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Strong Coproximinality in Bochner L^p-Spaces and in Köthe Spaces

Jamila Jawdat

Department of Mathematics, Faculty of Science, Zarqa University, Zarqa, Jordan

Abstract. In this paper, we study strong coproximinality in Bochner L^p-spaces and in the Köthe Bochner function space $E(X)$. We investigate some conditions to be imposed on the subspace G of the Banach space X such that $L^p(\mu, G)$ is strongly coproximinal in $L^p(\mu, X), 1 \leq p < \infty$. On the other hand, we prove that if G is a separable subspace of X then G is strongly coproximinal in X if and only if $E(G)$ is strongly coproximinal in $E(X)$, provided that E is a strictly monotone Köthe space. This generalizes some results in the literature. Some other results in this direction are also presented.

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1. Introduction And Some Preliminaries

Best approximation theory in normed linear spaces and that of best coapproximation are counterparts. Since 1970, [14], this topic had been intensively studied, and a huge work have been published, see for example [1, 2, 4, 6, 8–10, 12, 13]. If X is a Banach space with G a closed subspace, then G is called proximinal in X, if for each $x \in X$, there is g_0 in G satisfying

$$||g_0 - x|| \leq ||x - g||, \text{ for all } g \in G. \tag{1}$$

g_0 is called an element of best approximation to x from G. It is well-known that, $d(x, G) = inf\{||x - g||, \forall g \in G\}$. Hence, G is proximinal in X if for each x in X, there exists g_0 in G that satisfies,

$$||g_0 - x|| = d(x, G)$$

On the other hand, G is called coproximinal in X, if for each $x \in X$, there is g^0 in G satisfying

$$||g^0 - g|| \leq ||x - g||, \text{ for all } g \in G. \tag{2}$$

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Email address: `jjawdat@zu.edu.jo` (J. Jawdat)

Again, g^0 is called an element of best coapproximation to x from G. Let $P_G(x)$ (resp. $R_G(x)$) be the set of all elements in G that satisfy (1) (resp. (2)).

The notion of strong proximinality in general Banach spaces, was first studied by Godefroy and Indumathi, [3], and is defined as follows.

Definition 1. *A proximinal subspace G of X is called strongly proximinal at $x \in X$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $P_G(x,\delta) \subseteq P_G(x) + \varepsilon B_X$, where B_X is the unit ball of X and $P_G(x,\delta) = \{z \in G : ||x - z|| < d(x,G) + \delta\}$.*

Usually, $P_G(x,\delta)$ is referred to as the set of near best approximation points to x from G. In addition, if G is strongly proximinal at each $x \in X$ then it is called strongly proximinal in X.

An equivalent definition for strong proximinality in Banach spaces is given using the notion of minimizing sequences, defined as below.

Definition 2. *Let G be a subspace of X, that is proximinal in X. A sequence $\{y_n\}$ in G is called a minimizing sequence for an element x in X if*

$$lim_{n\to\infty}||\ x - y_n|| = d(x,G)$$

Definition 3. *A subset G, that is proximinal in X, is called strongly proximinal in X, if $\forall x \in X$ and any minimizing sequence $\{y_n\}$ in G for x, $\exists$ a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and a sequence $\{z_n\}$ in $P_G(x)$ satisfying $||y_{n_k} - z_n|| \to 0$.*

In other words, the sequence $\{y_{n_k}\}$ satisfies $d(y_{n_k}, P_G(x)) \to 0$ whenever $||x - y_n|| \to d(x,G)$.

In recent years, strong proximinality has become a topic of much interest, see [3, 5, 7, 15] and the references therein. It is well known that if G is finite dimensional in X then G is strongly proximinal in X. Moreover, if G is an M-ideal in X then G is also strongly proximinal in X. The question to be proposed here is that whether strong proximinality of G in X can be lifted to the L^p-space or to the Köthe space under certain conditions on G? In [15], the author proved that "If G is separable then G is strongly proximinal subspace in X if and only if $L^p(\mu, G)$ is strongly proximinal in $L^p(\mu, X), 1 \le p < \infty$". We proved a similar result for the case where $0 < p < 1$, see Theorem 3.3 in [7]. On the other hand, strong coproximinality in $L^p(\mu, X)$, was first studied in [5]. In this paper, we will study more properties in this direction and prove some new results. This will be done in section two, in which we are interested with the spaces of p-Bochner integrable functions $L^p(\mu, X), 1 \le p < \infty$, where (T, Σ, μ) is a finite measure space. The p-norm, defined on $L^p(\mu, X)$ for $1 \le p < \infty$, is given by:

$$||f||_p = (\int_T ||f||^p dt)^{1/p}.$$

In the third section, we study strong coproximinality in Köthe Bochner function spaces. First consider E to be the space of all "equivalence classes" of μ-measurable real-valued functions on T. This means for h and g in E then $h = g$ if and only if $h(t) = g(t)$, μ-almost everywhere t in T (for simplicity we write *a.e.* $t \in T$). When E is equipped with a norm $||.||_E$ under which it is complete then E is known as a real Köthe function space. Finally, E becomes a Banach Lattice [11], if it satisfies the two conditions below.

(i) For each measurable subset A of T, with $\mu(A) < \infty$, the characteristic function χ_A is again in E.

(ii) For any two functions h and g such that $|h| \leq |g|$ and $g \in E$ then $h \in E$ and $||h||_E \leq ||g||_E$.

A Köthe space E is said to be strictly monotone if the inequality in (ii) above is strict. In other words, if $h \geq g \geq 0$ in E and $||h||_E = ||g||_E$ imply $h = g$. For a real Banach space $(X, ||.||_X)$ and a real Köthe space E, consider $E(X)$ to be the space of (equivalence classes of) strongly-measurable functions $f : T \to X$ where $||f(.)||_X \in E$. Define a norm on $E(X)$ as follows.

$$|||f||| = || \, ||f(.)||_X ||_E.$$

Then $(E(X), |||.|||)$ is called the Köthe Bochner function space, see [11], which is a Banach space under the above norm. The Köthe Bochner function spaces that are most well-known classes are the Lebesgue-Bochner spaces $L^p(\mu, X)$, $1 \leq p < \infty$ and the Orlicz-Bochner spaces $L^\phi(\mu, X)$.

Let $E(X)$ be the Köthe Bochner function space on X. Several authors studied the problem under what conditions the subspace $E(G)$ is proximinal (resp. coproximinal) in $E(X)$, see for example, [9] and [6], but no work has been conducted in the direction of strong proximinality (resp. coproximinality) in these spaces. One of the main results of this paper is to prove that if G is a separable subspace of X then G is strongly coproximinal in X if and only if $E(G)$ is strongly coproximinal in $E(X)$, provided that E is a strictly monotone Köthe space. This generalizes the results for Bochner L^p-spaces.

2. Strong Coproximinality of $L^p(\mu, G)$ in $L^p(\mu, X)$

In this section, we first recall the definition of strong coproximinality in general Banach spaces, [4]. Some new results are also given. Let G be coproximinal in X, hence the set of best coapproximation points to x, which is denoted by $R_G(x)$, is nonempty for each x in X. For some $\delta > 0$, define $R_G(x, \delta)$ to be the set of "near best coapproximation points" to x from G, as follows.

$$R_G(x, \delta) = \{z \in G : ||z - g|| < ||x - g|| + \delta, \forall g \in G\}. \qquad (3)$$

Definition 4. *A coproximinal subset G in X, is called strongly coproximinal in X, if for each $x \in X$, the following is satisfied:*
For any $\varepsilon > 0$, there exists $\delta > 0$ such that $R_G(x, \delta) \subset R_G(x) + \varepsilon B_X$, where B_X again is the unit ball of X and $R_G(x, \delta)$ as defined above.

An alternative definition is the following.

Definition 5. *A coproximinal subset G of X is called strongly coproximinal at $x \in X$, if given $\varepsilon > 0$ there exists some $\delta > 0$, such that for each $z \in R_G(x, \delta)$ there exists $y^0 \in R_G(x)$ satisfying $||z - y^0|| < \varepsilon$. In addition, G is said to be strongly coproximinal in X, if it is so for all x in X.*

The main result in this section is that, given G separable in X and $1 \leq p < \infty$, then $L^p(\mu, G)$ is strongly coproximinal in $L^p(\mu, X)$ if and only if G is strongly coproximinal subspace of X. We first prove some results for the case $p = 1$, then these results can be extended easily to $L^p(\mu, X)$, for $1 < p < \infty$. The absence of the distance formula in the theory of best coapproximation leads to a variant way of dealing with the proofs. Since we cannot use the minimizing sequence definition (as for the case of strong proximinality) but, however, we will make use of the following Lemma.

Lemma 1. *Let $f \in L^1(\mu, X)$, and $g \in L^1(\mu, G)$. Let G be separable and coproximinal in X. Then $g \in R_{L^1(\mu,G)}(f, \delta')$ if and only if $g(t) \in R_G(f(t), \delta)$, a.e. $t \in T$, and for some $\delta,\ \delta' > 0$.*

Proof. Given $g \in L^1(\mu, G)$ such that $g(t) \in R_G(f(t), \delta)$, *a.e.* $t \in T$. Then from the definition of $R_G(f(t), \delta)$, we have

$$||g(t) - y|| < ||f(t) - y|| + \delta, \text{ for all } y \in G, \ a.e. \ t \in T.$$

This implies as a special case,

$$||g(t) - h(t)|| < ||f(t) - h(t)|| + \delta, \forall h \in L^1(\mu, G), \ a.e. \ t \in T.$$

Hence, we get

$$\int_T ||g(t) - h(t)||dt < \int_T ||f(t) - h(t)||dt + \delta \cdot \mu(T), \forall h \in L^1(\mu, G).$$

And since $\mu(T) < \infty$, we can take $\delta' = \delta \cdot \mu(T)$, hence, we obtain

$$||g - h|| < ||f - h|| + \delta', \ \forall h \in L^1(\mu, G).$$

So, we get $g \in R_{L^1(\mu,G)}(f, \delta')$.

For the other direction, we proceed as follows.
Given $g \in R_{L^1(\mu,G)}(f, \delta')$, for some $\delta' > 0$. Then, from (3), we have,

$$||g - k|| < ||f - k|| + \delta', \ \forall k \in L^1(\mu, G).$$

But since G is separable and coproximinal in X, then $L^1(\mu, G)$ is coproximinal in $L^1(\mu, X)$, see [3]. So, let $h \in R_{L^1(\mu,G)}(f)$ satisfying $||g - h|| < \delta'$. Again, from [4], $h(t)$ is a best coapproximation for $f(t)$, *a.e.* $t \in T$. Moreover,

$$||g(t) - h(t)|| < \delta'/\mu(T), \ a.e. \ t \in T.$$

Hence, for any $y \in G$, we have

$$\begin{aligned} ||g(t) - y|| &= ||g(t) - h(t) + h(t) - y|| \\ &\leq ||g(t) - h(t)|| + ||h(t) - y|| \\ &< ||f(t) - y|| + \delta'/\mu(T),\ \forall y \in G, a.e.\ t \in T. \end{aligned}$$

Finally, taking $\delta = \delta'/\mu(T)$, we get, $g(t) \in R_G(f(t), \delta)$, for *a.e.* $t \in T$.

Remark 1. *The result in Lemma 1 can be easily extended for the case of* $L^p(\mu, X)$, $1 < p < \infty$.

One main result in this paper, is the following.

Theorem 1. *If* G *is separable and strongly coproximinal in* X*, then* $L^1(\mu, G)$ *is strongly coproximinal in* $L^1(\mu, X)$.

Proof. Given G in X a strongly coproximinal subspace then G is coproximinal in X (by definition). Also G being separable then $L^1(\mu, G)$ is coproximinal in $L^1(\mu, X)$, see [4]. Now, let $f \in L^1(\mu, X)$ and $\varepsilon > 0$ be arbitrary. Let $g \in R_{L^1(\mu,G)}(f, \delta)$, for some $\delta > 0$, then by Lemma 1, $g(t) \in R_G(f(t), \delta_t)$, for some $\delta_t > 0$, *a.e.* t in T. Again, since G is strongly coproximinal in X then, from Definition 5, there exist $y_t \in R_G(f(t))$ satisfying $||g(t) - y_t|| < \varepsilon/\mu(T)$, *a.e.* $t \in T$. Since G separable, we may define a function h, such that $h(t) = y_t$, for all $t \in T$. So,

$$||g(t) - h(t)|| < \varepsilon/\mu(T),\ a.e.\ t \in T. \tag{4}$$

Then h can be proved to be a measurable function using a technique similar to that of Theorem 7 in [6]. Also, $h \in L^1(\mu, G)$ since $||h(t)|| \leq ||h(t) - g(t)|| + ||g(t)|| < \varepsilon/\mu(T) + ||g(t)||$, *a.e.* t in T. Finally, by the way h was defined, it follows that $h \in R_{L^1(\mu,G)}(f)$ and, from (4), h satisfies $||g - h|| < \varepsilon$. Hence, Definition 5 is satisfied and we get that $L^1(\mu, G)$ is strongly coproximinal in $L^1(\mu, X)$.

Theorem 2. *Let* $L^p(\mu, G)$ *be strongly coproximinal in* $L^p(\mu, X)$, $1 \leq p < \infty$*, then* G *is strongly coproximinal in* X.

Proof. By relating each x in X with a function $f_x = x \cdot \chi_T$, in $L^p(\mu, X)$, where χ_T is the characteristic function on T and since $L^p(\mu, G)$ is strongly coproximinal in $L^p(\mu, X)$ then by the definition of strong coproximinality, Lemma 1 and Remark 1, thereafter, the result follows.

Another main result is the following.

Theorem 3. *Let* $L^1(\mu, G)$ *be strongly coproximinal in* $L^1(\mu, X)$ *then* $L^p(\mu, G)$ *is strongly coproximinal in* $L^p(\mu, X)$*, for* $1 < p < \infty$.

Proof. It has been proved, in [3], that $L^1(\mu, G)$ is coproximinal in $L^1(\mu, X)$ if and only if $L^p(\mu, G)$ is coproximinal in $L^p(\mu, X)$, $1 < p < \infty$. Now, let $L^1(\mu, G)$ be strongly coproximinal in $L^1(\mu, X)$ and $f \in L^p(\mu, X)$. Take $h \in R_{L^p(\mu,G)}(f, \delta)$, for some $\delta > 0$. Since $\mu(T) < \infty$, then $L^p(\mu, X) \subset L^1(\mu, X)$, $1 < p < \infty$ and so $f \in L^1(\mu, X)$ and $h \in R_{L^1(\mu,G)}(f, \delta')$, for some $\delta' > 0$. But $L^1(\mu, G)$ is strongly coproxminal in $L^1(\mu, X)$, which implies that for any $\varepsilon > 0$, there exists $g^0 \in R_{L^1(\mu,G)}(f)$ such that

$$||h - g^0|| < \varepsilon.$$

But, by Theorem 2 and Lemma 1, we have G is strongly coproximinal in X, and

$$||h(t) - g^0(t)|| < \varepsilon/\mu(T), \ a.e.\ t \in T. \tag{5}$$

On the other hand, we have $g^0(t)$ is a best coapproximation for $f(t)$, $a.e.\ t \in T$. So, for w an arbitrary element of $L^p(\mu, G)$, we can write

$$||w(t) - g^0(t)|| \leq ||w(t) - f(t)||, \ a.e.\ t \in T. \tag{6}$$

This gives,

$$||g^0(t)|| \leq ||f(t)||, \ a.e.\ t \in T.$$

Therefore, $g^0 \in L^p(\mu, G)$ and consequently, from (6), we have, $||w - g^0||_p \leq ||w - f||_p$, for all w in $L^p(\mu, G)$. This implies that $g^0 \in R_{L^p(\mu,G)}(f)$. Equation (5) also gives that $||h - g^0||_p < \varepsilon$. Hence, $L^p(\mu, G)$ is strongly coproximinal in $L^p(\mu, X), 1 < p < \infty$.

The following corollary follows directly from Theorems 1, 2 and 3.

Corollary 1. *For G separable in X, then G is strongly coproximinal in X if and only if $L^p(\mu, G)$ is strongly coproximinal in $L^p(\mu, X), 1 \leq p < \infty$.*

3. Strong Coproximinality of $E(G)$ in $E(X)$

In this section, let $(X, \ ||.||_X)$ be a real Banach space and E a real Köthe space. Consider $E(X)$ as defined in the introduction section with the following norm,

$$|||f||| = || \ ||f(\cdot)||_X ||_E.$$

Then $(E(X), |||.|||)$ is a Banach space called the Köthe Bochner function space. For more on Köthe Bochner function spaces, see [11].

The second goal of this paper is to extend the main theorem in the previous section to the Köthe Bochner function spaces, as in the following Theorem.

Main Theorem (Theorem 5). Let G be a separable subspace of X such that E is strictly monotone. Then $E(G)$ is strongly coproximinal in $E(X)$ if and only if G is strongly coproximinal in X.

To prove our main Theorem, we need the following two results.

Theorem 4. *Let G be coproximinal in X and E is strictly monotone Köthe space. For f in $E(X)$ and g in $E(G)$ such that for each t, $g(t)$ is a near best coapproximation point in G to $f(t)$ in X, a.e. $t \in T$, then g is a near best coapproximation to f.*

Proof. Given f and g as above. Let $g(t)$ be a near best coapproximation point in G to $f(t)$ in X. Then from (3),

$$||g(t) - y|| < ||f(t) - y|| + \delta, \text{ for some } \delta > 0 \text{ and for all } y \in G.$$

So, if for any function h in $E(G)$, we have

$$||g(t) - h(t)|| < ||f(t) - h(t)|| + \delta, \text{ for some } \delta > 0.$$

This implies, from the strict monotonicity of E, that

$$|||g - h||| < |||f - h||| + \delta \ \mu(T), \forall h \in E(G).$$

Finally, since the measure space is finite then g is a near best coapproximation to f.

A simple function in $E(X)$ is a function $f : T \to X$ of the form $f = \sum_{k=1}^{n} a_k \chi_{A_k}$, where a_k's are in X (may or may not be distinct) and $\{A_1, \ldots, A_n\}$ is a finite collection of mutually disjoint measurable subsets of T such that $\cup A_k = T$.

The following lemma follows directly from Theorem 4 above and Lemma 3 in [6].

Lemma 2. *Let G be strongly coproximinal in X. Then $E(G)$ is strongly coproximinal at any simple function in $E(X)$.*

The following theorem is another main result in this paper.

Theorem 5. *Let G be a separable subspace of X and let E be a strictly monotone Köthe space. Then $E(G)$ is strongly coproximinal in $E(X)$ if and only if G is strongly coproximinal in X.*

Proof. $\Rightarrow$) Let x_0 in X. By taking $f = x_0 \chi_T$, then clearly f is a simple function in $E(X)$, since it can be represented as $f = \sum_{k=1}^{n} a_k \chi_{A_k}$, where $a_k = x_0$, for each k. The sequence $\{A_1, \ldots, A_n\}$ consists of mutually disjoint measurable subsets of T such that $\cup A_k = T$. Now, since $E(G)$ is strongly coproximinal in $E(X)$, then it is coproximinal in $E(X)$ and hence G is coproximinal in E, see [6]. Also, $E(G)$ is strongly coproximinal at f above. Hence, there exist $g_0 \in R_{E(G)}(f)$ and $h \in R_{E(G)}(f, \delta)$ such that $|||g_0 - h||| < \varepsilon$. Since g_0 and h can be taken to be simple functions, so for some $y \in R_G(x_0)$, and z_k in G, we set

$$g_0 = \sum_{k=1}^{n} z_k \ . \ \chi_{A_k} \text{ and } h = \sum_{k=1}^{n} y \ . \ \chi_{A_k}.$$

Now, both $|||g_0 - h||| < \varepsilon$ and the measure space being finite, imply that $||z_k - y|| < \varepsilon / \mu(T)$. Hence, the result follows.

$\Leftarrow$) Let G be strongly coproximinal in X. By Lemma 2, above $E(G)$ is strongly coproximinal at any simple function in $E(X)$. But since simple functions are dense in the whole space then one can deduce that $E(G)$ is strongly coproximinal at any function in $E(X)$.

Corollary 2. *Let G be separable in X. G is strongly coproximinal in X if and only if $L^p(\mu, G)$ is strongly coproximinal in $L^p(\mu, X)$, for $1 \leq p < \infty$.*

4. Conclusion

In this paper, strong coproximinality was studied for Bochner function spaces $L^p(\mu, X)$, for $1 \leq p < \infty$, and for the Köthe Bochner function space $E(X)$. The main result was: If G is separable in X, then $L^p(\mu, G)$ (resp. $E(G)$) is strongly coproximinal in $L^p(\mu, X)$ (resp. $E(X)$), if and only if G is strongly coproximinal subspace of X. Some other results were also given and proved for strong coproximinality in these spaces.

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