Note on Generalized Neighborhoods Structures in Fuzzy Bitopological Spaces

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Abstract. This article’s main aim is to study the concepts of the generalized neighborhood and generalized quasi-neighborhood in fuzzy bitopological spaces. It also introduces fundamental theorems for determining the relationships between them. Additionally, some significant examples were examined to demonstrate the significance of the interconnections, some theorems were also introduced to study some main properties of neighborhood structures. Finally, we also studied the concepts of closure, interior, and each of their critical theories and properties by generalized neighborhood systems in fuzzy bitopological spaces.

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1. Introduction

In this project, we have prioritized our study on fuzzy bitopology, which derived from fuzzy topology that was first introduced in 1965 by the scientist Zadeh [8]. Following this, many researchers applied fundamental ideas on fuzzy settings from general topology and improved the concept of fuzzy topology. Such as Chang, in 1968 introduced some fuzzy concepts in fuzzy topology [4]. In addition, in 1989 Kandil introduced fuzzy bitopological spaces [1]. Also, generalized fuzzy closed groups were established in fuzzy topology in 1997 by Balasubramanian and Sundaram [6]. After that, many scientists applied the notion of a generalized closed set in fuzzy space and in 2005 El-Shafei introduced some applications of it [9]. Also, in 2009 Xuzhu Wang et al presented a book that contains all the basic operations in fuzzy science [12]. As Zahran and El-Maghrabi studied in 2011 some operations on it in fuzzy space [13]. Then in 2017 Benchalli et al studied delta

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generalized beta closed in topological spaces [3]. After one year, Kandil et al defined the concept of locally pairwise closed sets and studied some of their properties [7]. In 2019 Ramaboopathi and Dharmalingam introduced a new class of generalized closed sets in bitopological spaces [11], as in the same year Andal and Thiripurasundari introduced a new concept of fuzzy generalized $\pi$ closed in fuzzy bitopological spaces [2]. Finally, in 2021 Das et al introduced the idea of $\gamma$ generalized fuzzy quasi neighborhood of a fuzzy point [5].

2. Preliminaries

In the following part, we go over important antecedent notions that are essential to the development of this paper.

Definition 1. [10] Suppose the set $X$ is not empty and the $I$ sign represents the unit period $[0, 1]$, then the following defined as:

1. an operator with $X$ domain and $I$ range is known as a fuzzy set $E$, where $E(x) \in (0, 1]$ when $x \in E$, and $E(x) = 0$ in case $x \notin E$.

2. a set $D$ is including $E$ indicated via $E \subseteq D$ if $E(x) \leq D(x)$, whenever $x \in X$.

3. $E$ and $D$ combination indicated by $E \lor D$ if $(E \lor D)(x) = \max\{E(x), D(x)\}$ $\forall x \in X$.

4. the intersection of $E$, $D$ indicated by $E \land D$ if $(E \land D)(x) = \min\{E(x), D(x)\}$ $\forall x \in X$.

5. the completeness of $E$ denoted via $E^c$ as $(E(x))^c = 1 - E(x)$, $\forall x \in X$.

The following definitions explain the meaning of fuzzy topology and fuzzy bitopological spaces.

Definition 2. [10] A fuzzy topology of $X$ is a class of fuzzy groups $\delta \in I$ which holds the coming three conditions:

1. 0 and 1 contained in $\delta$, where $0(x) = 0$, $1(x) = 1$, whenever $x \in X$.

2. For any $E, D \in \delta$, $E \land D \in \delta$.

3. For any $(E_{i \in I}) \in \delta$, $\lor_{i \in I} E_i \in \delta$.

The term "fuzzy topological space," or "fts," refers to the pair $(X, \delta)$. The components of $\delta$ are named fuzzy open sets. If $F^c \in \delta$, then $F$ is mean as fuzzy closed. The collection including whole fuzzy closed groups in fuzzy topology $\delta$ denote by $F_\delta$.

Definition 3. [1] A fuzzy bitopological spaces, or fbts for short, $(X, \delta_1, \delta_2)$ since $X$ is not empty, $\delta_1$, and $\delta_2$ are fuzzy topological spaces on $X$. Over this dissertation $X$ perform fuzzy bitopology $(X, \delta_1, \delta_2)$, and $Y$ to $(Y, \sigma_1, \sigma_2)$, where $i \neq j$, and $i, j \in \{1, 2\}$.

In the section which follows, the definitions of fuzzy set interiors and closings are covered.
Definition 4. [10] Closing and internal of any fuzzy set $M$ of $(X, \delta)$ are indicated also defined as follows:

\[
\text{cl}(M) = \wedge \{ F : M \leq F, F^c \in \delta \}
\]
\[
\text{int}(M) = \vee \{ O : O \leq M, O \in \delta \},
\]
respectively.

The closing, internal, and complements of $M$ of $X$ are indicated by $\delta_1-\text{cl}(M)$, $\delta_1-\text{int}(M)$, and $M^c_i$, respectively, with regard to fuzzy topology $\delta_i$. Additionally, we designate the class of all fuzzy $\delta_j$-closed by the mathematical symbol $F_{\delta_j}$.

One of the work’s core tenets is the definition of the fuzzy generalized closed set, which as following:

Definition 5. [6] Any fuzzy group $E$ of $X$ is termed fuzzy generalised closed when closure $E$ is subset of $W$, wherever $E$ is subset of $W$, $W$ is fuzzy open.

i.e., $E$ is fuzzy generalised closed in case of $\text{cl}(E) \leq W$, wherever $E \leq W$, $W$ is fuzzy open.

Definition 6. [10]

(1) A fuzzy point $x_\lambda$ is claimed that quasi-coincident with $E$, shown by $x_\lambda q E$ if $\lambda > E^c(x)$, or $\lambda + E(x) > 1$ and $x_\lambda$ is claimed does not quasi-coincident with $E$ if $\lambda + E(x) \leq 1$ and we write $E \overline{q} x_\lambda$.

(2) $E$ is claimed quasi-coincident with $C$ indicated as $E q C$ if there exists $x \in X$ so that $E(x) > C^c(x)$ or $E(x) + C(x) > 1$, and $E$ is claimed does not quasi-coincident with $C$ if there exists $x \in X$ so that $E(x) + C(x) \leq 1$ and we write $E \overline{q} C$.

If $E q C$ (resp, $E \overline{q} C$) is true, then $E$ and $C$ are quasi-coincident (resp, not quasi-coincident) with each other at $x$.

3. Generalized Neighborhoods Structures at Fuzzy Bitopological Spaces

This section introduces the idea of generalized neighborhoods concepts by using ($\in$) relationship and quasi coincident concept ($q$) in fuzzy bitopological spaces and characterize it in terms of important theorems and some properties.

Definition 7. A fuzzy subgroup $E$ of fbts $(X, \delta_1, \delta_2)$ is known as:

(1) Fuzzy $(i, j)-$generalized $\varphi-$closed (in sum, $(i, j) - g_{\varphi}-closed$) if $\delta_j - \varphi - \text{cl}(E) \leq U$ where $E \leq U$, $U \in \delta_i$, and $\varphi$ including the types (alpha ($\alpha$), semi ($s$), pre ($p$), and beta ($\beta$)).

(2) The supplement of the fuzzy $(i, j) - g_{\varphi}-closed$ set is referred to $(i, j) - g_{\varphi}-open$ set in $X$.

Remark 1. (1) The universal set of all fuzzy $(i, j) - g_{\varphi}-open$, and $(i, j) - g_{\varphi}-closed$ sets of fbts $(X, \delta_1, \delta_2)$ is represented by $O^{f_{g_{\varphi}}}_{(i,j)}$, $F^{f_{g_{\varphi}}}_{(i,j)}$, and so forth.
(2) Also, the family of all gφ-open, and gφ-closed subsets of X pertaining to the fuzzy topology δi is indicated Oiφ, and Fiφ, i = 1, 2.

**Proposition 1.** A fuzzy group E in fbsts (X, δ₁, δ₂) is fuzzy (i, j) − gφ-open ⇐⇒ F ≤ δj − φ − int(E) whenever Fc ∈ δi, F ≤ E.

**Proof.** Assume E is fuzzy (i, j) − gφ-open, Fc ∈ δi, when F ≤ E. Then E ≤ Fc. As Ec is fuzzy (i, j) − gφ-closed, thus δj − φ − cl(Ec) = (δj − φ − int(E))c ≤ Fc that indicates F ≤ δj − φ − int(E).

Conversely, assume E is fuzzy set of X, F ∈ Fᵢ so F ≤ δj − φ − int(E), F ≤ E. After adding the supplement to both sides, we find (δj − φ − int(E))c ≤ Fc so E ≤ Fc and Fc is fuzzy open in δi, thus Ec is fuzzy (i, j) − gφ-closed (Definition7). Hence E is fuzzy (i, j) − gφ-open.

**Definition 8.** A fuzzy group E in fbsts (X, δ₁, δ₂) is known as:

1. Fuzzy (i, j) − generalized φ−neighborhood (shortly, (i, j) − gφ−nbd) of fuzzy singleton set xᵣ if ∃ fuzzy (i, j) − gφ − open set C so xᵣ ∈ C ≤ E.
   
   The family of all fuzzy (i, j) − gφ−nbd of fuzzy singleton set xᵣ, will be denoted by Nᵣ(xᵣ).

2. Fuzzy (i, j) − generalized φ − Q−neighborhood (shortly, (i, j) − gφQ−nbd) of fuzzy singleton set xᵣ if ∃ fuzzy (i, j) − gφ − open set C so xᵣ ∉ C ≤ E.
   
   The family of all fuzzy (i, j) − gφQ−nbd of fuzzy singleton set xᵣ, will be denoted by NᵣQ(xᵣ).

**Remark 2.** In general, every fuzzy δ − Q−neighborhood of a fuzzy point does not include the point itself. The coming example show that:

**Example 1.** Assume x₀.₇ is fuzzy point of X = {a, b, c} and E is fuzzy set of X defined as E(a) = 0.4, E(b) = 0.5, E(c) = 0.3. Let δ = {0, 1, E} on X. Then E ∈ N₀(x₀.₇) but 0.7 ∉ E(x), and hence x₀.₇ ∉ E but x₁₋₀.₇₋₀.₃ ∈ E.

**Corollary 1.** In fbsts (X, δ₁, δ₂) every fuzzy (i, j) − gφQ − nbd of fuzzy point xᵣ of X is equivalent to (i, j) − gφ − nbd of fuzzy point x₁₋ᵣ.

**Theorem 1.** (1) Every fuzzy δᵣ − nbd of fuzzy point xᵣ is fuzzy (i, j) − g − nbd of xᵣ.

(2) Every fuzzy (i, j) − g − nbd of fuzzy point xᵣ is fuzzy (i, j) − gα − nbd of xᵣ.

(3) Every fuzzy (i, j) − gα − nbd of xᵣ is fuzzy (i, j) − gs − nbd and (i, j) − gp − nbd of xᵣ.

(4) Every fuzzy (i, j) − gs − nb or (i, j) − gp − nbd of xᵣ is fuzzy (i, j) − gβ − nbd of xᵣ.

**Proof.**

(1) Suppose that E ∈ Nᵣ(xᵣ), thus ∃ C ∈ δᵣ so xᵣ ∈ C ≤ E. As every δᵣ − open set is (i, j) − g − open set, then ∃ C is fuzzy (i, j) − g − open set, so xᵣ ∈ C ≤ E, and hence E ∈ Nᵣ(i,j)(xᵣ).
(2) Suppose that \( E \in N_{(i,j)}^{g}(x_r) \), thus \( \exists C \) is fuzzy \((i, j) - g - \) open set, so \( x_r \in C \leq E \), and since every fuzzy \((i, j) - g - \) open is fuzzy \((i, j) - g_{0} - \) open, then \( \exists C \) is fuzzy \((i, j) - g_{0} - \) open set, so \( x_r \in C \leq E \), and hence \( E \in N_{(i,j)}^{g_{0}}(x_r) \).

(3) Suppose that \( E \in N_{(i,j)}^{g_{0}}(x_r) \), thus \( \exists C \) is fuzzy \((i, j) - g_{0} - \) open set, so \( x_r \in C \leq E \), and since every fuzzy \((i, j) - g_{0} - \) open is fuzzy \((i, j) - g_{s} - \) open and \((i, j) - g_{p} - \) open, then \( \exists C \) is fuzzy \((i, j) - g_{s} - \) open and \((i, j) - g_{p} - \) open set, so \( x_r \in C \leq E \), and hence \( E \in N_{(i,j)}^{g_{s}}(x_r) \), and \( E \in N_{(i,j)}^{g_{p}}(x_r) \).

(4) Suppose that \( E \in N_{(i,j)}^{g_{s}}(x_r) \), or \( E \in N_{(i,j)}^{g_{p}}(x_r) \), thus \( \exists C \) is fuzzy \((i, j) - g_{s} - \) open or \((i, j) - g_{p} - \) open set, so \( x_r \in C \leq E \), and since every fuzzy \((i, j) - g_{s} - \) open or \((i, j) - g_{p} - \) open is fuzzy \((i, j) - g_{\beta} - \) open, then \( \exists C \) is fuzzy \((i, j) - g_{\beta} - \) open set, so \( x_r \in C \leq E \), and hence \( E \in N_{(i,j)}^{g_{\beta}}(x_r) \).

**Remark 3.** In fbts \((X, \delta_1, \delta_2)\) every fuzzy \(N_{(i,j)}^{g_{s}}(x_r)\), and \(N_{(i,j)}^{g_{p}}(x_r)\) are independents. The following example show that if \( X = \{a, b, c\}, \delta_1 = \{0, 1, E\}, \) and \( \delta_2 = \{0, 1, C, D\}\).

As \( E_{a,b,c} = \{0.7, 0.5, 0.6\}, C_{a,b,c} = \{0.5, 0.4, 0.3\}, \) and \( D_{a,b,c} = \{0.4, 0.3, 0.2\}, \) then \( S_{a,b,c} = \{0.5, 0.5, 0.6\} \in N_{(i,j)}^{g_{s}}(x_r), \) but \( S \notin N_{(i,j)}^{g_{p}}(x_r) \), as \( E^c \leq S, \) but \( E^c \notin \delta_2 - p - \text{int}(S) = C \).

On other hand for the same topologies above if \( E_{a,b,c} = \{0.3, 0.5, 0.4\}, C_{a,b,c} = \{0.6, 0.8, 0.9\}, \) and \( D_{a,b,c} = \{0.4, 0.3, 0.2\}, \) then \( S_{a,b,c} = \{0.8, 0.6, 0.5\} \in N_{(i,j)}^{g_{s}}(x_r), \) but \( S \notin N_{(i,j)}^{g_{p}}(x_r) \), as \( E^c \leq S, \) but \( E^c \notin \delta_2 - s - \text{int}(S) = C^c \).

The following Figure explaining the relation between nbds structures of all cases.

**Figure 1:** Explain the relations between all types of fuzzy \(N_{(i,j)}^{g_{s}}(x_r)\), and all types of \(N_{(i,j)}^{g_{p}}(x_r)\).
Definition 9. Suppose $E_{a,b,c} = (0.7, 0.5, 0.5)$, $H_{a,b,c} = (0.5, 0.4, 0.3)$, $R_{a,b,c} = (0.4, 0.3, 0.2)$, $S_{a,b,c} = (0.5, 0.5, 0.5)$. The conclusion is $S \in N_{(1,2)}^{g_{a,b,c}}(x_r)$, but never $S \notin N_{(1,2)}^{g_{a,b,c}}(x_r)$.

The following example clear that $N_{(1,2)}^{q_{a,b,c}}(x_r) \Rightarrow N_{(1,2)}^{g_{a,b,c}}(x_r)$.

Example 5. Suppose $E_{a,b,c} = (0.7, 0.5, 0.4)$, $H_{a,b,c} = (0.6, 0.8, 0.8)$, $R_{a,b,c} = (0.4, 0.3, 0.2)$, $S_{a,b,c} = (0.8, 0.6, 0.7)$. The conclusion is $S \in N_{(1,2)}^{g_{a,b,c}}(x_r)$, but never $S \notin N_{(1,2)}^{g_{a,b,c}}(x_r)$.

The example follow indicates that $N_{(1,2)}^{g_{a,b,c}}(x_r) \Rightarrow N_{(1,2)}^{g_{a,b,c}}(x_r)$.

Example 6. Suppose $E_{a,b,c} = (0.5, 0.7, 0.6)$, $H_{a,b,c} = (0.6, 0.5, 0.4)$, $R_{a,b,c} = (0.4, 0.3, 0.2)$, $S_{a,b,c} = (0.5, 0.5, 0.6)$. The conclusion is $S \in N_{(1,2)}^{g_{a,b,c}}(x_r)$, but never $S \notin N_{(1,2)}^{g_{a,b,c}}(x_r)$.

As well, the coming example demonstrates that $N_{(1,2)}^{g_{a,b,c}}(x_r) \Rightarrow N_{(1,2)}^{g_{a,b,c}}(x_r)$.

Example 7. Suppose $E_{a,b,c} = (0.5, 0.7, 0.6)$, $H_{a,b,c} = (0.4, 0.6, 0.7)$, $R_{a,b,c} = (0.3, 0.4, 0.5)$, $S_{a,b,c} = (0.5, 0.5, 0.6)$. The conclusion is $S \in N_{(1,2)}^{g_{a,b,c}}(x_r)$, but never $S \notin N_{(1,2)}^{g_{a,b,c}}(x_r)$.

Definition 9. A fuzzy singleton set $x_r$ is named fuzzy $(i,j)$ = generalized $\varphi$ = cluster point of fuzzy subset $E$ in fbits $(X, \delta_1, \delta_2)$ if and only if all $H \in N_{(i,j)}^{g_{(i,j)}}(x_r)$. Thus $H \notin N_{(i,j)}^{g_{(i,j)}}(x_r)$.

The following theorem examines some of the generalized neighborhood characteristics.

Theorem 2. If $(X, \delta_1, \delta_2)$ is fbits. Next, we find:

1. $\forall x_r \in X, N_{(i,j)}^{g_{(i,j)}}(x_r) \neq \emptyset$.
2. $\forall H \in N_{(i,j)}^{g_{(i,j)}}(x_r), x_r \in H$.
3. when $H, R \in N_{(i,j)}^{g_{(i,j)}}(x_r)$, then $H \cap R \in N_{(i,j)}^{g_{(i,j)}}(x_r)$.
4. when $H \in N_{(i,j)}^{g_{(i,j)}}(x_r)$ and $H \leq R$, then $R \in N_{(i,j)}^{g_{(i,j)}}(x_r)$.
5. when $H \in N_{(i,j)}^{g_{(i,j)}}(x_r)$, then there exists $R \in N_{(i,j)}^{g_{(i,j)}}(x_r)$ such that $R \leq H$ and $R \in N_{(i,j)}^{g_{(i,j)}}(x_r)$.

Proof.

From Definition 8 we conclude the prove of (1) and (2).

(3) Assume $H, R \in N_{(i,j)}^{g_{(i,j)}}(x_r)$. Thus $\exists S, T$ are fuzzy $(i,j) - g_{(i,j)}$-open, so $x_r \in S, x_r \in T$, then $x_r \in S \cap T$. As $S, T$ are fuzzy $(i,j) - g_{(i,j)}$-open, and hence we find $S \cap T$ is fuzzy $(i,j) - g_{(i,j)}$-open, $S \cap T \leq H \cap R$. As a result of that, $A \cap B \in N_{(i,j)}^{g_{(i,j)}}(x_r)$.  }

Theorem 3. When \((X, \delta_1, \delta_2)\) is fbt. Then we have:

1. \(\forall x_r q o r \in X, N^{g\varphi^Q}_{(i,j)}(x_r) \neq \phi\).

2. \(\forall E \in N^{g\varphi^Q}_{(i,j)}(x_r), x_r q E\).

3. when \(E, T \in N^{g\varphi^Q}_{(i,j)}(x_r)\), then \(E \land T \in N^{g\varphi^Q}_{(i,j)}(x_r)\).

4. when \(E \in N^{g\varphi^Q}_{(i,j)}(x_r)\), and \(E \leq T\), then \(T \in N^{g\varphi^Q}_{(i,j)}(x_r)\).

5. when \(E \in N^{g\varphi^Q}_{(i,j)}(x_r)\), then \(\exists T \in N^{g\varphi^Q}_{(i,j)}(x_r)\) so \(T \leq E\), and \(T \in N^{g\varphi^Q}_{(i,j)}(x_r) \forall x_h \in T\).

Proof. It resembles the earlier Theorem 2 proof.

Using the above-mentioned novel notion of fuzzy neighbourhood and quasi-neighborhood structure, we introduced the study of the degree of affiliation of a fuzzy element to fuzzy generalised closure in the subsequent theorem.

Theorem 4. If \(E\) is fuzzy set and \(x_r\) is fuzzy point of fbt \((X, \delta_1, \delta_2)\), then the following propositions are correct:

1. \(x_r \in (i, j) - g\varphi - \text{cl}(E) \iff \forall T \in N^{g\varphi^Q}_{(i,j)}(x_r), T q E\).

2. \(x_r \in (i, j) - g\varphi - \text{cl}(E) \iff \forall T \in N^{g\varphi^Q}_{(i,j)(x_1-r)}, T q E\).

3. If \(E\) is fuzzy \((i, j) - g\varphi - \text{closed}\), and hence \(\delta_i - \text{cl}(x_r) q E\) holds \(\forall x_r q \delta_j - \varphi - \text{cl}(E)\).

Proof.

Let \(x_r \in (i, j) - g\varphi - \text{cl}(E) \iff \forall F\) is fuzzy \((i, j) - g\varphi - \text{closed}, E \leq F, r \leq F(x)\)
\(\iff \forall F^c\) is fuzzy \((i, j) - g\varphi - \text{open}, F^c \leq E^c, F^c(x) \leq 1 - r\)
\(\iff \forall T\) is fuzzy \((i, j) - g\varphi - \text{open}, T \leq E^c, T(x) \leq 1 - r\)
\(\iff \forall T\) is fuzzy \((i, j) - g\varphi - \text{open}, 1 - r < T(x) \Rightarrow T \not\leq E^c\)
\(\iff \forall T\) is fuzzy \((i, j) - g\varphi - \text{open}, x_r q T, T q E\)
\(\iff \forall T \in N^{g\varphi^Q}_{(i,j)}(x_r), T q E\).
(2) From (1) we have \( x_r \in (i, j) - g\varphi - cl(E) \iff \forall T \in N^{g\varphi q}_{(i,j)}(x_r), T \cap q E \). So, we need to show that \( T \in N^{g\varphi q}_{(i,j)}(x_{1-r}) \iff T \in N^{g\varphi q}_{(i,j)}(x_r) \). Assume \( T \in N^{g\varphi q}_{(i,j)}(x_{1-r}) \). After that, \( \exists \) fuzzy \( (i, j) - g\varphi - \) open set \( V \) so \( x_{1-r} \in V \leq T \), then \( x_r, q V \leq T \). As a result of that, \( T \in N^{g\varphi q}_{(i,j)}(x_r) \).

In the opposite direction, assume \( T \in N^{g\varphi q}_{(i,j)}(x_r) \). Thus \( \exists \) fuzzy \( (i, j) - g\varphi - \) open set \( V \) so \( x_r, q V \leq T \), and hence \( x_{1-r} \in V \leq T \). As a result of that, \( T \in N^{g\varphi q}_{(i,j)}(x_{1-r}) \).

(3) Assume \( E \) be fuzzy \( (i, j) - g\varphi - \) closed. Suppose there \( \exists \) fuzzy point \( x_r \) so \( x_r, q \delta_j - \varphi - cl(E) \), but \( \delta_i - cl(x_r) \) \( \bar{q} E \). Thus \( E \leq (\delta_i - cl(x_r))^c \). As \( E \) is fuzzy \( (i, j) - g\varphi - \) closed, thus \( \delta_j - \varphi - cl(E) \leq (\delta_i - cl(x_r))^c \), hence \( \delta_j - \varphi - cl(E) \) \( \bar{q} \delta_i - cl(x_r) \). As \( x_r \in \delta_i - cl(x_r) \), thus \( x_r, q \delta_j - \varphi - cl(E) \) that is a contradiction. As a result of that, \( \delta_i - cl(x_r) q E \) holds \( \forall x_r, q \delta_j - \varphi - cl(E) \).

**Corollary 2.** If \( E \) is fuzzy \( (i, j) - g\varphi - \) closed, and \( x_r \) is fuzzy point in \( fbts \ (X, \delta_1, \delta_2) \), then \( \delta_i - cl(x_r) q E \) holds \( \forall x_r, q \delta_j - \beta - cl(E) \).

In the theory that follows, we studied the most fundamental generalized closure characteristics and demonstrated them using new neighborhood structure notions.

**Theorem 5.** If \( E, \) and \( T \) are fuzzy subsets of \( fbts \ (X, \delta_1, \delta_2) \), thus the following arguments are correct:

1. \( 0, \) and \( 1 \) are fuzzy \( (i, j) - g\varphi - \) closed.
2. When \( E \leq T \), then \( (i, j) - g\varphi - cl(E) \leq (i, j) - g\varphi - cl(T) \).
3. \( E \leq (i, j) - g\varphi - cl(E), \forall \) fuzzy set \( E \in I^N \).
4. When \( E \) is fuzzy \( (i, j) - g\varphi - \) closed, then \( (i, j) - g\varphi - cl(E) = E \). The converse is false, as the intersection of fuzzy \( (i, j) - g\varphi - \) closed sets need not be fuzzy \( (i, j) - g\varphi - \) closed.
5. \( (i, j) - g\varphi - cl((i, j) - g\varphi - cl(E)) = (i, j) - g\varphi - cl(E) \).
6. When \( v \) is \( (i, j) - g\varphi - \) open, then \( v q E \iff v q (i, j) - g\varphi - cl(E) \).
7. \( (i, j) - g\varphi - cl(E) \lor (i, j) - g\varphi - cl(T) \leq (i, j) - g\varphi - cl(E \lor T) \).

**Proof.** By using Definition 7 and Theorem 4 we can easily proved (1), (2), (3), and (4).

(5) Assume \( x_r \) is fuzzy point with \( x_r \notin (i, j) - g\varphi - cl(E) \). After that, \( \exists V \in N^{g\varphi q}_{(i,j)}(x_r) \) so \( x_r, q V, V \bar{q} E \), then \( \exists U \) is fuzzy \( (i, j) - g\varphi - \) open so \( x_r, q U \leq V \) and \( U \bar{q} E \). Thus from

6. \( U \bar{q} (i, j) - g\varphi - cl(E) \). As \( \exists U \) is fuzzy \( (i, j) - g\varphi - \) open so \( x_r, q U \) and \( U \bar{q} (i, j) - g\varphi - cl(E) \). Then \( x_r \notin (i, j) - g\varphi - cl((i, j) - g\varphi - cl(E)) \), after that

\( (i, j) - g\varphi - cl((i, j) - g\varphi - cl(E)) \leq (i, j) - g\varphi - cl(E) \). But

\( (i, j) - g\varphi - cl(E) \leq (i, j) - g\varphi - cl((i, j) - g\varphi - cl(E)) \). As a result of that,

\( (i, j) - g\varphi - cl(E) = (i, j) - g\varphi - cl((i, j) - g\varphi - cl(E)) \).
(6) Sufficiency, assume \( V q E \). After that, \( E \leq V^c, V^c \) is fuzzy \((i, j) - g \varphi = \text{closed}, \) then by applying \((i, j) - g \varphi - \text{clouser} \) for all sides and from (5) we find \( V \overline{q}(i, j) - g \varphi - \text{cl}(E) \). As a result of that, \( V q E \iff V q (i, j) - g \varphi - \text{cl}(E) \).

(7) As \( E \leq (E \lor T) \) and \( T \leq (E \lor T) \), then 
\[
(i, j) - g \varphi - \text{cl}(E) \lor (i, j) - g \varphi - \text{cl}(T) \leq (i, j) - g \varphi - \text{cl}(E \lor T).
\]

From the relationship between closure, interior, complement, and Theorem 5 we conclude the following:

**Theorem 6.** If \( E \) and \( T \) are fuzzy subsets of fbts \((X, \delta_1, \delta_2)\), then the coming statements are correct:

1. \( 0, \) and \( I \) are fuzzy \((i, j) - g \varphi = \text{open} \).
2. when \( E \leq T \), then \((i, j) - g \varphi - \text{int}(E) \leq (i, j) - g \varphi - \text{int}(T) \).
3. \((i, j) - g \varphi - \text{int}(E) \leq E, \forall \text{ fuzzy set } E \in I^X \).
4. when \( E \) is fuzzy \((i, j) - g \varphi = \text{open} \), then \((i, j) - g \varphi - \text{int}(E) = E \). The converse is false, as the combination of fuzzy \((i, j) - g \varphi = \text{open} \) sets not necessary to be fuzzy \((i, j) - g \varphi = \text{open} \).
5. \((i, j) - g \varphi - \text{int}((i, j) - g \varphi - \text{int}(E)) = (i, j) - g \varphi - \text{int}(E) \).
6. when \( v \) is \((i, j) - g \varphi = \text{closed} \), then \( v \overline{q}E \iff v \overline{q}(i, j) - g \varphi - \text{int}(E) \).
7. \((i, j) - g \varphi - \text{int}(E \land T) \leq (i, j) - g \varphi - \text{int}(E) \land (i, j) - g \varphi - \text{int}(T) \).

**Theorem 7.** If \( x_r \) is fuzzy point, and \( E \) is fuzzy subset of fbts \((X, \delta_1, \delta_2)\), then \( x_r \in (i, j) - g \varphi - \text{int}(E) \iff \exists \text{ fuzzy } (i, j) - g \varphi - \text{open} \) set \( G \), so \( x_r \in G \leq E \).

**Theorem 8.** Suppose \( E \) is fuzzy set in fbts \((X, \delta_1, \delta_2)\). If \( E \) is fuzzy \((i, j) - g \varphi = \text{open} \), then \( E \in N_{(i,j)}^{g \varphi}(x_r) \) for each \( x_r \in E \).

**Theorem 9.** If \((X, \delta_1, \delta_2)\) is fbts, \( E \) is fuzzy \((i, j) - g \varphi = \text{closed} \), and \( E \leq T \leq \delta_j - \varphi - \text{cl}(E) \), then \( T \) is fuzzy \((i, j) - g \varphi = \text{closed} \).

Proof. Assume \( T \leq U \), and \( U \) is fuzzy open of \( \delta_i \). As \( E \leq T \), thus \( E \leq U \), after that \( \delta_j - \varphi - \text{cl}(E) = \delta_j - \varphi - \text{cl}(T) \), which implies \( \delta_j - \varphi - \text{cl}(T) \leq U \). As a result of that, \( T \) is fuzzy \((i, j) - g \varphi = \text{closed} \).

From the above we conclude the following:

**Corollary 3.** Assume \((X, \delta_1, \delta_2)\) is fbts, \( E \) is fuzzy \((i, j) - g \varphi = \text{closed} \), and \( E \leq T \leq \delta_j - \beta - \text{cl}(E) \). Then \( T \) is fuzzy \((i, j) - g \varphi = \text{closed} \).

**Corollary 4.** Assume \((X, \delta_1, \delta_2)\) is fbts, \( E \) is fuzzy \((i, j) - g \varphi = \text{open} \), and \( \delta_j - \varphi - \text{int}(E) \leq T \leq E \). Then \( T \) is fuzzy \((i, j) - g \varphi = \text{open} \).
Corollary 5. Assume \((X, \delta_1, \delta_2)\) is fbts, \(E\) is fuzzy \((i, j) - g\varphi - open\), and \(\delta_j - \beta - \text{int}(E) \leq T \leq E\). Then \(T\) is fuzzy \((i, j) - g\varphi - open\).

The following important study demonstrates when equivalence between the types of generalized closed sets in fuzzy bitopology and types of fuzzy sets from one topology is attained.

Theorem 10. In fbts \((X, \delta_1, \delta_2)\) the following statements are equivalents:

(i) \(\delta_i \subseteq \mathcal{F}_{\varphi}^{(X,\delta_j)}\)

(ii) All fuzzy groups of \(X\) are fuzzy \((i, j) - g\varphi - closed\).

Proof.

(i) \(\rightarrow\) (ii) Assume \(E\) is fuzzy subset of \(X\), so \(E \subseteq U \in \delta_i\). Then from (i) we find \(U \in \mathcal{F}_{\varphi}^{(X,\delta_j)}\), after that \(\delta_j - \varphi - cl(E) \leq U\). As a result of that, \(E\) is fuzzy \((i, j) - g\varphi - closed\).

(ii) \(\rightarrow\) (i) Let \(E\) be fuzzy \((i, j) - g\varphi - closed\), \(E \in \delta_i\). Since \(E \subseteq E\), then \(\delta_j - \varphi - cl(E) \leq E\), thus \(E\) is fuzzy \(\delta_j - \varphi - closed\). Therefore \(\delta_i \subseteq \mathcal{F}_{\varphi}^{(X,\delta_j)}\).

From the above and the complement relation we conclude the following:

Corollary 6. In fbts \((X, \delta_1, \delta_2)\) the following statements are equivalents:

1. \(\mathcal{F}_{\delta_i} \subseteq \mathcal{O}_{\varphi}^{(X,\delta_j)}\)

2. All fuzzy subset of \(X\) is fuzzy \((i, j) - g\varphi - open\).

Corollary 7. Assume \(E\), and \(T\) are fuzzy \((i, j) - g\varphi - closed\) sets in fbts \((X, \delta_1, \delta_2)\) with \(E \lor \delta_i - \text{int}(T) = T \lor \delta_i - \text{int}(E) = 1\), then \(E \land T\) is fuzzy \((i, j) - g\varphi - closed\).

4. Conclusion

In this study, we introduced and studied the definition of some types of generalized neighborhood and generalized quasi-neighborhood ideas fuzzy bitopology space, and we prove some relations and inclusion relation between them by listing some examples, then applied them to, closure, interior, and studied some key properties of them.

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References


