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Strongly 2-Nil Clean Rings with Units of Order Two

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Abstract. A ring R is considered a strongly 2-nil clean ring, or (strongly 2-NC ring for short), if each element in R can be expressed as the sum of a nilpotent and two idempotents that commute with each other. In this paper, further properties of strongly 2-NC rings are given. Furthermore, we introduce and explore a special type of strongly 2-NC ring where every unit is of order 2, which we refer to as a strongly 2-NC rings with $U(R) = 2$. It was proved that the Jacobson radical over a strongly 2-NC ring is a nil ideal, here, we demonstrated that the Jacobson radical over strongly 2-NC ring with $U(R) = 2$ is a nil ideal of characteristic 4. We compare this ring with other rings, since every SNC ring is strongly 2-NC, but not every unit of order 2, and if R is a strongly 2-NC with $U(R) = 2$, then R need not be SNC ring. In order to get $Nil(R) = 0$, we added one more condition involving this ring.

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Key Words and Phrases: Clean, Nil clean, Strongly 2-nil clean, Tripotent

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1. Introduction

In [1] W.K. Nicholson defined a clean ring as having an $\Sigma = \Sigma^2$ and a unit u with $a = \Sigma + u$. In [2], an element $a \in R$ is said to be strongly clean if $a = \Sigma + u$ with $u \in U(R), \Sigma \in Id(R)$ and $u\Sigma = \Sigma u$. While the ring R is strongly clean if every element of R is strongly clean. Clearly, Z_9 is a strongly clean ring.

A nil-clean ring is defined as a ring with each element is the sum of an idempotent and a nilpotent was first proposed by Diesl in [3], R is considered a strongly nil clean (SNC for short) if the idempotent and nilpotent commute [4]. The structure of SNC rings and related topics was given for example in [5] and [6]. Clearly, Z_8 is an SNC ring.

A strongly 2-NC ring was defined by Chen and Sheibani in [7] as a ring R with each element is a sum of two idempotents and a nilpotent that commute with each other. Many authors have been working on these topics see for example [8] a ring R is called strongly

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2-nil-*-clean if every element in R is the sum of two projections and a nilpotent that commute, [9] if every element in R is the sum of an idempotent and two nilpotents , then R is called 2-nil-clean and [10] a ring R is defined to be 2-nil-good if every element in R is the sum of two units and a nilpotent. The purpose of this paper is to present new properties of strongly 2-NC rings, and their connection with other related rings. We prove that if R is a strongly 2-NC ring, with $n^2 + 2n = 0$ for every $n \in Nil(R)$. Then R is of characteristic 48 with every unit is of order 4. Additionally, we introduce and investigate a strongly 2-NC rings with $U(R) = 2$, providing their fundamental properties and their connection with tripotent rings and other related rings. Among other results we prove that: If R is a strongly 2-NC ring with $2 \in U(R)$. Then $24 = 0$, and the Jacobson radical over a strongly 2-NC ring is a nil ideal of characteristic 4. In addition, we show that if R is a strongly 2-NC ring with $U(R) = 2$ and $2 \in U(R)$, then $Nil(R) = 0$. In this paper, we define R as an associative ring containing an identity element. Finally, it is worth mentioning that ring theory has several applications in many field, see for example [11], [12] and [13]. To represent the set of units, idempotents and nilpotents in R, we will use the symbols $U(R), Id(R)$ and $Nil(R)$, respectively. Additionally, we will use $J(R)$ to denote the Jacobson radical and Z_n for the ring of integers modulo n.

Recall that:

Definition 1. [14]. A ring R is considered to be n-good if each element is a sum of n units.

Definition 2. [15]. If $t = t^3$ is referred to as a tripotent. R is called a tripotent ring if every element of R is tripotent.
Clearly, Z_6 is a tripotent ring.

Definition 3. For any $a \in R$, we define $Ann(a) = \{b \in R : ab = ba = 0\}$.

Theorem 1. *[7]. Let R be a ring. Then the following are equivalent:*

1. *R is strongly 2-NC.*
2. *For all $a \in R, a - a^3 \in Nil(R)$.*
3. *For all $a \in R, a^2$ is SNC element.*

Theorem 2. *[7]. A ring R is strongly 2-NC if and only if*

1. *$J(R)$ is nil.*
2. *$R/J(R)$ is tripotent.*

Theorem 3. *[16] The following are equivalent for a ring R:*

1. *Every element of R is a sum of a nilpotent and two tripotents that commute with one another.*
2. *$a^5 - a$ is nilpotent for all $a \in R$.*

2. Fundamental properties of strongly 2-NC rings

This section presents new properties of strongly 2-NC rings, and we provide a condition for strongly 2-NC rings to be tripotent rings.

Example 1. Consider the ring Z_{18}.
Note that: $Nil(Z_{18}) = \{0, 6, 12\}$, and $Id(Z_{18}) = \{0, 1, 9, 10\}$. By direct calculation, we may find that Z_{18} is a strongly 2-NC.

Chen and Sheibani in [7] proved that:

Lemma 1. *The following two issues are equivalent:*

1. *R is a strongly 2-NC ring.*
2. *$a = \Sigma_1 - \Sigma_2 + n$, for each $a \in R$, and some $\Sigma_1, \Sigma_2 \in Id(R), n \in Nil(R)$, that commute.*

Next, we shall record the following two lemmas, that will be used extensively throughout our current work.

Lemma 2. *[17]. If $u \in U(R)$ and $n \in Nil(R)$, and if $un = nu$, then $1 + n$ and $u + n$ are units.*

Lemma 3. *Suppose that Σ_1 and Σ_2 are two commuting idempotents. Then:*

1. *$(\Sigma_1 - \Sigma_2)^2$ is an idempotent.*
2. *$(\Sigma_1 - \Sigma_2)^3$ is tripotent.*
3. *$(\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) - 1$ is a unit of order 2.*
4. *$2(\Sigma_1 - \Sigma_2)^2 - 1$ is a unit of order 2.*

Proof.

1. $(\Sigma_1 - \Sigma_2)^4 = \Sigma_1^4 - 4\Sigma_1^3\Sigma_2 + 6\Sigma_1^2\Sigma_2^2 - 4\Sigma_1\Sigma_2^3 + \Sigma_2^4$
$$= \Sigma_1 - 4\Sigma_1\Sigma_2 + 6\Sigma_1\Sigma_2 - 4\Sigma_1\Sigma_2 + \Sigma_2 = (\Sigma_1 - \Sigma_2)^2.$$
2. $(\Sigma_1 - \Sigma_2)^3 = \Sigma_1^3 - 3\Sigma_1^2\Sigma_2 + 3\Sigma_1\Sigma_2^2 - \Sigma_2^3$
$$= \Sigma_1 - 3\Sigma_1\Sigma_2 + 3\Sigma_1\Sigma_2 - \Sigma_2 = (\Sigma_1 - \Sigma_2).$$
3. $((\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) - 1)((\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) - 1)$
$= (\Sigma_1 - \Sigma_2)^4 + (\Sigma_1 - \Sigma_2)^3 - (\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2)^3 + (\Sigma_1 - \Sigma_2)^2$
$- (\Sigma_1 - \Sigma_2) - (\Sigma_1 - \Sigma_2)^2 - (\Sigma_1 - \Sigma_2) + 1$
$= (\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) - (\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) + (\Sigma_1 - \Sigma_2)^2$
$- (\Sigma_1 - \Sigma_2) - (\Sigma_1 - \Sigma_2)^2 - (\Sigma_1 - \Sigma_2) + 1 = 1.$

4. $(2(\Sigma_1 - \Sigma_2)^2 - 1)(2(\Sigma_1 - \Sigma_2)^2 - 1)$
$= 4(\Sigma_1 - \Sigma_2)^4 - 2(\Sigma_1 - \Sigma_2)^2 - 2(\Sigma_1 - \Sigma_2)^2 + 1$
$= 4(\Sigma_1 - \Sigma_2)^2 - 2(\Sigma_1 - \Sigma_2)^2 - 2(\Sigma_1 - \Sigma_2)^2 + 1 = 1.$

Next, we shall give the following results.

Proposition 1. Let R be a strongly 2-NC ring, then for any $a \in R$ we have:

1. a^2 is an SNC.
2. a^2 is the sum of a tripotent and a nilpotent that commute.
3. a^2 is the sum of an idempotent, a unit of order 2, and a nilpotent that commutes.

Proof.

1. Given $a \in R$, there existing some $\Sigma_1, \Sigma_2 \in Id(R)$ and $n \in Nil(R)$, that commute with one another, such that $a = \Sigma_1 - \Sigma_2 + n$. Thus, $a^2 = (\Sigma_1 - \Sigma_2)^2 + 2(\Sigma_1 - \Sigma_2)n + n^2$. But $2(\Sigma_1 - \Sigma_2)n + n^2 = n_1 \in Nil(R)$, so $a^2 = (\Sigma_1 - \Sigma_2)^2 + n_1$. According to Lemma 3(1) $(\Sigma_1 - \Sigma_2)^2$ is an idempotent. Yielding a^2 is an SNC element.
2. Follows from Lemma 3(2).
3. By (1) a^2 is a SNC element, then $a^2 = \Sigma + n$ where $\Sigma \in Id(R), n \in Nil(R)$ that commute, we may write $a^2 = (1 - \Sigma) + (2\Sigma - 1) + n$. Clearly, $(1 - \Sigma)^2 = 1 - \Sigma, (2\Sigma - 1)^2 = 1$. Thus, a^2 is the sum of an idempotent, a unit of order 2, and a nilpotent.

Proposition 2. Suppose R is a ring, and let $a \in R$. Then:

1. If a^2 is a strongly 2-NC, then a and $-a$ are strongly clean.
2. If a^2 is a strongly 2-NC, then a is the sum of two tripotents and a nilpotent commute one another.

Proof.

1. Take $a^2 = \Sigma_1 - \Sigma_2 + n$, by Proposition 1(1), a^4 is a SNC element, so $a^4 = \Sigma + n$ where $\Sigma \in Id(R), n \in Nil(R)$ that commute. Write $a^4 = (1 - \Sigma) + (2\Sigma - 1) + n$. But $(2\Sigma - 1)^2 = 1$, then $(2\Sigma - 1) + n = u_1 \in U(R)$. So $a^4 = (1 - \Sigma) + u_1$, implies $a^4 - (1 - \Sigma) = u_1$, but $(1 - \Sigma)^4 = 1 - \Sigma$, yields $(a^2 - (1 - \Sigma))(a^2 + (1 - \Sigma)) = u_1$. and hence, $(a - (1 - \Sigma))(a + (1 - \Sigma))(a^2 + (1 - \Sigma)) = u_1$. Thus, $a - (1 - \Sigma) \in U(R)$ and $-a - (1 - \Sigma) \in U(R)$.
2. Let a in R. Applying Theorem 1, $(a^2)^3 - a^2 \in Nil(R)$. Hence $a(a^5 - a) \in Nil(R)$, so $(a^4 - 1)a(a^5 - a) = (a^5 - a)^2 \in Nil(R)$. Using Theorem 3, a is a sum of two tripotents and a nilpotent that commute.

Proposition 3. Suppose R is a strongly 2-NC ring, and $a = \Sigma_1 - \Sigma_2 + n$ for any $a \in R$. Then:

1. $Ann(a) \cap (\Sigma_1 - \Sigma_2)R = 0$.
2. If $2 \in U(R)$, then a is 3-good element.
3. If $a \in U(R)$, then $(\Sigma_1 - \Sigma_2)^2 = 1$.
4. If a is a non-zero divisor, then $a \in U(R)$.

Proof.

1. Let $c \in Ann(a) \cap (\Sigma_1 - \Sigma_2)R$. Then $ac = ca = 0$ and $c = (\Sigma_1 - \Sigma_2)r$, for some $r \in R$. Hence $a(\Sigma_1 - \Sigma_2)r = 0$, so $(\Sigma_1 - \Sigma_2 + n)(\Sigma_1 - \Sigma_2)r = 0$, $(\Sigma_1 - \Sigma_2)^2 + n(\Sigma_1 - \Sigma_2) = 0$. Applying Lemma 3, we get $((\Sigma_1 - \Sigma_2)^2 + n(\Sigma_1 - \Sigma_2)^3)r = 0, (\Sigma_1 - \Sigma_2)^2(1 + n(\Sigma_1 - \Sigma_2))r = 0$. But $1 + n(\Sigma_1 - \Sigma_2) \in U(R)$, say u, then we have $(\Sigma_1 - \Sigma_2)^2 ur = 0$, so $(\Sigma_1 - \Sigma_2)^2 r = 0$. Multiply by $(\Sigma_1 - \Sigma_2)$, we have $(\Sigma_1 - \Sigma_2)r = c = 0$. Therefore, $Ann(a) \cap (\Sigma_1 - \Sigma_2)R = 0$.
2. We may write $a = \Sigma_1 + 1 + \Sigma_2 + 1 + n - 2$. Consider $(\Sigma_1 + 1)(2 - \Sigma_1) = 2\Sigma_1 - \Sigma_1 + 2 - \Sigma_1 = 2$. Since $2 \in U(R)$, then $\Sigma_1 + 1 = u_1 \in U(R)$. Similarly $\Sigma_2 + 1 = u_2 \in U(R)$. Furthermore, $n - 2 \in U(R)$, say u_3. Thus, $a = u_1 + u_2 + u_3$.
3. Let $a = (\Sigma_1 - \Sigma_2) + n$, and let $a \in U(R)$. Then $a - n = (\Sigma_1 - \Sigma_2) \in U(R)$. Applying Lemma 3(2), then $(\Sigma_1 - \Sigma_2) = (\Sigma_1 - \Sigma_2)^3$. Thus $(\Sigma_1 - \Sigma_2)^2 = 1$.
4. Let a be a non-zero divisor element. Applying Theorem 1, $a^3 - a \in Nil(R)$, this gives $a(a^2 - 1) \in Nil(R)$, thus, $a^r(a^2 - 1)^r = 0$, for some positive integer r. Since a^r is a non-zero divisor, then $(a^2 - 1)^r = 0$, so $a^2 - 1 = n_1 \in Nil(R)$, implies $a^2 = 1 + n_1 \in U(R)$, then $a \in U(R)$.

It was proved in [18], that.

Proposition 4. [18, Proposition 1]. Assume R is a nil clean ring with every nilpotent is the difference between two commuting idempotents, then R is a Boolean ring.

We here extend this result as follows:

Theorem 4. *Suppose R is a strongly 2-NC ring, with any nilpotent is the difference between two commuting idempotents. Then R is a tripotent ring.*

Proof. Let a in R, then $a = \Sigma_1 - \Sigma_2 + n$ for some existing $\Sigma_1, \Sigma_2 \in Id(R), n \in Nil(R)$, that commute which each other. Then $n = \Sigma_3 - \Sigma_4$ for some $\Sigma_3, \Sigma_4 \in Id(R)$ and $\Sigma_3\Sigma_4 = \Sigma_4\Sigma_3$. So $n + \Sigma_4 = \Sigma_3$, this implies $(n + \Sigma_4)^2 = (n + \Sigma_4)$, then $n^2 + 2n\Sigma_4 + \Sigma_4^2 = n + \Sigma_4$, this gives $n^2 + 2n\Sigma_4 - n = 0$, so $n^2 + n(2\Sigma_4 - 1) = 0$, but $(2\Sigma_4 - 1)^2 = 1$, then we have $n = -n^2(2\Sigma_4 - 1)^{-1}$. As n is nilpotent, then $n = 0$. Thus, $a = \Sigma_1 - \Sigma_2$. Applying Lemma 3(2), $(\Sigma_1 - \Sigma_2)^3 = \Sigma_1 - \Sigma_2$. Hence, $a = a^3$ therefore, R is a tripotent ring.

3. Strongly 2-NC rings with units of order two

In this section, we introduce and investigate a strongly 2-NC rings with every unit is of order 2, we refer to this type of ring as strongly 2-NC rings with $U(R) = 2$.

Definition 4. A ring R is called strongly 2-NC with $U(R) = 2$ if for every $a \in R$, existing two idempotents Σ_1, Σ_2 and a nilpotent n, that commute and every unit is of order 2, such that $a = \Sigma_1 + \Sigma_2 + n$.

Example 2. The rings $Z_4, Z_6, Z_8, Z_{12}, Z_{24}$ are all strongly 2-NC with $U(R) = 2$, while the ring Z_9 is not strongly 2-NC with $U(R) = 2$.

We start this section with some fundamental properties of a strongly 2-NC ring with $U(R) = 2$.

Proposition 5. Homomorphic images of strongly 2-NC ring with $U(R) = 2$ is again strongly 2-NC ring with every unit is of order 2.

Proof. Let $f : R \to R'$ be a homomorphism from a strongly 2-NC ring R with $U(R) = 2$ onto R'. Then for any $b \in R'$, there exists $a \in R$, such that $b = f(a), a = \Sigma_1 + \Sigma_2 + n$ and $U(R) = 2$, where $\Sigma_1, \Sigma_2 \in Id(R), n \in Nil(R)$ that commute of with one another. Now, $b = f(a) = f(\Sigma_1 + \Sigma_2 + n) = f(\Sigma_1) + f(\Sigma_2) + f(n)$. Clearly, $f(\Sigma_1), f(\Sigma_2) \in Id(R')$ and $f(n) \in Nil(R')$. On the other hand for any $u \in (R)$, where u is a unit, $(f(u))^2 = f(u^2) = f(1)$, this shows that $f(u)$ is a unit of order 2. Therefore R' is a strongly 2-NC ring with $U(R') = 2$.

Proposition 6. If R is a strongly 2-NC ring with $U(R) = 2$. Then $24 = 0$.

Proof. Assume that a in R, then existing two idempotents Σ_1, Σ_2 and a nilpotent n that commute with one another, such that $a = \Sigma_1 - \Sigma_2 + n$. By Theorem 1, $a^3 - a \in Nil(R)$, this gives $2^3 - 2 = 6 \in Nil(R)$. Since every unit is of order 2, and since 6 is nilpotent, then $6 - 1 = 5 \in U(R)$. This gives $5^2 = 1$, so $24 = 0$.

Example 3. Consider the ring Z_{24}. Clearly, Z_{24} is a strongly 2-NC, with
$U(Z_{24}) = \{1, 5, 7, 11, 13, 17, 19, 23\}$. Observe that
$1^2 = 5^2 = 7^2 = 11^2 = 13^2 = 17^2 = 19^2 = 23^2 = 1$.

Observe that every SNC ring is strongly 2-NC, but not every unit of order 2.

Example 4. The ring Z_{16} is an SNC which is strongly 2-NC, but Z_{16} is not strongly 2-NC ring with $U(R) = 2$, since the units $3, 5, 11, 13$ are not of order 2.

Note that: If R is a strongly 2-NC with $U(R) = 2$, then R need not to be SNC ring.

Example 5. In the ring Z_{12}. Then
$U(Z_{12}) = \{1, 5, 7, 11\}$ and
$1^2 = 5^2 = 7^2 = 11^2 = 1$. Clearly, Z_{12} is a strongly 2-NC with $U(Z_{12}) = 2$, but (Z_{12}) is not SNC ring. Since 2 is not SNC element.

Proposition 7. If a ring R is a strongly 2-NC ring with $U(R) = 2$, for which $3 \in U(R)$, then R is SNC ring of characteristic 8.

Proof. Assume R is a strongly 2-NC ring with $U(R) = 2$. Then By Proposition 1(1), a^2 is a SNC element for every $a \in R$. Applying Proposition 2(1), a is strongly clean. Then a may be written $a - 1 = \Sigma + u$, where $\Sigma \in Id(R)$ and $u^2 = 1$. Then $a = \Sigma + u + 1$. Since $3 \in U(R)$, then $3^2 = 1$, gives $8 = 0$ thus, $2 \in Nil(R)$. So $(u + 1)^2 = u^2 + 2u + 1 = 2(u + 1) \in Nil(R)$. Thus, $u + 1 \in Nil(R)$. Therefore R is an SNC ring.

Example 6. Consider the ring Z_8. Then
$U(Z_8) = \{1, 3, 5, 7\}$. So
$1^2 = 3^2 = 5^2 = 7^2 = 1$. Clearly, Z_8 is a strongly 2-NC with $U(Z_8) = 2$. Observe that $3 \in U(Z_8)$. Then Z_8 is an SNC ring.

It was proved in Theorem 2, if a ring R is a strongly 2-NC, then $J(R)$ is nil. In the next result, we consider $J(R)$ over a strongly 2-NC ring with $U(R) = 2$.

Theorem 5. *If R is a strongly 2-NC ring with $U(R) = 2$, then $J(R)$ is nil of characteristic 4.*

Proof. Given $a \in J(R)$, then $a = \Sigma_1 - \Sigma_2 + n$, where $\Sigma_1, \Sigma_2 \in Id(R)$ and $n \in Nil(R)$, that commute with one another. Write $a = 1-(\Sigma_1-\Sigma_2)^2+(\Sigma_1-\Sigma_2)^2+(\Sigma_1-\Sigma_2)-1+n$. According to Lemma 3(3), $(\Sigma_1 - \Sigma_2)^2 + (\Sigma_1 - \Sigma_2) - 1 = u_1$ is a unit of order 2, then $a = 1-(\Sigma_1-\Sigma_2)^2+u_1+n$ implies $a = 1-(\Sigma_1-\Sigma_2)^2+u_2$, where $u_2 = u_1+n$. Since $a \in J(R)$, so $a - u_2 \in U(R)$, applying to Proposition 3(3), we conclude that $1 - (\Sigma_1 - \Sigma_2)^2 = 1$, gives $(\Sigma_1 - \Sigma_2)^2 = 0$, whence it follows that $a = 1 + u_2$, with $u_2^2 = 1$. Now consider $a^2 = (1 + u_2)^2 = 2(1 + u_2) = 2a$, and $a^3 = 2^2(1 + u_2) = 4a$. Choose $a = 2b$, then $(2b)^3 = 4(2b)$, so $8b^3 = 8b$, implies $8b(1 - b^2) = 0$, but $b \in J(R)$, gives $1 - b^2 \in U(R)$, gives $8b = 0$. Thus $4a = a^3 = 0$.

Example 7. Consider the ring Z_{24}. Then Z_{24} is a strongly 2-NC with $U(Z_{24}) = 2$. Now $J(Z_{24}) = \{0, 6, 12, 18\}$. So $J(Z_{24})$ is a nil ideal of characteristic 4.

Corollary 1. If R is a strongly 2-NC ring with $U(R) = 2$ and if $2 \in U(R)$, then $J(R) = 0$.

Proof. Let $a \in J(R)$, then by Theorem 5, $4a = 0$, since $2 \in U(R)$, then $a = 0$.

Proposition 8. If R is a strongly 2-NC ring with $U(R) = 2$, and if $2 \in U(R)$, then $Nil(R) = 0$.

Proof. Given $a \in R$, then $a - 1 = \Sigma_1 - \Sigma_2 + n$, so $a = \Sigma_1 - \Sigma_2 + n + 1$, but $n+1 \in U(R)$, say u, then $a = \Sigma_1 - \Sigma_2 + u$. Let $n \in Nil(R)$, then $n = \Sigma_1 - \Sigma_2 + u$, implies $\Sigma_1 - \Sigma_2 = n - u$, since $n - u \in U(R)$, according to Proposition 3(3), $(\Sigma_1 - \Sigma_2)^2 = 1$. Furthermore, $n^2 = (\Sigma_1-\Sigma_2)^2+2(\Sigma_1-\Sigma_2)u+u^2 = 1+2(\Sigma_1-\Sigma_2)u+1 = 2(1+(\Sigma_1-\Sigma_2)u)$. Observe that $nu = (\Sigma_1 - \Sigma_2)u + 1$. Thus, $n^2 = 2nu$, so $n(n - 2u) = 0$. since $2 \in U(R)$, by assumption then $n - 2u \in U(R)$. Whence it follows that $n = 0$.

Next, we shall explore the relationship between strongly 2-NC ring with $U(R) = 2$ and a tripotent ring.

Theorem 6. *A ring R with $2 \in U(R)$ is strongly 2-NC with $U(R) = 2$ if and only if R is a tripotent.*

Proof. *Let R be a strongly 2-NC ring with $U(R) = 2$, and let $a \in R$, then $a = \Sigma_1 - \Sigma_2 + n$, where $\Sigma_1, \Sigma_2 \in Id(R), n \in Nil(R)$, that commute with one another. According to Proposition 8, $n = 0$. Thus, $a = \Sigma_1 - \Sigma_2 = (\Sigma_1 - \Sigma_2)^3 = a^3$.*
Conversely, assume that R is a tripotent ring, and $t = t^3 \in R$, since $2 \in U(R)$, then t may be written as $t = \frac{t^2+t}{2} - \frac{t^2-t}{2}$. Note that:

$(\frac{t^2+t}{2})^2 = \frac{t^2+2t+t^2}{4} = \frac{t^2+t}{2}$, *and*

$(\frac{t^2-t}{2})^2 = \frac{t^2-2t+t^2}{4} = \frac{t^2-t}{2}$, *so* $(\frac{t^2+t}{2}), (\frac{t^2-t}{2}) \in Id(R)$. *Observe that for any unit $u, u^3 = u$ thus, $u^2 = 1$. Therefore, R is a strongly 2-NC ring with $U(R) = 2$.*

To end this section, we consider a strongly 2-NC ring, with every unit is of order 4.

Proposition 9. Suppose R is a strongly 2-NC ring, and if $n^2 + 2n = 0$ for every nilpotent n. Then every unit of R is of order 4, and $48 = 0$.

Proof. Given $a \in R$, then by Proposition 1(1), a^2 is an SNC element. Write $a^2 = \Sigma + n$, where $\Sigma \in Id(R), n \in Nil(R)$ and $\Sigma n = n\Sigma$. Let $u \in U(R)$, then $u^2 = \Sigma + n$, implies $\Sigma = u^2 - n = v \in U(R)$. Thus, $\Sigma = 1$. Hence $u^2 = 1+n$, implies $u^4 = (1+n)^2 = 1+2n+n^2$. By assumption $n^2 + 2n = 0$, then $u^4 = 1$. On the other hand $6 \in Nil(R)$ Theorem 1. Thus, $6^2 + 2(6) = 0$, gives $48 = 0$.

Example 8. In the ring Z_{48}. Then
$U(Z_{48}) = \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47\}$,
$Nil(Z_{48}) = \{0, 6, 12, 18, 24, 30, 36, 42\}$,
$Id(Z_{48}) = \{0, 1, 16, 33\}$.
By direct calculation, one easily check that Z_{48} is a strongly 2-NC ring, with every unit is of order 4.

4. Conclusion

In this article, new properties of a strongly 2-NC rings are given. Additionally, we added certain conditions for strongly 2-NC ring with each unit must be present of order four. We also introduce and investigated a strongly 2-NC ring with every unit of order two. We discuss some of the fundamental properties and present several examples. It was proved that the Jacobson radical over a strongly 2-NC ring is a nil ideal, here, we demonstrated that the Jacobson radical over strongly 2-NC ring with $U(R) = 2$ is a nil ideal of characteristic 4. In order to get $Nil(R) = 0$, we added one more condition involving this ring. The relationships between these rings, tripotent rings, and other related rings are given. Future goals include obtaining a deeper outcome on issues raised in this article, such as

1. The SNC ring with $U(R) = 2, 3$ or 4.

2. The strongly 2-NC ring with $U(R) = 3$ or 4.

3. The divisor graph of a strongly 2-NC ring with $U(R) = 2$.

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