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Another Look at Geodetic Hop Domination in a Graph

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Abstract. Let G be an undirected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A subset S of vertices of G is a geodetic hop dominating set if it is both a geodetic set and a hop dominating set. The geodetic hop domination number of G is the minimum cardinality among all geodetic hop dominating sets in G. In this paper, we characterize the geodetic hop dominating sets in the join of two graphs. These characterizations which use the concept of pointwise non-dominating 2-path closure absorbing set are, in turn, used to determine the geodetic hop domination number of the join of graphs. Moreover, a realization result involving the hop domination number and geodetic hop domination number is also obtained.

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1. Introduction

Over the years, a number of studies dealing with the topic on hop domination, a concept introduced and initially studied by Natarajan and S. K. Ayyaswamy [13], had been done. In particular, some variations of hop domination had been introduced and considered in many studies (see [2], [3], [4], [5], [6], [7], [10], [11], [12], [14], [15], and [16]). Henning and Rad [9] gave a probabilitics upper bound of the hop domination number of a graph and showed that the hop dominating set problem is NP-complete for planar bipartite graphs and planar chordal graphs. In a recent study, Henning et al. [8] presented a linear time algorithm for computing a minimum hop dominating set in bipartite permutation graphs.

The idea of combining the concepts of hop domination and geodetic has led to the introduction of the notion of geodetic hop domination. This hop domination variant was first defined and examined by Anusha and Robin [1]. Motivated by the new concept, Saromines and Canoy [16] gave characterizations of the geodetic hop dominating sets in the corona and lexicographic product of two graphs. In this present paper, we revisit the concept of geodetic hop domination and give further results of this new parameter.

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2. Terminology and Notation

For any two vertices u and v in an undirected connected graph G, the distance $d_G(u,v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u,v)$ is called a u-v *geodesic*. The interval $I_G[u,v]$ consists u,v and all vertices lying on a u-v geodesic. The interval $I_G(u,v) = I_G[u,v] \setminus \{u,v\}$. The *open neighborhood* of a vertex u is the set $N_G(u)$ consisting of all vertices v which are adjacent to u. The *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G(A) = \bigcup_{v \in A} N_G(v)$ is called the *open neighborhood of* A and $N_G[A] = N_G(A) \cup A$ is called the *closed neighborhood of* A. The *open hop neighborhood* of a vertex u is the set $N_G^2(u) = \{v \in V(G) : d_G(v,u) = 2\}$. The *closed hop neighborhood* of u is $N_G^2[u] = N_G^2(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G^2(A) = \bigcup_{v \in A} N_G^2(v)$ is called the *open hop neighborhood of* A and $N_G^2[A] = N_G^2(A) \cup A$ is called the *closed hop neighborhood of* A.

A set $S \subseteq V(G)$ is a *dominating set* in G if $N_G[S] = V(G)$. The smallest cardinality of a dominating set in G, denoted by $\gamma(G)$ is called the *domination number* of G. The *geodetic closure* of a set $S \subseteq V(G)$, denoted by $I_G[S]$, is the union of the intervals $I_G[u,v]$, where $u,v \in S$. Set S is *geodetic set* in G if $I_G[S] = V(G)$. The smallest cardinality among all geodetic sets in G, denoted by $g(G)$, is called the *geodetic number* of G. A geodetic set of cardinality $g(G)$ is called a g-set of G. A set $S \subseteq V(G)$ is a *geodetic dominating set* in G if it is both a dominating and a geodetic set.

A set $S \subseteq V(G)$ is a *hop dominating* set if $N_G^2[S] = V(G)$. The minimum cardinality of a hop dominating set of a graph G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. A subset S of $V(G)$ is a *total hop dominating* set of G if for every $v \in V(G)$, there exists $u \in S$ such that $d_G(u,v) = 2$. The smallest cardinality of a total hop dominating set of G, denoted by $\gamma_{th}(G)$ is called the *total hop domination number* of G. Any total hop dominating set of G with cardinality $\gamma_{th}(G)$ is called a γ_{th}-set.

A subset S of vertices of G is a *geodetic hop dominating* set if it is both a geodetic and a hop dominating set. The *geodetic hop domination number* $\gamma_{hg}(G)$ of G is the minimum cardinality among all geodetic hop dominating sets in G. Any geodetic hop dominating set of G with cardinality $\gamma_{hg}(G)$ is called a γ_{hg}-set.

A set $S \subseteq V(G)$ of a graph G is called a 2-*path closure absorbing* if for each $x \in V(G) \setminus S$ there exist $u,v \in S$ such that $d_G(u,v) = 2$ and $x \in I_G(u,v)$. The minimum cardinality of a 2-path closure absorbing set in G is denoted by $\rho_2(G)$. Any 2-path closure absorbing set of G with cardinality $\rho_2(G)$ is called a ρ_2-set.

A set $D \subseteq V(G)$ is a *pointwise non-dominating* set of G if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $v \notin N_G(u)$. The smallest cardinality of a pointwise non-dominating set of G, denoted by $pnd(G)$, is called the *pointwise non-domination number* of G. A pointwise non-dominating set $S \subseteq V(G)$ of a graph G is called a 2-*path closure absorbing pointwise non-dominating set* if it is a 2-path closure absorbing set. The minimum cardinality of a 2-path closure absorbing pointwise non-dominating set in G is denoted by $\rho_{2pnd}(G)$. Any 2-path closure absorbing pointwise non-dominating set of G

with cardinality $\rho_{2pnd}(G)$ is called a ρ_{2pnd}-set.

Let G and H be two graphs. The *join* $G+H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

3. Results

Since every geodetic hop dominating set is a hop dominating set, we have the following remark.

Remark 1. *Let G be any connected graph on n vertices. Then $\gamma_h(G) \leq \gamma_{hg}(G)$.*

Remark 2. *The bound given in Remark 1 is tight. Moreover, strict inequality can also be attained.*

To see this, consider $G = K_4$ and $H = K_{1,3}$. It can easily be verified that $\gamma_h(G) = \gamma_{hg}(G) = 4$ and $\gamma_h(H) = 2 < 4 = \gamma_{hg}(H)$.

Theorem 1. *Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_h(G) = a$ and $\gamma_{hg}(G) = b$.*

Proof. Consider the following cases:

Case 1. $a = b$.

Let $G = K_a$. Then $\gamma_h(G) = a = \gamma_{hg}(G)$.

Case 2. $a < b$.

Consider the following subcases:

Subcase 2.1. a is even.

Suppose $a = 2$ and let $m = b - a$. Consider the graph G in Figure 1. Let $S_1 = \{x_1, x_2\}$ and $S_2 = \{y_1, y_2, z_1, z_2, ..., z_m\}$. Then S_1 and S_2 are, respectively, γ_h-set and γ_{hg}-set of G. Hence, $\gamma_h(G) = a$ and $\gamma_{hg}(G) = a + m = b$.

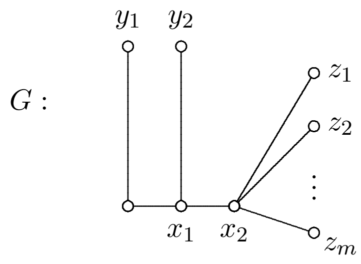

Figure 1

Suppose $a \geq 4$. Consider the graph G' in Figure 2. Let $S_3 = \{x_1, x_2, ..., x_{a-1}, x_a\}$ and $S_4 = \{y_1, y_2, ..., y_{a-1}, y_a, z_1, z_2, ..., z_m\}$. Then S_3 and S_4 are, respectively, γ_h-set and γ_{hg}-set of G'. Hence, $\gamma_h(G') = a$ and $\gamma_{hg}(G') = a + m = b$.

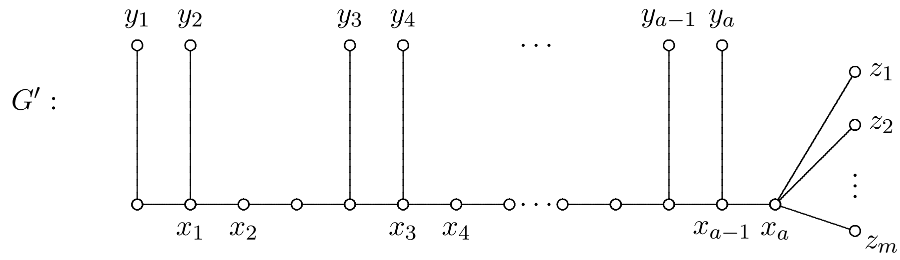

Figure 2

Subcase 2.2. a is odd.

Suppose $a = 3$ and let $m = b - a + 1$. Consider the graph H in Figure 3. Let $S_5 = \{x_1, x_2, x_3\}$ and $S_6 = \{y_1, y_2, z_1, z_2, ..., z_m\}$. Then S_5 and S_6 are, respectively, γ_h-set and γ_{hg}-set of H. Hence, $\gamma_h(H) = a$ and $\gamma_{hg}(H) = m + a - 1 = b$.

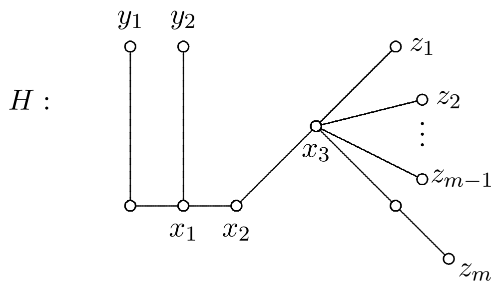

Figure 3

Suppose $a \geq 5$ and let $m = b - a + 1$. Consider the graph H' in Figure 4. Let $S_7 = \{x_1, x_2, ..., x_{a-1}, x_a\}$ and $S_8 = \{y_1, y_2, ..., y_{a-1}, z_1, z_2, ..., z_m\}$. Then S_7 and S_8 are, respectively, γ_h-set and γ_{hg}-set of H'. Hence, $\gamma_h(H') = a$ and $\gamma_{hg}(H') = m + a - 1 = b$.

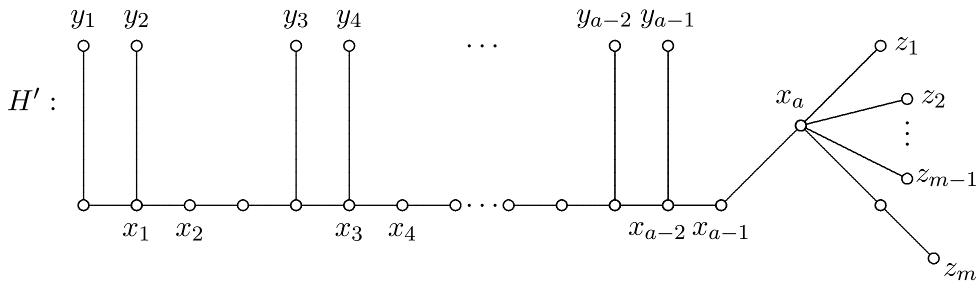

Figure 4

This proves the assertion.

Corollary 1. *Let n be a positive integer. Then there exists a connected graph such that $\gamma_{hg}(G) - \gamma_h(G) = n$. In other words, the difference $\gamma_{hg}(G) - \gamma_h(G)$ can be made arbitrarily large.*

The next few results deal with the concept of pointwise non-dominating 2-path closure absorbing sets.

Remark 3. *Every pointwise non-dominating* 2*-path closure absorbing is both a pointwise non-dominating set and a* 2*-path closure absorbing set in* G*. Hence,*

$$\rho_{2pnd}(G) \geq max\left\{pnd(G), \rho_2(G)\right\}.$$

Theorem 2. *Let* G *be a graph on* $n \geq 3$ *vertices. Then*

$$3 \leq \rho_{2pnd}(G) \leq n.$$

Moreover,

(i) $\rho_{2pnd}(G) = 3$ *if and only if* $n = 3$ *or* $n > 3$ *and there exists* $S \subseteq V(G)$ *with* $|S| = 3$ *such that for each* $v \in V(G) \setminus S$*,* $|N_G(v) \cap S| = 2$ *and* $d_G(a, b) = 2$ *for* $a, b \in N_G(v) \cap S$*.*

(ii) $\rho_{2pnd}(G) = n$ *if and only if one of the following holds:*

- (a) G *is connected and for every pair of vertices* x, y *with* $d_G(x, y) = 2$*, the set* $N_G(x) \cap N_G(y)$ *contains dominating vertices of* G *only or*
- (b) G *is disconnected such that every component* H *of* G *is complete.*

Proof.
Let S be a ρ_{2pnd}-set of G. Suppose $|S| \leq 2$ and let $v \in V(G) \setminus S$. Since S is a 2-path closure absorbing set, there exist $p, q \in S$ such that $d_G(p, q) = 2$ and $v \in I_G(p, q)$. Hence, S cannot be a pointwise non-dominating set, contradicting our assumption that S is a ρ_{2pnd}-set of G. Therefore, $3 \leq \rho_{2pnd}(G)$.

(i) Suppose $\rho_{2pnd}(G) = 3$. Suppose further that $n > 3$ and let S be a ρ_{2pnd}-set of G. Then $|S| = 3$. Let $v \in V(G) \setminus S$. Then there exist vertices $a, b \in S$ such that $d_G(a, b) = 2$ and $v \in I_G(a, b)$ because S is a 2-path closure absorbing set. Also, since S is a pointwise non-dominating set, there exists $z \in S \setminus \{a, b\}$ such that $v \notin N_G(z)$. Therefore, $|N_G(v) \cap S| = 2$.

For the converse, suppose that $n > 3$ and there exists $S \subseteq V(G)$ with $|S| = 3$ that satisfies the given conditions. Let $v \in V(G) \setminus S$. Then, by assumption, there exist $a, b \in S$ with $d_G(a, b) = 2$ and $v \in I_G(a, b)$. This implies that S is a 2-path closure absorbing set of G. Since $|N_G(v) \cap S| = 2$, S is also a pointwise non-dominating set of G.

Finally, suppose that $n = 3$. Then $S = V(G)$ is both a pointwise non-dominating and 2-path closure absorbing set. Since $3 \leq \rho_{2pnd}(G)$, it follows that $\rho_{2pnd}(G) = 3$.

(ii) Suppose $\rho_{2pnd}(G) = n$. Consider the following cases:

Case 1. G is connected.

Suppose there exist $p, q \in V(G)$ with $d_G(p,q) = 2$ such that $N_G(p) \cap N_G(q)$ contains a non-dominating vertex, say z. Then there is a vertex $w \in V(G) \setminus N_G(z)$. This implies that $V(G) \setminus \{z\}$ is a pointwise non-dominating and 2-path closure absorbing set of G, contrary to the assumption that $\rho_{2pnd}(G) = n$. Thus, $N_G(x) \cap N_G(y)$ contains dominating vertices of G only for every pair of vertices x, y with $d_G(x,y) = 2$, showing that (a) holds.

Case 2. G is disconnected.

Suppose there exists a component H of G that is not complete. Then there exist $v, w \in V(H)$ such that $d_G(v,w) = 2$. Let $u \in N_G(v) \cap N_G(w)$. Then $V(G) \setminus \{u\}$ is a 2-path closure absorbing set of G. Let $H^{'}$ be a component of G with $H^{'} \neq H$ and pick $u^{'} \in V(H)$. Then $u^{'} \in V(G) \setminus \{u\}$ and $uu^{'} \in E(G)$. Hence, $V(G) \setminus \{u\}$ is also a pointwise non-dominating set of G. This gives a contradiction. Thus, every component of G is complete.

For the converse, suppose first that (a) holds. Let S be a ρ_{2pnd}-set of G. Suppose $S \neq V(G)$, say $v \in V(G) \setminus S$. Since S is a 2-path closure absorbing set of G, there exist $x, y \in S$ such that $d_G(x,y) = 2$ and $v \in I_G(x,y)$. By assumption, v is a dominating vertex of G. Therefore, S is not a pointwise non-dominating set, a contradiction. Hence, $S = V(G)$ and $\rho_{2pnd}(G) = n$.

Next, suppose that (b) holds. Then the only 2-path closure absorbing set of G is $V(G)$. Therefore, $V(G)$ is the only pointwise non-dominating and 2-path closure absorbing set of G. Accordingly, $\rho_{2pnd}(G) = n$.

The next result follows from Theorem 2.

Corollary 2. *Let n be a positive integer and $n \geq 2$. Then $\rho_{2pnd}(K_n) = \rho_{2pnd}(\overline{K_n}) = \rho_{2pnd}(K_{1,n-1}) = n$.*

Proposition 1. *Let m and n be positive integers with $m, n \geq 2$. Then*

$$\rho_{2pnd}(K_{m,n}) = \begin{cases} 3 & \text{if } m = 2 \text{ or } n = 2 \\ 4 & \text{if } m \geq 3 \text{ and } n \geq 2 \end{cases}.$$

Proof. Suppose $m = 2$ or $n = 2$, say $m = 2$. Choose any $w \in V(\overline{K}_n)$. Then $S = V(\overline{K}_m) \cup \{w\}$ is a pointwise non-dominating and 2-path closure absorbing set of $K_{m,n}$. By Theorem 2, $\rho_{2pnd}(K_{m,n}) = |S| = 3$.

Next, suppose that $m \geq 3$ and $n \geq 3$. Pick any $x, y \in V(\overline{K}_m)$ and $p, q \in V(\overline{K}_n)$. Then $\{x, y, p, q\}$ is a pointwise non-dominating and 2-path closure absorbing set of $K_{m,n}$. This implies that $\rho_{2pnd}(K_{m,n}) \leq 4$. Let $S_\circ$ be a ρ_{2pnd}-set of $K_{m,n}$. Suppose further that $|S_\circ| = 3$. Since $S_\circ$ is a pointwise non-dominating set, $S_1 = S_\circ \cap V(\overline{K}_m) \neq \varnothing$ and $S_2 = S_\circ \cap V(\overline{K}_n) \neq \varnothing$. We may assume that $|S_1| = 1$. Then $|S_2| = 2$. Let $z \in V(\overline{K}_n) \setminus S_\circ$. Then $z \notin I_{K_{m,n}}(u,v)$ for all $u, v \in S_\circ$, a contradiction. Therefore, $|S_\circ| \geq 4$. Accordingly, $\rho_{2pnd}(K_{m,n}) = 4$.

Proposition 2. *For each positive integer $n \geq 2$,*

(i) $\rho_{2pnd}(P_n) = \begin{cases} 2 & \text{if } n = 2 \\ 3 & \text{if } n = 3, 4 \\ \lceil \frac{n+1}{2} \rceil & \text{if } n \geq 5 \end{cases}$

(ii) $\rho_{2pnd}(C_n) = \begin{cases} 3 & \text{if } n = 3, 4 \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 5 \end{cases}$

Proof.

(i) Clearly, $\rho_{2pnd}(P_2) = 2$ and $\rho_{2pnd}(P_3) = \rho_{2pnd}(P_4) = 3$. Let $n \geq 5$. If n is odd, then $S_1 = \{v_1, v_3, ..., v_{n-2}, v_n\}$ is the only ρ_{2pnd}-set of P_n. Hence, $\rho_{2pnd}(P_n) = \frac{n+1}{2}$. If n is even, then $S_2 = \{v_1, v_3, ..., v_{n-3}, v_{n-1}, v_n\}$ and $S_3 = \{v_1, v_3, v_4..., v_{n-2}, v_n\}$ are the only ρ_{2pnd}-sets of P_n. It follows that $\rho_{2pnd}(P_n) = \frac{n+2}{2}$.

(ii) By Theorem 2(i), $\rho_{2pnd}(C_3) = \rho_{2pnd}(C_4) = 3$. Let $n \geq 5$. If n is odd, then $\{v_1, v_3, v_5, ..., v_{n-2}, v_n\}$ is a ρ_{2pnd}-set of C_n. If n is even, then $\{v_1, v_3, v_5, ..., v_{n-1}\}$ is a ρ_{2pnd}-set of C_n. Therefore, $\rho_{2pnd}(C_n) = \lceil \frac{n}{2} \rceil$.

From the proof of Proposition 2, the next result follows.

Corollary 3. *Let n be a positive integer. Then*

(i) $\rho_{2pnd}(P_n) = \rho_2(P_n)$ *for all* $n \neq 3$ *and*

(ii) $\rho_{2pnd}(C_n) = \rho_2(C_n)$ *for all* $n \geq 3$.

The next result is found in [11].

Theorem 3. *Let G and H be any two graphs. A set $S \subseteq V(G + H)$ is hop dominating set of $G + H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets of G and H, respectively.*

Theorem 4. *Let G and H be any two graphs. A set $S \subseteq V(G + H)$ is geodetic hop dominating set of $G + H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets of G and H, respectively, such that*

(i) *S_G is a 2-path closure absorbing set in G whenever $\langle S_H \rangle$ is a complete subgraph of H and*

(ii) *S_H is a 2-path closure absorbing set in H whenever $\langle S_G \rangle$ is a complete subgraph of G.*

Proof. Suppose that S is a geodetic hop dominating set of $G + H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Since S is a hop dominating set, by Theorem 3, S_G and S_H are pointwise non-dominating sets of G and H, respectively. Next, suppose that $\langle S_H \rangle$ is a complete subgraph of H. If $S_G = V(G)$, then we are done. Suppose $S_G \neq V(G)$. Let $x \in V(G) \setminus S_G$. Since S is a geodetic set of $G + H$, there exist $y, z \in S$ such that $x \in I_{G+H}(y, z)$. Since $\langle S_H \rangle$ is complete, $y, z \in S_G$. Thus, $d_G(y, z) = 2$. Hence, S_G is 2-path closure absorbing set in G. Similarly, (ii) holds.

Conversely, suppose S satisfies the given conditions. By Theorem 3, S is a hop dominating set of $G + H$. Let $u \in V(G + H) \setminus S$. Suppose $u \in V(H) \setminus S_H$. If $\langle S_G \rangle$ is non-complete, then there exist $v, w \in S_G \subseteq S$ such that $d_{G+H}(v, w) = 2$. Hence, $u \in I_{G+H}(v, w)$. If $\langle S_G \rangle$ is complete, then there exist $s, t \in S_H \subseteq S$ such that $d_H(s, t) = 2$ and $u \in I_G(s, t) = I_{G+H}(s, t)$ by (ii). Similarly, there exist $p, q \in S$ such that $[p, u, q]$ is a geodesic in $G + H$ if $u \in V(G) \setminus S_G$. Therefore, S is a geodetic hop dominating set of $G + H$.

Lemma 1. *Let G be a non-complete graph. Then the following hold:*

(i) If D is a pnd-set of G such that $\langle D \rangle$ is complete, then $D \cup \{v\}$ is a pointwise non-dominating set and $\langle D \cup \{v\} \rangle$ is non-complete for every $v \in V(G) \setminus D$.

(ii) If E is a ρ_{2pnd}-set of G, then $\langle E \rangle$ is non-complete.

Proof.

(i) Let $v \in V(G) \setminus D$. Since D is a pointwise non-dominating set, $D \cup \{v\}$ is a pointwise non-dominating set and there exists $w \in D \setminus N_G(v)$. Therefore, $\langle D \cup \{v\} \rangle$ is non-complete.

(ii) If $E = V(G)$, then we are done. Suppose $E \neq V(G)$. Let $x \in V(G) \setminus E$. Since E is a 2-path closure absorbing, there exist $p, q \in E$ such that $d_G(p, q) = 2$ and $x \in I_G(p, q)$. Therefore, $\langle E \rangle$ is non-complete.

Before proceeding to the next result, we denote the family $\mathcal{C}$ of graphs by

$$\mathcal{C} = \{G : G \text{ has a } pnd\text{-set which induces a non-complete graph}\}.$$

Lemma 2. *Let G be a non-complete graph. If $G \notin \mathcal{C}$, then $pnd(G) < \rho_{2pnd}(G)$, that is, $pnd(G) + 1 \leq \rho_{2pnd}(G)$.*

Proof. Let S be a pnd-set of G. Then $\langle S \rangle$ is complete because $G \notin \mathcal{C}$. Therefore, $pnd(G) < \rho_{2pnd}(G)$ by Remark 3 and Lemma 1(ii).

Corollary 4. *Let G and H be any two non-complete graphs of orders m and n, respectively. Then*

$$\gamma_{hg}(G) = \begin{cases} pnd(G) + pnd(H), & \text{if } G, H \in \mathcal{C} \\ min\,\{\rho_{2pnd}(G) + pnd(H), pnd(G) + pnd(H) + 1\} & \text{if } G \in \mathcal{C} \text{ and } H \notin \mathcal{C} \\ min\,\{pnd(G) + \rho_{2pnd}(H), pnd(G) + pnd(H) + 1\} & \text{if } G \notin \mathcal{C}, H \in \mathcal{C} \\ min\{\rho_{2pnd}(G) + pnd(H), pnd(G) + \rho_{2pnd}(H) & \\ \qquad pnd(G) + pnd(H) + 2\} & \text{if } G, H \notin \mathcal{C}. \end{cases}$$

Proof. Let S be a γ_{hg}-set of $G+H$. Then $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$ are pointwise non-dominating sets of G and H, respectively, by Theorem 4. Hence, if G and H are in $\mathcal{C}$, then S_G and S_H are *pnd*-sets of G and H, respectively. Therefore, $\gamma_{hg}(G+H) = pnd(G) + pnd(H)$ if $G, H \in \mathcal{C}$.

Next, suppose that $G \in \mathcal{C}$ and $H \notin \mathcal{C}$. Let D_G be a ρ_{2pnd}-set of G and let D_H be a *pnd*-set of H. Then $\langle D_G \rangle$ is non-complete by Lemma 1(ii). Since $H \notin \mathcal{C}$, $\langle D_H \rangle$ is complete. Hence, $D_H \neq V(H)$ because H is non-complete. Let $w \in V(H) \setminus D_H$. By Lemma 1(i), $D_H' = D_H \cup \{w\}$ is a pointwise non-dominating set of H and $\left\langle D_H' \right\rangle$ is non-complete. Let D_G' be a *pnd*-set of G such that $\left\langle D_G' \right\rangle$ is non-complete. Then $S_1 = D_G \cup D_H$ and $S_2 = D_G' \cup D_H'$ are geodetic hop dominating sets of $G+H$ by Theorem 4. Thus, $\gamma_{hg}(G+H) \leq |S_1| = \rho_{2pnd}(G) + pnd(H)$ and $\gamma_{hg}(G+H) \leq |S_2| = pnd(G) + pnd(H) + 1$. Consequently,

$$\gamma_{hg}(G+H) \leq min\left\{\rho_{2pnd}(G) + pnd(H), pnd(G) + pnd(H) + 1\right\}.$$

Now, suppose that $S^* = S_G^* \cup S_H^*$ is a γ_{hg}-set of $G+H$. Then S_G^* and S_H^* satisfy the conditions in Theorem 4. Suppose $\rho_{2pnd}(G) + pnd(H) \leq pnd(G) + pnd(H) + 1$. If $\langle S_H^* \rangle$ is complete, then S_G^* is pointwise non-dominating 2-path closure absorbing set of G by Theorem 4. It follows that $\gamma_{hg}(G+H) = |S_G^*| + |S_H^*| \geq \rho_{2pnd}(G) + pnd(H)$. Suppose $\langle S_H^* \rangle$ is non-complete. Since $H \notin \mathcal{C}$, H is non-complete, and S^* is γ_{hg}-set of $G+H$, $|S_H^*| \geq pnd(H) + 1$ (see Lemma 1(i)). It follows that

$$\begin{aligned}
\gamma_{hg}(G+H) &= |S^*| \\
&= |S_G^*| + |S_H^*| \\
&\geq pnd(G) + pnd(H) + 1 \geq \rho_{2pnd}(G) + pnd(H).
\end{aligned}$$

Similar arguments may be used to show that $\gamma_{hg}(G+H) \geq pnd(G) + pnd(H) + 1$ if $pnd(G) + pnd(H) + 1 \leq \rho_{2pnd}(G) + pnd(H)$. Therefore,

$$\gamma_{hg}(G+H) = min\left\{\rho_{2pnd}(G) + pnd(H), pnd(G) + pnd(H) + 1\right\} \text{ if } G \in \mathcal{C} \text{ and } H \notin \mathcal{C}.$$

Similarly, $\gamma_{hg}(G+H) = min\left\{\rho_{2pnd}(H) + pnd(G), pnd(G) + pnd(H) + 1\right\}$ if $G \notin \mathcal{C}$ and $H \in \mathcal{C}$.

Suppose $G, H \notin \mathcal{C}$. Let

$$R = min\{\rho_{2pnd}(G) + pnd(H), pnd(G) + \rho_{2pnd}(H), pnd(G) + pnd(H) + 2\}.$$

Clearly,

$$\gamma_{hg}(G+H) \leq min\left\{\rho_{2pnd}(G) + pnd(H), \rho_{2pnd}(H) + pnd(G))\right\}.$$

Let S_1 and S_2 be *pnd*-sets of G and H, respectively. Let $S_1' = S_1 \cup \{p\}$ and $S_2' = S_2 \cup \{q\}$, where $p \in V(G) \setminus S_1$ and $q \in V(H) \setminus S_2$. Then S_1' and S_2' are pointwise non-dominating sets of G and H, respectively, and $\left\langle S_1' \right\rangle$ and $\left\langle S_2' \right\rangle$ are non-complete by Lemma 1(i). Hence,

$S^{'} = S_1^{'} \cup S_2^{'}$ is a geodetic hop dominating set of $G+H$ by Theorem 5. It follows that

$$\gamma_{hg}(G+H) \leq \left|S^{'}\right| = pnd(G) + pnd(H) + 2.$$

Therefore, $\gamma_{hg}(G+H) \leq R$.

Let $S_\circ = S_G^\circ \cup S_H^\circ$ be a γ_{hg}-set of $G+H$. Then S_H° and S_H° satisfy the conditions in Theorem 5. Consider the following cases:

Case 1. $\langle S_H^\circ \rangle$ is complete.

By Theorem 4, S_G° is a pointwise non-dominating 2-path closure absorbing set of G and $|S_G^\circ| \geq \rho_{2pnd}(G)$. Hence,

$$\begin{aligned} \gamma_{hg}(G+H) &= |S_\circ| = |S_G^\circ| + |S_H^\circ| \\ &\geq \rho_{2pnd}(G) + pnd(H) \geq R. \end{aligned}$$

Case 2. $\langle S_G^\circ \rangle$ is complete.

Then S_H° is a pointwise non-dominating and 2-path closure absorbing set of H. Hence,

$$\begin{aligned} \gamma_{hg}(G+H) &= |S_\circ| = |S_G^\circ| + |S_H^\circ| \\ &\geq \rho_{2pnd}(H) + pnd(G) \geq R. \end{aligned}$$

Case 3. $\langle S_G^\circ \rangle$ and $\langle S_H^\circ \rangle$ are non-complete.

Then $|S_G^\circ| \geq pnd(G) + 1$ and $|S_H^\circ| \geq pnd(H) + 1$ by Lemma 1(i). Hence,

$$\begin{aligned} \gamma_{hg}(G+H) &= |S_\circ| = |S_G^\circ| + |S_H^\circ| \\ &\geq pnd(G) + pnd(H) + 2 \geq R. \end{aligned}$$

Accordingly, $\gamma_{hg}(G+H) = R$.

Corollary 5. *Let G be a non-complete graph and n a positive integer. Then $S \subseteq V(K_n + G)$ is a geodetic hop dominating set of $K_n + G$ if and only if $S = V(K_n) \cup S_G$, where S_G is a pointwise non-dominating and 2-path closure absorbing set in G. In particular, $\gamma_{hg}(K_n + G) = n + \rho_{2pnd}(G)$.*

Corollary 6. *Let G and H be any two graphs of orders m and n respectively. Then*

(i) $\gamma_{hg}(G+H) = m+n$ *if G and H are complete;*

(ii) $\gamma_{hg}(K_{1,n-1}) = \gamma_{hg}(K_1 + \overline{K}_{n-1}) = n$ *for $n \geq 2$;*

(iii) $\gamma_{hg}(F_n) = 1 + \rho_{2pnd}(P_n)$;

(iv) $\gamma_{hg}(W_n) = 1 + \rho_{2pnd}(C_n)$; *and*

(v) $\gamma_{hg}(K_{m,n}) = \begin{cases} 3 & \text{if } m = 2 \text{ or } n = 2. \\ 4 & \text{otherwise.} \end{cases}$

4. Conclusion

A realization result involving the hop domination number and the geodetic hop domination number was obtained. This result shows that the difference of these two parameters can be made arbitrarily large. The concept of pointwise non-dominating 2-path closure absorbing set was defined and studied for some graphs. The geodetic hop dominating sets in the join of two graphs were characterized using the concept of 2-path closure absorbing pointwise non-dominating set. Complexity of the geodetic hop domination problem may be investigated and the parameter may studied for other graphs.

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