EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 1260-1273
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

CC-Tychonoffness, CCT_3 and CC-Almost Regularity

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Abstract. Following the notion of so-called C-normality - a weaker version of normality in topological spaces as proposed by A. V. Arhangel'skii, further weaker version called CC-normality is studied by Kalantan et al [14]. In this paper, we investigate various type of properties such as CC-complete regularity, CC-almost complete regularity, CC-regularity, CC-almost regularity, CCT_3 and CC-Tychonoffness. A space $(X, \mathcal{T})$ is called a *CC-completely regular* (resp. *CC-almost completely regular, CC-regular, CC-almost regular, CCT_3, CC-Tychonoff*) space if there exist a completely regular (resp. almost completely regular, regular, almost regular, T_3, Tychonoff) space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$. We study these properties and present some examples to illustrate the relationships among them with other forms of topological properties.

2020 Mathematics Subject Classifications: 54C10, 54D10, 54D20, 54D15,54D70

Key Words and Phrases: C-normal, CC-normal, C-regular, C-Tychonoff, L-normal, L-regular and L-Tychonoff

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1. Introduction

The notion of C-normality has been studied by Alzahrani and Kalantan in [7]. The notion of L-normality has been studied by Kalantan and Saeed in [12]. Then, Alzahrani studied the notions of C-regularity, L-regularity, C-Tychonoff and L-Tychonoff in [5, 6]. At the end of 2022, Al-Awadi and others studied the notions of C-mild normality and C-κ-normality [1]. Thabit studied the notion of epi-partial normality in [26]. At the end of 2021, Thabit and others studied the notion of epi-quasi normality in [25]. Thabit and Alqurashi studied the notions of C-almost normality and L-almost normality in [3]. Thabit and others studied the notions of C-complete regularity and CT_3 and C-almost regularity in [24]. The notions of LT_3, L-complete regularity and L-almost regularity

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DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4776

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have been studied in [2]. The notion of CC-normality have been studied by Kalantan and others in [14]. The notions of C, C_2-paracompactness are studied in [19] and the notions of L, L_2-paracompactness are studied in [13]. In this paper, we investigate the properties, CC-complete regularity, CC-regularity, CC-almost regularity, CC-almost complete regularity, CCT_3 and CC-Tychonoffness. We present some examples to illustrate the relationships among these properties with other kinds of normality, complete regularity and regularity. We need to recall that: a subset A of a space X is said to be a *closed domain* subset if $A = \overline{\text{int}(A)}$ [15]. A subset A of a space X is called *π-closed* if it is a finite intersection of closed domain subsets [27]. Two subsets A and B of a space X are said to be *separated* if there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$ [9, 10, 17]. If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T}'$ is called a topology *coarser* than $\mathcal{T}$, and $\mathcal{T}$ is called *finer* [10]. A T_4-space is a T_1 normal space, a T_3-space is a T_1 regular space and a Tychonoff space is a T_1 completely regular space. A space X is said to be *Hausdorff* or *a T_2-space*, if for each distinct two points $x, y \in X$ there exist two open subsets U and V of X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ [10]. A space X is said to be *completely Hausdorff* or *Urysohn* [10, 23], if for each distinct two points $x, y \in X$ there exist two open subsets U and V of X such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. A space X is said to be *almost completely-regular* if for each $x \in X$ and each closed domain subset F of X such that $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$ [21]. A space X is said to be *almost-regular* if for each $x \in X$ and each closed domain subset F of X such that $x \notin F$, there exist two disjoint open subsets U and V such that $x \in U$ and $F \subseteq V$ [20]. A space X is said to be *sub-metrizable* [11], if there exists a metric d on X such that the topology $\mathcal{T}_d$ on X generated by d is coarser than $\mathcal{T}$. The topology on X generated by the family of all open domain subsets of X, denoted by $\mathcal{T}_s$, is coarser than $\mathcal{T}$, and $(X, \mathcal{T}_s)$ is called the *semi-regularization* of X and the space $(X, \mathcal{T})$ is called *semi-regular* if $\mathcal{T} = \mathcal{T}_s$ [16]. A space X is called *CC-normal* [14] if there exist a normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$. The basic definitions and any undefined terms in this article can be found in [25] and [26].

2. Preliminaries

First, we present the main definitions of this work.

Definition 1. Let X be a space, then:

(1) A space X is called a *CC-regular* (resp. *CC-almost regular*) space if there exist a regular (resp. almost regular) space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$.

(2) A space X is called a *CC-completely regular* (resp. *CC-almost completely regular*) space if there exist a completely regular (resp. almost completely regular) space Y

and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$.

(3) A space X is called a *CC-Tychonoff* (resp. CCT_3) space if there exist a Tychonoff (resp. T_3) space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$.

From Definition 1, clearly that: every completely regular (resp. regular, almost completely regular, almost regular, T_3, Tychonoff) space is CC-completely regular (resp. CC-regular, CC-almost completely regular, CC-almost regular, CCT_3, CC-Tychonoff), just by taking $X = Y$ and the identity function, but the converses need not be true. The next example is of a CC-Tychonoff, CCT_3, CC-completely regular and CC-regular space which is neither Tychonoff, T_3, completely regular nor regular.

Example 1. *The Smirnov's deleted sequence topology:* [23, Example 64], is a Urysohn, Lindelöf first countable separable space which is not paracompact [23]. Since X is a sub-metrizable space, by Corollary 1 and Theorem 1 we get: X is CC-Tychonoff, CCT_3, CC-completely regular, CC-regular, CC-almost regular and CC-almost completely regular, but it is neither almost normal, Tychonoff, completely regular, T_3 nor regular. Also, the half disc topology [23, Example 78], is a CC-normal, CC-regular, CC-completely regular, CC-Tychonoff, CCT_3 and CC-almost completely regular space being sub-metrizable, but it is neither regular, normal, completely regular, T_3 nor Tychonoff.

The following examples are CC-almost regular and CC-almost completely regular spaces which are neither almost regular, L-almost regular nor almost completely regular:

Example 2. *The relatively prime integer topology* [23, Example 60], is a Hausdorff, semi regular, Lindelöf, first countable separable space that is neither Urysohn, quasi normal, almost regular nor regular [25, Example 2.9]. The space X is epi-mildly normal space which is neither epi-quasi normal, epi-regular nor epi-completely regular [4, 25]. Since the space X is Lindelöf non Urysohn, we conclude: it is neither C-normal, C-regular, C-completely regular nor C-Tychonoff [24]. Thus, it is neither L-almost regular nor L-almost completely regular [2]. By Theorem 2, it is neither CC-regular, CC-completely regular, CCT_3, CC-Tychonoff nor CC-normal. Since the space X is a Hausdorff first countable space, by Theorem 17 and Corollary 10 we obtain that: the space X is CC-almost regular and CC-almost completely regular. Observe that: any Hausdorff first countable Lindelöf space is not necessary to be CC-regular, CCT_3, CC-normal, CC-Tychonoff, epi-regular nor Urysohn. This example also shows that CC-almost regularity does not imply L-almost regularity.

Now, we present the following basic results.

Theorem 1. *Every epi-completely regular space is CC-Tychonoff.*

Proof. By assumption, there exist a topology $\mathcal{T}'$ on X coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is Tychonoff [4]. Thus, the identity mapping $I_X : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is a bijective continuous function. Let M be any countably compact subspace of $(X, \mathcal{T})$. Since a continuous image of a countably compact subset is countably compact [10], we get: $I_X(M)$ is a countably compact subspace of $(X, \mathcal{T}')$ as $I_X(M) = M$ is countably compact subspace of both $(X, \mathcal{T})$ and $(X, \mathcal{T}')$. Thus, the restriction of the identity function $(I_X)|_M : M \to I_x(M)$ is bijective continuous. Let U be any open set in $(M, \mathcal{T}_M)$. Since M is a countably compact subset of $(X, \mathcal{T}')$, there exists an open set V in $(X, \mathcal{T}')$ and hence in $(X, \mathcal{T})$ such that $U = V \cap M$. Thus, $(I_X)|_M(U) = (I_X)|_M(V \cap M) = V \cap M = U$, which is an open set in $(I_X(M), \mathcal{T}'_M)$. Hence, $(I_X)|_M$ is open and hence a homeomorphism. Therefore, X is CC-Tychonoff.

Note that: every epi-normal space is epi-almost normal and epi-almost normal space is epi-completely regular [4]. Obviously, every epi-regular space is CCT_3, every epi-normal space is CC-normal and every epi-regular space is CC-regular.

Corollary 1. Every sub-metrizable (resp. epi-almost normal, epi-normal) space is CC-Tychonoff.

The converses of Theorem 1 and Corollary 1 are not true in general as shown by the next example:

Example 3. *The countable complement topology:* [23, Example 20], $(\mathbb{R}, \mathcal{CC})$ is a T_1-Lindelöf C-regular space, which is neither Hausdorff, regular, normal, first countable, separable, paracompact nor L-regular [5, 23]. Also, $(\mathbb{R}, \mathcal{CC})$ is a CC-normal space, which is not L-normal [14]. Since X is not Hausdorff, it is neither epi-regular, epi-normal nor epi-mildly normal. Since the only countably compact subsets in X are finite subsets, by Theorem 4 and Corollary 2 $(\mathbb{R}, \mathcal{CC})$ is CC-Tychonoff, CCT_3, CC-completely regular and CC-regular. This example shows that: CC-complete regularity, CC-normality, CCT_3 and CC-Tychonoffness do not imply epi-regularity (resp. epi-complete regularity, epi-mild normality, sub-metrizable, L-regularity, LT_3, L-normality nor Hausdorffness). Also, it is a CC-Tychonoff space which is not L-regular.

Theorem 2. *Every CC-completely regular space is C-completely regular.*

Proof. By assumption, there exist a completely regular space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subsets $A \subseteq X$. Let C be any compact subset of X. Since every compact space is countably compact [10], we have: C is countably compact subset of X. Thus, the restriction function $f|_C : C \to f(C)$ is a homeomorphism. Since C was arbitrary compact subset of X, we conclude that: X is C-completely regular.

Similarly, it is easy to prove that: every CC-regular space is C-regular, every CCT_3-space is CT_3, every CC-Tychonoff space is C-Tychonoff, every CC-almost regular space is C-almost regular and every CC-almost completely regular space is C-almost completely regular.

Theorem 3. *Every CC-completely regular space is CC-almost completely regular.*

Proof. By assumption, there exist a completely regular space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subsets $A \subseteq X$. Since every completely regular space is almost completely regular [21], we obtain: Y is an almost completely regular space. Therefore, X is CC-almost completely regular.

Similarly, every CC-completely regular space is CC-regular, every CC-regular space is CC-almost regular, every CC-almost completely regular space is CC-almost regular, every CC-Tychonoff space is CC-completely regular, every CCT_3-space is CC-regular and every CC-Tychonoff space is CCT_3. The converses of Theorem 3 and stated facts are not true in general. Here is an example of a CC-normal and CC-almost completely regular space, which is neither CC-completely regular, CCT_3, CC-Tychonoff nor CC-regular.

Example 4. The left ray topology $(\mathbb{R}, \mathcal{L})$ is a normal second countable and almost completely regular space [23]. Therefore, $(\mathbb{R}, \mathcal{L})$ is a CC-normal and CC-almost completely regular space, which is neither CCT_3, CC-regular, CC-Tychonoff nor CC-completely regular because it is not C-regular [5].

The next example is of a CC-completely regular space which is neither CCT_3 nor CC-Tychonoff.

Example 5. *The odd-even topology* [23, Example 6], is a regular, completely regular, normal, locally compact, paracompact, separable, second countable space, which is neither T_0, compact, countably compact nor semi regular [23]. So, the odd-even topology is a CC-regular, CC-completely regular, CC-normal and CC-almost completely regular space, which is neither epi-regular nor epi-mildly normal. Observe that: the odd even topology is neither CT_3 nor LT_3 [2, 24]. Hence, it is neither C-Tychonoff nor L-Tychonoff. Therefore, it is neither CC-Tychonoff nor CCT_3. Therefore, the odd-even topology is a CC-completely regular and CC-normal space, which is neither CC-Tychonoff, CCT_3 nor epi-regular. Note that: the odd even topology is C-paracompact space which is not CC-regular.

Note that: CC-regularity does not imply CC-complete regularity, CCT_3 does not imply CC-Tychonoff and CC-almost regularity does not imply CC-almost complete regularity. Here is a counterexample:

Example 6. *The Tychonoff corkscrew topology:* [23, Example 90], is a T_3, regular, semi regular and countably compact space, which is neither completely regular, normal, locally compact, Lindelöf, second countable nor paracompact [23]. Since X is a T_3-space, it is epi-regular, CCT_3 and CC-regular. Since X is countably compact space which is neither almost completely regular nor epi-completely regular [4], we conclude that: it is neither CC-completely regular, CC-Tychonoff nor CC-almost completely regular. Therefore, the Tychonoff corkscrew topology is a CC-regular, CCT_3 and CC-almost regular space, which is neither CC-completely regular, CC-Tychonoff nor CC-almost completely regular.

Observed that: any uncountable indiscrete space X is a CC-normal, CC-regular, CC-completely regular and CC-almost completely regular space which is neither epi-regular, CC-Tychonoff nor CCT_3. The following example is a CC-almost completely regular space, which is neither CCT_3, CC-normal nor CC-regular.

Example 7. *The particular point topology:* [23, Example 10], $(\mathbb{R}, \mathcal{T}_p)$ is a separable first countable space which is neither Hausdorff, paracompact, regular nor normal [23]. $(\mathbb{R}, \mathcal{T}_p)$ is neither a C-regular nor C-normal space [5, 7]. Then, it is neither CC-regular nor CC-normal. Since the particular point topology $(\mathbb{R}, \mathcal{T}_p)$ is an almost completely regular space, it is both CC-almost regular and CC-almost completely regular. Therefore, $(\mathbb{R}, \mathcal{T}_p)$ is a CC-almost regular and CC-almost completely regular space, which is neither CC-regular, CCT_3, CC-completely regular, CC-normal, CC-Tychonoff nor epi-regular.

In view of the fact that: if X is a T_1-space such that the only countably compact subsets of X are the finite subsets, then X is CC-normal [14]. Then, we conclude:

Theorem 4. *If X is a T_1-space such that the only countably compact subsets of X are the finite subsets, then X is CC-Tychonoff.*

Proof. Let X be a T_1-space. Let $X = Y$ and consider Y with the discrete topology. Then, the identity function $I_X : X \to Y$ is a bijective function. If M is any countably compact subspace of $(X, \mathcal{T})$, then by assumption M is a finite subspace of X and Y is with a discrete topology. Since any finite countably compact subspace of a T_1-space is discrete, the restriction function $(I_X)|_M : M \to I_X(M) = M$ is a homeomorphism because both the domain and the co-domain are discrete, and they have the same cardinality. Since Y is a Tychonoff space, we have: X is CC-Tychonoff.

Corollary 2. If X is a T_1-space such that the only countably compact subsets of X are the finite subsets, then X is CC-completely regular, CC-regular, CCT_3, CC-almost regular and CC-almost completely regular.

Theorem 5. *If X is a countably compact CC-completely regular (resp. CC-Tychonoff, CCT_3, CC-regular, CC-almost completely regular, CC-almost regular) space, then X is completely regular (resp. Tychonoff, T_3, regular, almost completely regular, almost regular).*

Proof. Let X be a countably compact CC-completely regular (resp. CC-Tychonoff, CCT_3, CC-regular, CC-almost completely regular, CC-almost regular) space. Then, there exist a completely regular (resp. Tychonoff, T_3, regular, almost completely regular, almost regular) space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subspace A of X. Since X is a countably compact space, put $A = X$. Since f is bijective, the function $f : X \to Y$ is a homeomorphism. Since $X \cong Y$, we get X is completely regular (resp. Tychonoff, T_3, regular, almost completely regular, almost regular).

Corollary 3. If X is a countably compact non-completely regular (resp. non-Tychonoff, non T_3, non-regular, non-almost completely regular, non-almost regular) space, then X cannot be CC-completely regular (resp. CC-Tychonoff, CCT_3, CC-regular, CC-almost completely regular, CC-almost regular).

Recall that: a space X is called *locally compact* if X is Hausdorff and for each $x \in X$ and each open neighborhood V of x there exists an open neighborhood U of x such that $x \in U \subseteq \overline{U} \subseteq V$ and $\overline{U}$ is compact [10]. In view of the fact that: every locally compact space is Tychonoff [10], we get the following corollary:

Corollary 4. Every locally compact space is CC-Tychonoff.

Recall that: a space X is said to be *mildly normal* [22], if any pair of disjoint closed domain subsets A and B of X can be separated. The converse of Corollary 4 is not true in general as shown by the next example:

Example 8. *The modified Dieudonné plank topology:* [14, Example 2.4, Example 3.3], is a Tychonoff, L-normal and CC-normal space, which is neither mildly normal nor locally compact [14]. Thus, the modified Dieudonné plank is a CC-Tychonoff, CCT_3, CC-completely regular and CC-regular space, which is neither locally compact nor mildly normal.

Note that: if X is a CC-almost completely regular (resp. CC-almost regular, CC-complete regularity, CC-regular, CCT_3, CC-Tychonoff) space and $f : X \to Y$ is a witness of the CC-almost complete regularity (resp. CC-almost regularity, CC-complete regularity, CC-regularity, CCT_3, CC-Tychonoffness) of X, then f is not necessary to be continuous. Here is a counterexample:

Example 9. Consider the countable complement topology on $\mathbb{R}$, $(\mathbb{R}, \mathcal{CC})$. The only countably compact subspaces are finite subspaces and $(\mathbb{R}, \mathcal{CC})$ is T_1-space. Hence, $(\mathbb{R}, \mathcal{CC})$ is CC-Tychonoff (hence CC-completely regular, CC-almost completely regular, CC-almost regular, CCT_3 and CC-regular). It is well known that the finite countably-compact subspaces in a T_1-space are discrete. If we let $\mathcal{D}$ be the discrete topology on $\mathbb{R}$, then the identity function from $(\mathbb{R}, \mathcal{CC})$ onto $(\mathbb{R}, \mathcal{D})$ is a witness of the CC-Tychonoffness (resp. CC-complete regularity, CC-almost complete regularity, CC-almost regularity, CCT_3, CC-regularity) of $(\mathbb{R}, \mathcal{CC})$, which is not continuous.

Recall that: a space X is called a *Fréchet* if for any subset B of X and any $x \in \overline{B}$, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of points of B such that $a_n \longrightarrow x$ [10]. Thus, we conclude:

Theorem 6. *If X is a CC-completely regular Fréchet space, then any function bears the CC-complete regularity of X is continuous.*

Proof. Similar to the proof of Theorem 2.9 in [14].

The proof of the next theorem is also similar to the proof of Theorem 2.9 in [14]..

Theorem 7. *If X is a CC-Tychonoff (resp. CC-regular, CCT_3, CC-almost regular, CC-almost completely regular) Fréchet space, then any function bears the CC-Tychonofness (resp. CC-regularity, CCT_3, CC-almost regularity, CC-almost complete regularity) of X is continuous.*

Since every first countable space is Fréchet [10], we get the next corollary:

Corollary 5. If X is a CC-almost regular first countable space and $f : X \to Y$ is a witness of the CC-almost regularity of X, then f is continuous.

Next, we introduce the following results:

Proposition 1. *If X is a T_1 CC-completely regular space, then the witness Y is Tychonoff.*

Proof. Let X be a T_1 CC-completely regular space. Since X is a CC-completely regular space, there exist a completely regular space Y and a bijective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ such that $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subset $A \subseteq X$. Suppose Y is not Tychonoff, then Y cannot be T_1 because it is completely regular. Then, there exist two distinct elements x and y in Y such that if U is any open neighborhood of x, then $y \in U$ or if V is any open neighborhood of y , then $x \in V$. Thus, the set $M = \{f^{-1}(\{x\}), f^{-1}(\{y\})\}$ is a T_1 countably compact subspace of X. Then, $f|_M : M \to f(M)$ is a homeomorphism. But $f(M) = \{x, y\}$ cannot be T_1, which is a contradiction. Hence, Y must be T_1 and thus Tychonoff.

Similarly, we can prove the next proposition:

Proposition 2. *If X is a T_1 CC-regular (resp. CC-normal) space, then the witness Y is T_3 (resp. T_4).*

Thus, we get the next corollary:

Corollary 6. Every T_1 CC-completely regular (resp. CC-regular, CC-normal) space is CC-Tychonoff (resp. CCT_3, CC-Tychonoff).

Theorem 8. *Every T_1 CC-completely regular Fréchet (resp. first countable) is epi-completely regular.*

Proof. Let X be a T_1 CC-completely regular Fréchet space (resp. first countable). Then, there exist a completely regular space Y and a bijective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ such that $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subset $A \subseteq X$. Since X is Fréchet (resp. first countable), we have f is continuous. Since X is T_1 CC-completely regular, by Proposition 1 we obtain Y is Tychonoff. Now, define a topology $\mathcal{T}^\star$ on X as follows: $\mathcal{T}^\star = \{f^{-1}(U) : U \in \mathcal{T}'\}$. Clearly, $\mathcal{T}^\star$ is a topology on X coarser than $\mathcal{T}$ such that $f : (X, \mathcal{T}^\star) \to (Y, \mathcal{T}')$ is continuous. If $W \in \mathcal{T}^\star$, then $W = f^{-1}(U)$ for some open set U in $\mathcal{T}'$. So, $f(W) = f(f^{-1}(U)) = U$, which is open set in $(Y, \mathcal{T}')$. Thus, $f : (X, \mathcal{T}^\star) \to (Y, \mathcal{T}')$ is open and hence a homoeomorphism. Therefore, $(X, \mathcal{T}^\star)$ is a Tychonoff space. Since $\mathcal{T}^\star \subseteq \mathcal{T}$, we get: $(X, \mathcal{T})$ is epi-completely regular.

Similarly, every T_1 CC-regular Fréchet (resp. first countable) is epi-regular, every CCT_3-Fréchet (resp. first countable) is epi-regular, every CC-Tychonoff Fréchet (resp. first countable) is epi-completely regular and every T_1 CC-normal Fréchet (resp. first countable) is epi-normal.

Corollary 7.

(1) Every CC-regular T_1-first countable space is Urysohn.

(2) Every CCT_3-first countable space is Urysohn.

By using Theorem 8 and Proposition 1, we can prove the next result as follows:

Theorem 9. *Every T_1 CC-regular Fréchet (first countable) Lindelöf space is epi-normal.*

Proof. Let X be a CC-regular T_1 Fréchet (resp. first countable) Lindelöf space. Then, there exist a regular space Y and a bijective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ such that $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subset $A \subseteq X$. Since X is Fréchet (resp. first countable), we get f is continuous. Since the continuous image of a Lindelöf space is Lindelöf [10], we obtain: Y is Lindelöf. Since Y is a regular Lindelöf space, we have $(Y, \mathcal{T}')$ is normal. By Proposition 2, $(Y, \mathcal{T}')$ is a T_3-space. Thus, $(Y, \mathcal{T}')$ is a Hausdorff normal space and hence a T_4-space. Define a topology $\mathcal{T}^\star$ on X as follows: $\mathcal{T}^\star = \{f^{-1}(U) : U \in \mathcal{T}'\}$. By using the same arguments to the proof of Theorem 8, we obtain: $f : (X, \mathcal{T}^\star) \to (Y, \mathcal{T}')$ is a homoeomorphism. Since $(Y, \mathcal{T}')$ is a T_4-space, we have: $(X, \mathcal{T}^\star)$ is T_4. Since $\mathcal{T}^\star \subseteq \mathcal{T}$, we get: $(X, \mathcal{T})$ is epi-normal.

Corollary 8.

(1) Every T_1 CC-completely regular Fréchet (first countable) Lindelöf space is epi-normal.

(2) Every CCT_3-Fréchet (first countable) Lindelöf space is epi-normal.

Theorem 10. *Every CC-regular Fréchet (resp. first countable) Lindelöf space is CC-normal.*

Proof. It is similar to that of Theorem 9.

Corollary 9. Every CC-completely regular (resp. CC-Tychonoff, CCT_3) first countable Lindelöf space is CC-normal.

It is obvious that every CC-completely regular (resp. CC-regular, CCT_3, CC-Tychonoff) countably compact Lindelöf space is CC-normal. The proof of the next results is similar to that of Theorem 3.5 in [14]:

Theorem 11. *If X is a CC-completely regular (resp. CC-regular, CC-Tychonoff, CCT_3) space, then the Alexandroff duplicate $A(X)$ of X is CC-completely regular (resp. CC-regular, CC-Tychonoff, CCT_3).*

3. Some properties and relationships

Now, we present the next results: the proof of the next theorem is similar to that of Theorem 2.7 in [14].

Theorem 12. *CC-Tychonoffness, CCT_3, CC-complete regularity, CC-regularity, CC-almost regularity and CC-almost complete regularity are topological properties.*

The proof of the following results is similar to the proof of Theorem 2.8 in [14]:

Theorem 13. *CC-Tychonoffness, CCT_3, CC-complete regularity, CC-regularity, CC-almost regularity and CC-almost complete regularity are additive properties.*

Theorem 14. *CC-complete regularity, CC-Tychonoffness, CCT_3 and CC-regularity are hereditary properties.*

Proof. Let X be a CC-completely regular (resp. CC-Tychonoff, CCT_3, CC-regular) space. Pick a completely regular (resp. Tychonoff, T_3, regular) space Y and a bijective function $f : X \to Y$ such that $f|_A : A \to f(A)$ is a homoeomorphism for each countably compact subspace $A \subseteq X$. Let M be any subspace of X and let $N = f(M) \subseteq Y$. Then, N is a completely regular (resp. Tychonoff, T_3, regular) space because it is a subspace of a completely regular (resp. Tychonoff, T_3, regular) space Y. Now, we have: $f|_M : M \to f(M)$ is a bijective function. Since any countably compact subspace K of M is countably compact subset in X and $(f|_M)|_K = f|_K$, we conclude that: M is CC-completely regular (resp. CC-Tychonoff, CCT_3, CC-regular).

Theorem 15. *If X is an L-Tychonoff space such that each countably compact subspace is contained in a Lindelöf subspace, then X is CC-Tychonoff.*

Proof. Let X be an L-Tychonoff space such that if A is a countably compact subspace of X, there exists a Lindelöf subspace B of X such that $A \subseteq B$. Let Y be a Tychonoff space and $f : X \to Y$ be a bijective function such that $f|_C : C \to f(C)$ is a homeomorphism for each Lindelöf subspace $C \subseteq X$. Now, let A be a countably compact subspace of X. Pick a Lindelöf subspace B of X such that $A \subseteq B$. Then, $f|_B : B \to f(B)$ is a homeomorphism. Thus, $f|_A : A \to f(A)$ is a homeomorphism as $(f|_B)|_A = f|_A$. Hence, X is CC-Tychonoff.

We can find some statements analogous to that of Theorem 15. Here are some of them:

Theorem 16.

(1) *If X is a C-Tychonoff (resp. C-completely regular, CT_3, C-regular, C-almost regular, C-almost completely regular) space such that each countably compact subspace is contained in a compact subspace, then X is CC-Tychonoff (resp. CC-completely regular, CCT_3, CC-regular, CC-almost regular, CC-almost completely regular).*

(2) *If X is a CC-Tychonoff (resp. CC-completely regular, CCT_3, CC-regular, CC-almost regular, CC-almost completely regular) space such that each Lindelöf subspace is contained in a countably compact subspace, then X is L-Tychonoff (resp. L-completely regular, LT_3, L-regular, L-almost regular, L-almost completely regular).*

(3) *If X is an L-completely regular (resp. L-completely regular, LT_3, L-regular, L-almost regular, L-almost completely regular) space such that each countably compact subspace is contained in a Lindelöf subspace, then X is CC-completely regular (resp. CC-completely regular, CCT_3, CC-regular, CC-almost regular, CC-almost completely regular).*

Theorem 17. *If X is a Hausdorff Fréchet (resp. first countable) space, then X is CC-almost completely regular.*

Proof. Let X be a Hausdorff Fréchet (resp. first countable) space. Then, there exists a topology $\mathcal{T}'$ coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is T_1-almost completely regular [4]. The identity function $I_X : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is a bijective continuous function. Let M be any countably compact subspace of $(X, \mathcal{T})$. Then, $(I_X)|_M : M \to I_X(M)$ is a bijective continuous function and $I_X(M) = M$ is a countably compact subset in both $(X, \mathcal{T})$ and $(X, \mathcal{T}')$. Let U be any open set in $(M, \mathcal{T}_M)$. Since M is a countably compact subset of $(X, \mathcal{T}')$, there exists an open set V in $(X, \mathcal{T}')$ such that $U = V \cap M$. Thus, $(I_X)|_M(U) = (I_X)|_M(V \cap M) = V \cap M = U$, which is an open set in $(I_X(M), \mathcal{T}'_M)$. Hence, $(I_X)|_M$ is open and hence a homeomorphism. Therefore, X is CC-almost completely regular.

Corollary 10. If X is a Hausdorff Fréchet (resp. first countable) space, then X is CC-almost completely regular.

Theorem 18. *Every Hausdorff almost completely regular space is CC-Tychonoff.*

Proof. Let $(X, \mathcal{T})$ be a Hausdorff almost completely regular space. Let $(X, \mathcal{T}_s)$ be the semi-regularization of $(X, \mathcal{T})$. Then, $(X, \mathcal{T}_s)$ is a Hausdorff completely regular space because a semi-regularization of a Hausdorff almost completely regular space is Hausdorff completely regular [16]. Hence, $(X, \mathcal{T}_s)$ is Tychonoff. Since $\mathcal{T}_s \subseteq \mathcal{T}$, we obtain that $(X, \mathcal{T})$ is epi-completely regular. By Theorem 1, we conclude that $(X, \mathcal{T})$ is CC-Tychonoff.

Similarly, we can prove the next result:

Theorem 19. *Every Hausdorff almost regular space is a CCT_3-space.*

Observed that: CC-normality does not imply CC-almost regularity. Here is a counterexample.

Example 10. *The excluded point topology:* [23, Example 15], $(X, \mathcal{E}_p)$ is a T_0, compact, paracompact, first countable and normal space, which is neither T_1, regular, separable nor semi regular [23]. $(X, \mathcal{E}_p)$ is not almost regular [24]. Hence, it is not almost completely regular. Since X is a countably compact space which is not almost regular, by Corollary 3, we obtain: X is neither CC-almost regular, CC-regular, CC-completely regular, CC-almost completely regular, CCT_3 nor CC-Tychonoff. Since X is a normal space, it is CC-normal. Since X is not T_1, we obtain: X is neither epi-regular nor epi-mildly normal. Therefore, the space $(X, \mathcal{E}_p)$ is a CC-normal space, which is neither CC-almost regular, CC-regular, CC-Tychonoff nor CCT_3.

Here is another example of a CC-Tychonoff space, which is not sub-metrizable.

Example 11. *The deleted Tychonoff plank:* [23, Example 87], is a Hausdorff and locally compact space [23]. By Corollary 4, the deleted Tychonoff plank is a CC-Tychonoff, CCT_3, CC-completely regular and CC-regular space. Hence, it is CC-almost completely regular and CC-almost regular. The deleted Tychonoff plank is also neither almost-normal nor sub-metrizable [5, 7].

Any CC-completely regular (resp. CC-regular, CCT_3, CC-Tychonoff) space is not necessarily locally compact nor CC-normal as shown by the next example:

Example 12. Consider the Example 10 in [18], let $G = D^{\omega_1}$, where $D = \{0, 1\}$ with the discrete topology. Let H be a subspace of G consisting of all points of G with at most countably many non zero coordinates. Put $X = G \times H$. Raushan Buzyakova proved that X cannot be mapped onto a normal space Y by a bijective continuous function [8]. Observe that: H is T_2-Fréchet and hence H is a k-space. The space G is also a T_2-compact space. Hence, $X = G \times H$ is a k-space [18]. Since X is Tychonoff, we get X is CC-Tychonoff. Hence, it is a CC-completely regular, CCT_3, CC-regular and CC-almost completely regular space, which is not C-normal [18]. Since X is not C-normal, we obtain X is neither CC-normal, sub-metrizable, C_2-paracompact nor epi-normal. Note that: every C_2-paracompact space is C-normal [19]. The space X is also not locally compact. Thus, the space X is a CCT_3, CC-regular, CC-completely regular and CC-Tychonoff space, which is neither CC-normal, C_2-paracompact, epi-normal, sub-metrizable nor locally compact.

Since every Hausdorff paracompact space is T_4, the proof of the next result is similar to that of Theorem 9:

Theorem 20. *Every C_2-paracompact first countable space is epi-normal (hence epi-completely regular).*

Thus, we obtain the next corollary:

Corollary 11. Every C_2-paracompact first countable space is CC-Tychonoff.

Note that: the space presented in Example 2.25 in [19], is a C-paracompact first countable space which is neither CC-regular nor L-almost regular because it is a Lindelöf space that is neither almost regular nor C-regular. The following problems are still open in this research.

Problems:

(1) Is there an example of a C-Tychonoff space which is not CC-almost regular?.

(2) Is there an example of a C_2-paracompact space which is not CC-regular?.

(3) Is there an example of an L-Tychonoff space which is not CC-regular?.

(4) Are CC-complete regularity, CC-Tychonoffness, CCT_3 and CC-regularity multiplicative properties?.

4. Conclusion

New topological properties, called CC-complete regularity, CC-almost complete regularity, CC-almost regularity, CCT_3, CC-Tychonoffness and CC-regularity have been studied in this work. Some results, properties, relationships and counterexamples have been given and discussed.

Acknowledgements

The authors would like to thank the anonymous referee for his/her comments that will help us improve this article.

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