = \{\{a_s\}_{s\in S}, V\}$ such that each a_s is distinct from all the points $(0, 0), (1, 0), ..., (n, 0)$, and V is an open neighborhood of $(0, 0)$ which is distinct from U in the following sense. If $D = \{(l, d)\ : l\ is\ fixed\ and\ l \neq 0\} \subset U$, then we require that $D \not\subset V$. Now, $\mathcal{U}$ is an open cover of the closure $\overline{U}$ which does not have a finite open subcover. For any point $x \in X^\bullet$ one can take X as a neighborhood. Now, X can be written

as a union of singletons. Clearly, each one-point set in X is compact. Also, note that the partition of singletons is locally finite except for a finite number of points. Therefore, by using Theorem 2.15 we see that X is a Tychonoff space. We can also apply Theorem 2.13 to see that this space is a Tychonoff space. Namely, $X^\bullet$ is finite, then is compact. Observe that partition of singletons of $X\backslash(X^\bullet \cup (X\backslash X)$ is locally finite.

Proposition 2.17. *Let $\{X_1, X_2, ..., X_n\}$ be a collection of l.c.w.d. topological spaces. Suppose that for each topological space X_i we have $X^\bullet \neq \phi \neq X_i$. Then for $X = \prod_{i=1}^n X_i$, we have $X^\bullet \neq \phi \neq X$.*

Proof. It is straightforward.

Proposition 2.18. *Let each of $\{X_s\}_{s\in S}$ be a collection of l.c.w.d spaces such each point of X_s has an open neighborhood with closure being a union of pairwise disjoint compact subsets, then so does each point of the cartesian product $\prod_{s\in S} X_s$.*

Proof. It is straightforward.

Proposition 2.19. *Let X be l.c.w.d such that any $x \in X^\bullet$ has a σ-compact neighborhood. Then for any closed subspace $F \subset X$, $x \in X^\bullet \cap F$ has a σ-compact neighborhood of the subspace F, i.e., this space is hereditarily with respect to closed subspaces.*

Proof. Take any $x \in X^\bullet \cap F$, then there is an open neighborhood $U \subset X$ of x such that $\overline{U} = \bigcup_{s=1}^\infty F_s$ where each F_s is compact as a subset of X. Observe that $U \cap F$ is an open of x in F such that its closure, $\overline{(U \cap F)} \cap F$, in F σ-compact.

3. Conclusion

A well-known result in general topology states that any locally compact space is a Tychonoff space. In this paper we investigate a weaker version of local compactness. Instead of assuming that all points in a space have compact neighborhoods, we allow a possibility of having some points which do not possess compact neighborhoods. We denote by $X^\bullet$ a set of points which do not have open neighborhoods with compact closures. One of the results we obtain is that by requiring the set $X^\bullet$ to be compact, we show that if each point $x \in X^\bullet$ has an open neighborhood $U \neq X$ such that the partition of singletons of the complement of $X^\bullet \cup (\overline{U}\backslash U)$ is locally finite, then the space is a Tychonoff space. The following questions are still not answered. Could we assume that $X^\bullet$ is locally compact instead of being compact in Theorem 2.13? Could we drop the requirement of compact subsets need to be closed in Theorem 2.15?

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