Strongly 2-Nil Clean Rings with Units of Order Two

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Abstract. A ring $R$ is considered a strongly 2-nil clean ring, or (strongly 2-NC ring for short), if each element in $R$ can be expressed as the sum of a nilpotent and two idempotents that commute with each other. In this paper, further properties of strongly 2-NC rings are given. Furthermore, we introduce and explore a special type of strongly 2-NC ring where every unit is of order 2, which we refer to as a strongly 2-NC rings with $U(R) = 2$. It was proved that the Jacobson radical over a strongly 2-NC ring is a nil ideal, here, we demonstrated that the Jacobson radical over strongly 2-NC ring with $U(R) = 2$ is a nil ideal of characteristic 4. We compare this ring with other rings, since every SNC ring is strongly 2-NC, but not every unit of order 2, and if $R$ is a strongly 2-NC with $U(R) = 2$, then $R$ need not be SNC ring. In order to get $Nil(R) = 0$, we added one more condition involving this ring.

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1. Introduction

In [1] W.K. Nicholson defined a clean ring as having an $\Sigma = \Sigma^2$ and a unit $u$ with $a = \Sigma + u$. In [2], an element $a \in R$ is said to be strongly clean if $a = \Sigma + u$ with $u \in U(R)$, $\Sigma \in Id(R)$ and $u\Sigma = \Sigma u$. While the ring $R$ is strongly clean if every element of $R$ is strongly clean. Clearly, $\mathbb{Z}_9$ is a strongly clean ring.

A nil-clean ring is defined as a ring with each element is the sum of an idempotent and a nilpotent was first proposed by Diesl in [3], $R$ is considered a strongly nil clean (SNC for short) if the idempotent and nilpotent commute [4]. The structure of SNC rings and related topics was given for example in [5] and [6]. Clearly, $\mathbb{Z}_8$ is an SNC ring.

A strongly 2-NC ring was defined by Chen and Sheibani in [7] as a ring $R$ with each element is a sum of two idempotents and a nilpotent that commute with each other. Many authors have been working on these topics see for example [8] a ring $R$ is called strongly...
2-nil*-clean if every element in $R$ is the sum of two projections and a nilpotent that commute, [9] if every element in $R$ is the sum of an idempotent and two nilpotents, then $R$ is called 2-nil-clean and [10] a ring $R$ is defined to be 2-nil-good if every element in $R$ is the sum of two units and a nilpotent. The purpose of this paper is to present new properties of strongly 2-NC rings, and their connection with other related rings. We prove that if $R$ is a strongly 2-NC ring, with $n^2 + 2n = 0$ for every $n \in \text{Nil}(R)$. Then $R$ is of characteristic 48 with every unit is of order 4. Additionally, we introduce and investigate strongly 2-NC rings with $U(R) = 2$, providing their fundamental properties and their connection with tripotent rings and other related rings. Among other results we prove that: If $R$ is a strongly 2-NC ring with $2 \in U(R)$. Then $24 = 0$, and the Jacobson radical over a strongly 2-NC ring is a nil ideal of characteristic 4. In addition, we show that if $R$ is a strongly 2-NC ring with $U(R) = 2$ and $2 \in U(R)$, then $\text{Nil}(R) = 0$. In this paper, we define $R$ as an associative ring containing an identity element. Finally, it is worth mentioning that ring theory has several applications in many field, see for example [11], [12] and [13]. To represent the set of units, idempotents and nilpotents in $R$, we will use the symbols $U(R)$, $\text{Id}(R)$ and $\text{Nil}(R)$, respectively. Additionally, we will use $J(R)$ to denote the Jacobson radical and $\mathbb{Z}_n$ for the ring of integers modulo $n$.

Recall that:

**Definition 1.** [14]. A ring $R$ is considered to be $n$-good if each element is a sum of $n$ units.

**Definition 2.** [15]. If $t = t^3$ is referred to as a tripotent. $R$ is called a tripotent ring if every element of $R$ is tripotent.

Clearly, $\mathbb{Z}_6$ is a tripotent ring.

**Definition 3.** For any $a \in R$, we define $\text{Ann}(a) = \{b \in R : ab = ba = 0\}$.

**Theorem 1.** [7]. Let $R$ be a ring. Then the following are equivalent:

1. $R$ is strongly 2-NC.
2. For all $a \in R, a - a^3 \in \text{Nil}(R)$.
3. For all $a \in R, a^2$ is SNC element.

**Theorem 2.** [7]. A ring $R$ is strongly 2-NC if and only if

1. $J(R)$ is nil.
2. $R/J(R)$ is tripotent.

**Theorem 3.** [16] The following are equivalent for a ring $R$:

1. Every element of $R$ is a sum of a nilpotent and two tripotents that commute with one another.
2. $a^5 - a$ is nilpotent for all $a \in R$. 

2. Fundamental properties of strongly 2-NC rings

This section presents new properties of strongly 2-NC rings, and we provide a condition for strongly 2-NC rings to be tripotent rings.

Example 1. Consider the ring $\mathbb{Z}_{18}$.
Note that: $\text{Nil}(\mathbb{Z}_{18}) = \{0, 6, 12\}$, and $\text{Id}(\mathbb{Z}_{18}) = \{0, 1, 9, 10\}$. By direct calculation, we may find that $\mathbb{Z}_{18}$ is a strongly 2-NC.

Chen and Sheibani in [7] proved that:

Lemma 1. The following two issues are equivalent:

1. $R$ is a strongly 2-NC ring.
2. $a = \Sigma_1 - \Sigma_2 + n$, for each $a \in R$, and some $\Sigma_1, \Sigma_2 \in \text{Id}(R), n \in \text{Nil}(R)$, that commute.

Next, we shall record the following two lemmas, that will be used extensively throughout our current work.

Lemma 2. [17]. If $u \in U(R)$ and $n \in \text{Nil}(R)$, and if $un = nu$, then $1 + n$ and $u + n$ are units.

Lemma 3. Suppose that $\Sigma_1$ and $\Sigma_2$ are two commuting idempotents. Then:

1. $(\Sigma_1 - \Sigma_2)^2$ is an idempotent.
2. $(\Sigma_1 - \Sigma_2)^3$ is tripotent.
3. $(\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) - 1$ is a unit of order 2.
4. $2(\Sigma_1 - \Sigma_2)^2 - 1$ is a unit of order 2.

Proof.

1. $(\Sigma_1 - \Sigma_2)^4 = \Sigma_1^4 - 4\Sigma_1^3\Sigma_2 + 6\Sigma_1^2\Sigma_2^2 - 4\Sigma_1\Sigma_2^3 + \Sigma_2^4$

$$= \Sigma_1 - 4\Sigma_1\Sigma_2 + 6\Sigma_1\Sigma_2 - 4\Sigma_1\Sigma_2 + \Sigma_2 = (\Sigma_1 - \Sigma_2)^2.$$

2. $(\Sigma_1 - \Sigma_2)^3 = \Sigma_1^3 - 3\Sigma_1^2\Sigma_2 + 3\Sigma_1\Sigma_2^2 - \Sigma_2^3$

$$= \Sigma_1 - 3\Sigma_1\Sigma_2 + 3\Sigma_1\Sigma_2 - \Sigma_2 = (\Sigma_1 - \Sigma_2).$$

3. $((\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) - 1)((\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) - 1)$

$$= (\Sigma_1 - \Sigma_2)^4 + (\Sigma_1 - \Sigma_2)^3 - (\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2)^3 + (\Sigma_1 - \Sigma_2)^2$$

$$- (\Sigma_1 - \Sigma_2) - (\Sigma_1 - \Sigma_2)^2 - (\Sigma_1 - \Sigma_2) + 1$$

$$= (\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) - (\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) + (\Sigma_1 - \Sigma_2)^2$$

$$- (\Sigma_1 - \Sigma_2) - (\Sigma_1 - \Sigma_2)^2 - (\Sigma_1 - \Sigma_2) + 1 = 1.$$
Proposition 1. Let $R$ be a strongly 2-NC ring, then for any $a \in R$ we have:

1. $a^2$ is an SNC.
2. $a^2$ is the sum of a tripotent and a nilpotent that commute.
3. $a^2$ is the sum of an idempotent, a unit of order 2, and a nilpotent that commutes.

Proof.

1. Given $a \in R$, there existing some $\Sigma_1, \Sigma_2 \in Id(R)$ and $n \in Nil(R)$, that commute with one another, such that $a = \Sigma_1 - \Sigma_2 + n$. Thus, $a^2 = (\Sigma_1 - \Sigma_2)^2 + 2(\Sigma_1 - \Sigma_2)n + n^2$.

But $2(\Sigma_1 - \Sigma_2)n + n^2 = n_1 \in Nil(R)$, so $a^2 = (\Sigma_1 - \Sigma_2)^2 + n_1$. According to Lemma 3(1), $(\Sigma_1 - \Sigma_2)^2$ is an idempotent. Yielding $a^2$ is an SNC element.

2. Follows from Lemma 3(2).

3. By (1) $a^2$ is a SNC element, then $a^2 = \Sigma + n$ where $\Sigma \in Id(R), n \in Nil(R)$ that commute, we may write $a^2 = (1 - \Sigma) + (2\Sigma - 1) + n$. Clearly, $(1 - \Sigma)^2 = 1 - \Sigma, (2\Sigma - 1)^2 = 1$. Thus, $a^2$ is the sum of an idempotent, a unit of order 2, and a nilpotent.

Proposition 2. Suppose $R$ is a ring, and let $a \in R$. Then:

1. If $a^2$ is a strongly 2-NC, then $a$ and $-a$ are strongly clean.
2. If $a^2$ is a strongly 2-NC, then $a$ is the sum of two tripotents and a nilpotent commute one another.

Proof.

1. Take $a^2 = \Sigma_1 - \Sigma_2 + n$, by Proposition 1(1), $a^4$ is a SNC element, so $a^4 = \Sigma + n$ where $\Sigma \in Id(R), n \in Nil(R)$ that commute. Write $a^4 = (1 - \Sigma) + (2\Sigma - 1) + n$. But $(2\Sigma - 1)^2 = 1$, then $(2\Sigma - 1) + n = u_1 \in U(R)$. So $a^4 = (1 - \Sigma) + u_1$, implies $a^4 - (1 - \Sigma) = u_1$, but $(1 - \Sigma)^4 = 1 - \Sigma$, yields $(a^2 - (1 - \Sigma))(a^2 + (1 - \Sigma)) = u_1$, and hence, $(a - (1 - \Sigma))(a + (1 - \Sigma))(a^2 + (1 - \Sigma)) = u_1$. Thus, $a - (1 - \Sigma) \in U(R)$ and $-a - (1 - \Sigma) \in U(R)$.

2. Let $a$ in $R$. Applying Theorem 1, $(a^2)^3 - a^2 \in Nil(R)$. Hence $a(a^5 - a) \in Nil(R)$, so $(a^4 - 1)a(a^5 - a) = (a^3 - a)^2 \in Nil(R)$. Using Theorem 3, $a$ is a sum of two tripotents and a nilpotent that commute.
**Proposition 3.** Suppose $R$ is a strongly 2-NC ring, and $a = \Sigma_1 - \Sigma_2 + n$ for any $a \in R$. Then:

1. $\text{Ann}(a) \cap (\Sigma_1 - \Sigma_2)R = 0$.
2. If $2 \in U(R)$, then $a$ is a 3-good element.
3. If $a \in U(R)$, then $(\Sigma_1 - \Sigma_2)^2 = 1$.
4. If $a$ is a non-zero divisor, then $a \in U(R)$.

**Proof.**

1. Let $c \in \text{Ann}(a) \cap (\Sigma_1 - \Sigma_2)R$. Then $ac = ca = 0$ and $c = (\Sigma_1 - \Sigma_2)r$, for some $r \in R$. Hence $a(\Sigma_1 - \Sigma_2)r = 0$, so $(\Sigma_1 - \Sigma_2 + n)(\Sigma_1 - \Sigma_2)r = 0$, $(\Sigma_1 - \Sigma_2)^2 + n(\Sigma_1 - \Sigma_2) = 0$. Applying Lemma 3, we get $((\Sigma_1 - \Sigma_2)^2 + n(\Sigma_1 - \Sigma_2)^3)r = 0$. So $1 + n(\Sigma_1 - \Sigma_2) \in U(R)$, say $u$, then we have $(\Sigma_1 - \Sigma_2)^2ur = 0$, so $(\Sigma_1 - \Sigma_2)^2r = 0$. Multiply by $(\Sigma_1 - \Sigma_2)$, we have $(\Sigma_1 - \Sigma_2)r = c = 0$. Therefore, $\text{Ann}(a) \cap (\Sigma_1 - \Sigma_2)R = 0$.

2. We may write $a = \Sigma_1 + 1 + \Sigma_2 + 1 + n - 2$. Consider $(\Sigma_1 + 1)(2 - \Sigma_1) = 2\Sigma_1 - \Sigma_1 + 2 - \Sigma_1 = 2$. Since $2 \in U(R)$, then $\Sigma_1 + 1 = u_1 \in U(R)$. Similarly $\Sigma_2 + 1 = u_2 \in U(R)$. Furthermore, $n - 2 \in U(R)$, say $u_3$. Thus, $a = u_1 + u_2 + u_3$.

3. Let $a = (\Sigma_1 - \Sigma_2) + n$, and let $a \in U(R)$. Then $a - n = (\Sigma_1 - \Sigma_2) \in U(R)$. Applying Lemma 3(2), then $(\Sigma_1 - \Sigma_2) = (\Sigma_1 - \Sigma_2)^3$. Thus $(\Sigma_1 - \Sigma_2)^2 = 1$.

4. Let $a$ be a non-zero divisor element. Applying Theorem 1, $a^3 - a \in \text{Nil}(R)$, this gives $a(a^2 - 1) \in \text{Nil}(R)$, thus, $a^r(a^2 - 1)^r = 0$, for some positive integer $r$. Since $a^r$ is a non-zero divisor, then $(a^2 - 1)^r = 0$, so $a^2 - 1 = n_1 \in \text{Nil}(R)$, implies $a^2 = 1 + n_1 \in U(R)$, then $a \in U(R)$.

It was proved in [18], that.

**Proposition 4.** [18, Proposition 1]. Assume $R$ is a nil clean ring with every nilpotent is the difference between two commuting idempotents, then $R$ is a Boolean ring.

We here extend this result as follows:

**Theorem 4.** Suppose $R$ is a strongly 2-NC ring, with any nilpotent is the difference between two commuting idempotents. Then $R$ is a tripotent ring.

**Proof.** Let $a$ in $R$, then $a = \Sigma_1 - \Sigma_2 + n$ for some existing $\Sigma_1, \Sigma_2 \in \text{Id}(R), n \in \text{Nil}(R)$, that commute which each other. Then $n = \Sigma_3 - \Sigma_4$ for some $\Sigma_3, \Sigma_4 \in \text{Id}(R)$ and $\Sigma_3 \Sigma_4 = \Sigma_4 \Sigma_3$. So $n + \Sigma_4 = \Sigma_3$, this implies $(n + \Sigma_4)^2 = (n + \Sigma_4)$, then $n^2 + 2n\Sigma_4 + \Sigma_4^2 = n + \Sigma_4$, this gives $n^2 + 2n\Sigma_4 - n = 0$, so $n^2 + n(2\Sigma_4 - 1) = 0$, but $(2\Sigma_4 - 1)^2 = 1$, then we have $n = -n^2(2\Sigma_4 - 1)^{-1}$. As $n$ is nilpotent, then $n = 0$. Thus, $a = \Sigma_1 - \Sigma_2$. Applying Lemma 3(2), $(\Sigma_1 - \Sigma_2)^3 = \Sigma_1 - \Sigma_2$. Hence, $a = a^3$ therefore, $R$ is a tripotent ring.
3. Strongly 2-NC rings with units of order two

In this section, we introduce and investigate a strongly 2-NC rings with every unit is of order 2, we refer to this type of ring as strongly 2-NC rings with $U(R) = 2$.

**Definition 4.** A ring $R$ is called strongly 2-NC with $U(R) = 2$ if for every $a \in R$, existing two idempotents $\Sigma_1, \Sigma_2$ and a nilpotent $n$, that commute and every unit is of order 2, such that $a = \Sigma_1 + \Sigma_2 + n$.

**Example 2.** The rings $Z_4, Z_6, Z_8, Z_{12}, Z_{24}$ are all strongly 2-NC with $U(R) = 2$, while the ring $Z_9$ is not strongly 2-NC with $U(R) = 2$.

We start this section with some fundamental properties of a strongly 2-NC ring with $U(R) = 2$.

**Proposition 5.** Homomorphic images of strongly 2-NC ring with $U(R) = 2$ is again strongly 2-NC ring with every unit is of order 2.

**Proof.** Let $f : R \rightarrow R'$ be a homomorphism from a strongly 2-NC ring $R$ with $U(R) = 2$ onto $R'$. Then for any $b \in R'$, there exists $a \in R$, such that $b = f(a), a = \Sigma_1 + \Sigma_2 + n$ and $U(R) = 2$, where $\Sigma_1, \Sigma_2 \in Id(R), n \in Nil(R)$ that commute of with one another. Now, $b = f(a) = f(\Sigma_1 + \Sigma_2 + n) = f(\Sigma_1) + f(\Sigma_2) + f(n)$. Clearly, $f(\Sigma_1), f(\Sigma_2) \in Id(R')$ and $f(n) \in Nil(R')$. On the other hand for any $u \in (R)$, where $u$ is a unit, $(f(u))^2 = f(u^2) = f(1)$, this shows that $f(u)$ is a unit of order 2. Therefore $R'$ is a strongly 2-NC ring with $U(R') = 2$.

**Proposition 6.** If $R$ is a strongly 2-NC ring with $U(R) = 2$. Then $24 = 0$.

**Proof.** Assume that $a$ in $R$, then existing two idempotents $\Sigma_1, \Sigma_2$ and a nilpotent $n$ that commute with one another, such that $a = \Sigma_1 - \Sigma_2 + n$. By Theorem 1, $a^3 - a \in Nil(R)$, this gives $2^3 - 2 = 6 \in Nil(R)$. Since every unit is of order 2, and since 6 is nilpotent, then $6 - 1 = 5 \in U(R)$. This gives $5^2 = 1$, so $24 = 0$.

**Example 3.** Consider the ring $Z_{24}$. Clearly, $Z_{24}$ is a strongly 2-NC, with $U(Z_{24}) = \{1, 5, 7, 11, 13, 17, 19, 23\}$. Observe that $1^2 = 5^2 = 7^2 = 11^2 = 13^2 = 17^2 = 19^2 = 23^2 = 1$.

Observe that every SNC ring is strongly 2-NC, but not every unit of order 2.

**Example 4.** The ring $Z_{16}$ is an SNC which is strongly 2-NC, but $Z_{16}$ is not strongly 2-NC ring with $U(R) = 2$, since the units $3, 5, 11, 13$ are not order 2.

Note that: If $R$ is a strongly 2-NC with $U(R) = 2$, then $R$ need not to be SNC ring.

**Example 5.** In the ring $Z_{12}$. Then $U(Z_{12}) = \{1, 5, 7, 11\}$ and $1^2 = 5^2 = 7^2 = 11^2 = 1$. Clearly, $Z_{12}$ is a strongly 2-NC with $U(Z_{12}) = 2$, but $(Z_{12})$ is not SNC ring. Since 2 is not SNC element.
Proposition 7. If a ring $R$ is a strongly 2-NC ring with $U(R) = 2$, for which $3 \in U(R)$, then $R$ is SNC ring of characteristic 8.

Proof. Assume $R$ is a strongly 2-NC ring with $U(R) = 2$. Then By Proposition 1(1), $a^2$ is a SNC element for every $a \in R$. Applying Proposition 2(1), $a$ is strongly clean. Then $a$ may be written $a = 1 + u$, where $u \in \text{Id}(R)$ and $u^2 = 1$. Then $a = 1 + u + 1$. Since $3 \in U(R)$, then $3^2 = 1$, gives $8 = 0$ thus, $2 \in \text{Nil}(R)$. So $(u + 1)^2 = u^2 + 2u + 1 = 2(u + 1) \in \text{Nil}(R)$. Thus, $u + 1 \in \text{Nil}(R)$. Therefore $R$ is an SNC ring.

Example 6. Consider the ring $Z_8$. Then $U(Z_8) = \{1, 3, 5, 7\}$. So $1^2 = 3^2 = 5^2 = 7^2 = 1$. Clearly, $Z_8$ is a strongly 2-NC with $U(Z_8) = 2$. Observe that $3 \in U(Z_8)$. Then $Z_8$ is an SNC ring.

It was proved in Theorem 2, if a ring $R$ is a strongly 2-NC, then $J(R)$ is nil. In the next result, we consider $J(R)$ over a strongly 2-NC ring with $U(R) = 2$.

Theorem 5. If $R$ is a strongly 2-NC ring with $U(R) = 2$, then $J(R)$ is nil of characteristic 4.

Proof. Given $a \in J(R)$, then $a = \Sigma_1 - \Sigma_2 + n$, where $\Sigma_1, \Sigma_2 \in \text{Id}(R)$ and $n \in \text{Nil}(R)$, that commute with one another. Write $a = 1 - (\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2)^2 - 1 + n$. According to Lemma 3(3), $(\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) - 1 = u_1$ is a unit of order 2, then $a = 1 - (\Sigma_1 - \Sigma_2)^2 + u_1 + n$ implies $a = 1 - (\Sigma_1 - \Sigma_2)^2 + u_2$, where $u_2 = u_1 + n$. Since $a \in J(R)$, so $a - u_2 \in U(R)$, applying to Proposition 3(3), we conclude that $1 - (\Sigma_1 - \Sigma_2)^2 = 1$, gives $(\Sigma_1 - \Sigma_2)^2 = 0$, whence it follows that $a = 1 + u_2$, with $u_2^2 = 1$. Now consider $a^2 = (1 + u_2)^2 = 2(1 + u_2) = 2a$, and $a^3 = 2^2(1 + u_2) = 4a$. Choose $a = 2b$, then $(2b)^3 = 4(2b)$, so $8b^3 = 8b$, implies $8b(1 - b^2) = 0$, but $b \in J(R)$, gives $1 - b^2 \in U(R)$, gives $8b = 0$. Thus $4a = a^3 = 0$.

Example 7. Consider the ring $Z_{24}$. Then $Z_{24}$ is a strongly 2-NC with $U(Z_{24}) = 2$. Now $J(Z_{24}) = \{0, 6, 12, 18\}$. So $J(Z_{24})$ is a nil ideal of characteristic 4.

Corollary 1. If $R$ is a strongly 2-NC ring with $U(R) = 2$ and if $2 \in U(R)$, then $J(R) = 0$.

Proof. Let $a \in J(R)$, then by Theorem 5, $4a = 0$, since $2 \in U(R)$, then $a = 0$.

Proposition 8. If $R$ is a strongly 2-NC ring with $U(R) = 2$, and if $2 \in U(R)$, then $\text{Nil}(R) = 0$.

Proof. Given $a \in R$, then $a = 1 - \Sigma_1 - \Sigma_2 + n$, so $a = \Sigma_1 - \Sigma_2 + n + 1$, but $n + 1 \in U(R)$, say $u$, then $a = \Sigma_1 - \Sigma_2 + u$. Let $n \in \text{Nil}(R)$, then $n = \Sigma_1 - \Sigma_2 + u$, implies $\Sigma_1 - \Sigma_2 = n - u$, since $n - u \in U(R)$, according to Proposition 3(3), $(\Sigma_1 - \Sigma_2)^2 = 1$. Furthermore, $n^2 = (\Sigma_1 - \Sigma_2)^2 + 2(\Sigma_1 - \Sigma_2)u + u^2 = 1 + 2(\Sigma_1 - \Sigma_2)u + 1 = 2(1 + (\Sigma_1 - \Sigma_2)u)$. Observe that $nu = (\Sigma_1 - \Sigma_2)u + 1$. Thus, $n^2 = 2nu$, so $n(n - 2u) = 0$. since $2 \in U(R)$, by assumption then $n - 2u \in U(R)$. Whence it follows that $n = 0$.

Next, we shall explore the relationship between strongly 2-NC ring with $U(R) = 2$ and a tripotent ring.
Theorem 6. A ring $R$ with $2 \in U(R)$ is strongly 2-NC with $U(R) = 2$ if and only if $R$ is a tripotent.

Proof. Let $R$ be a strongly 2-NC ring with $U(R) = 2$, and let $a \in R$, then $a = \Sigma_1 - \Sigma_2 + n$, where $\Sigma_1, \Sigma_2 \in Id(R), n \in \text{Nil}(R)$, that commute with one another. According to Proposition 8, $n = 0$. Thus, $a = \Sigma_1 - \Sigma_2 = (\Sigma_1 - \Sigma_2)^3 = a^3$.

Conversely, assume that $R$ is a tripotent ring, and $t = t^3 \in R$, since $2 \in U(R)$, then $t$ may be written as $t = \frac{t^2 + t}{2} = \frac{t^2 - t}{2}$. Note that:

$$\left(\frac{t^2 + t}{2}\right)^2 = \frac{t^2 + 2t^2}{4} = \frac{t^2 + t}{2},$$

and

$$\left(\frac{t^2 - t}{2}\right)^2 = \frac{t^2 - 2t + t^2}{4} = \frac{t^2 - t}{2},$$

so $(\frac{t^2 + t}{2}), (\frac{t^2 - t}{2}) \in Id(R)$. Observe that for any unit $u, u^3 = u$ thus, $u^2 = 1$. Therefore, $R$ is a strongly 2-NC ring with $U(R) = 2$.

To end this section, we consider a strongly 2-NC ring, with every unit is of order 4.

Proposition 9. Suppose $R$ is a strongly 2-NC ring, and if $n^2 + 2n = 0$ for every nilpotent $n$. Then every unit of $R$ is of order 4, and $48 = 0$.

Proof. Given $a \in R$, then by Proposition 1(1), $a^2$ is an SNC element. Write $a^2 = \Sigma + n$, where $\Sigma \in Id(R), n \in \text{Nil}(R)$ and $\Sigma n = n\Sigma$. Let $u \in U(R)$, then $u^2 = \Sigma + n$, implies $\Sigma = u^2 - n = v \in U(R)$. Thus, $\Sigma = 1$. Hence $u^2 = 1 + n$, implies $u^4 = (1 + n)^2 = 1 + 2n + n^2$.

By assumption $n^2 + 2n = 0$, then $u^4 = 1$. On the other hand $6 \in \text{Nil}(R)$ Theorem 1. Thus, $6^2 + 2(6) = 0$, gives $48 = 0$.

Example 8. In the ring $Z_{48}$. Then

$U(Z_{48}) = \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47\}$,

$\text{Nil}(Z_{48}) = \{0, 6, 12, 18, 24, 30, 36, 42\}$,

$Id(Z_{48}) = \{0, 1, 16, 33\}$.

By direct calculation, one easily check that $Z_{48}$ is a strongly 2-NC ring, with every unit is of order 4.

4. Conclusion

In this article, new properties of a strongly 2-NC rings are given. Additionally, we added certain conditions for strongly 2-NC ring with each unit must be present of order four. We also introduce and investigated a strongly 2-NC ring with every unit of order two. We discuss some of the fundamental properties and present several examples. It was proved that the Jacobson radical over a strongly 2-NC ring is a nil ideal, here, we demonstrated that the Jacobson radical over strongly 2-NC ring with $U(R) = 2$ is a nil ideal of characteristic 4. In order to get $\text{Nil}(R) = 0$, we added one more condition involving this ring. The relationships between these rings, tripotent rings, and other related rings are given. Future goals include obtaining a deeper outcome on issues raised in this article, such as

1. The SNC ring with $U(R) = 2, 3$ or 4.
2. The strongly 2-NC ring with $U(R) = 3$ or 4.

3. The divisor graph of a strongly 2-NC ring with $U(R) = 2$.

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