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$(\mathcal{H}, \Omega_b)$-Interpolative Contractions in Ω_b-Distance Mappings with Applications

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Abstract. Interpolative Kannan contractions are a refinement of Kannan contraction, which is considered as one of the significant notions in fixed point theory. G_b-metric spaces is considered as a generalized concept of both concepts b-metric and G-metric spaces therefore, the significant fixed and common fixed point results of the contraction based on this concept is generalized results for both concepts. The purpose of this manuscript, is to take advantage to interpolative Kannan contraction together with the notion of Ω_b which equipped with G_b-metric spaces and $\mathcal{H}$ simulation functions to formulate two new interpolative contractions namely, $(\mathcal{H}, \Omega_b)$-interpolative contraction for self mapping f and generalized $(\mathcal{H}, \Omega_b)$-interpolative contraction for pair of self mappings (f_1, f_2). We discuss new fixed and common fixed point theorems. Moreover, to demonstrate the applicability and novelty of our theorems, we formulate numerical examples and applications to illustrate the importance of fixed point theory in applied mathematics and other sciences.

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Key Words and Phrases: Ω_b distance mappings, Interpolative Kannan contractions, $\mathcal{H}$-simulation functions, G_b-metric spaces

1. Introduction and Mathematical Preliminaries

The study of fixed point theory has gained increasing importance and interest in pure and applied mathematics [8]–[17] ever since Banach came up with his result (Banach contraction principle) [4] which is considered to be one of the most important results in mathematics as well as other sciences. Since then, many mathematicians refined the result of Banach in two directions; some by replacing the frame of distance space such as b, G-metric spaces, modified ω, Ω-distance mappings (see [5]–[16]), and the others refined the contraction condition (for example see [18]–[15]).

Kannan contraction principle [12] is the first outstanding result after Banach contraction principle, and it is important to mention that this contraction characterizes the metric completeness. Many mathematicians improved this contraction; an interesting example of this improving is interpolative Kannan contractions [10, 13]. Since then, many significant

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contractions formulated based on interpolative contractions which utilized in the literature to investigate significant fixed and common fixed point results such as Debnath et.al. [7, 9]. In this study, our purpose is to formulate two significant interpolative contractions in the framework of Ω_b distance mappings which equipped with G_b-metric spaces where nontrivial generalisations are possible and as such, application of the results in relevant fields becomes feasible and easier.

Definition 1. *[10, 13] Suppose $(\mathcal{C}, d)$ is a metric space and f, g are two self mappings on $\mathcal{C}$ and $\lambda \in [0,1)$, $\alpha, \beta \in (0,1)$ where $\beta + \alpha < 1$. Then*

1. *We call f a (λ, α, β)-interpolative Kannan contraction if*
$$d(fc_1, fc_2) \leq \lambda d(c_1, fc_1)^{\alpha} d(c_2, fc_2)^{\beta}, \tag{1}$$
$\forall\ c_1, c_2 \in \mathcal{C}$ *with* $fc_1 \neq c_1$ *and* $fc_2 \neq c_2$.
2. *We call the pair (f, g) a (λ, α, β)-interpolative Kannan contraction pair if*
$$d(fc_1, gc_2) \leq \lambda d(c_1, fc_2)^{\alpha} d(c_2, gc_2)^{\beta}, \tag{2}$$
$\forall\ c_1, c_2 \in \mathcal{C}$ *with* $fc_1 \neq c_1$ *and* $gc_2 \neq c_2$.

The concept of G_b space has been formulated by a pioneer mathematician, Aghajani et al. [2], providing a generalization of the standard concepts of G-metric space which are formulated by Mustafa and Sims [14] and b-metric space , which is formulated by Bakhtin [3] as follows:

Definition 2. *[2] Let $\mathcal{C}$ be a non-empty set and $b \in [1, +\infty)$. Assume that the function $G_b : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to [0, +\infty)$ fulfills the following conditions:*

1. $G_b(c, c^{'}, c^{''}) = 0$ *if and only if* $c = c^{'} = c^{''}$;
2. $G_b(c, c, c^{'}) \geq 0$ *for all* $c, c^{'} \in \mathcal{C}$ *with* $c \neq c^{'}$;
3. $G_b(c, c^{'}, c^{'}) \leq G_b(c, c^{'}, c^{''})$ *for all* $c, c^{'}, c^{''} \in \mathcal{C}$ *with* $c^{'} \neq c^{''}$;
4. $G_b(c, c^{'}, c^{''}) = G_b(p\{c, c^{'}, c^{''}\})$ *where p is a permutation of* $c, c^{'}, c^{''}$;
5. $G_b(c, c^{'}, c^{''}) \leq b[G_b(c, a, a) + G_b(a, c^{'}, c^{''})]$ *for all* $c, c^{'}, c^{''}, a \in \mathcal{C}$.

Then G_b is called G_b-metric on $\mathcal{C}$ and the pair $(\mathcal{C}, G_b)$ is called G_b-metric spaces.

Example 1. *[2] If $(\mathcal{C}, G)$ is G-metric space and $p \in (1, +\infty)$. Define $G_b : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to [0, +\infty)$ via $G_b(c_1, c_2, c_3) = (G(c_1, c_2, c_3))^p$. Then G_b is G_b-metric space with the base $b = 2^{p-1}$.*

Henceforth, $(\mathcal{C}, G_b)$ refers to G_b-metric spaces on the set $\mathcal{C}$.
In the sub-sequence, $\mathcal{C}$ refers to non empty set and Λ_f refers to the set of all fixed points of f in $\mathcal{C}$.
The concepts of G_b-completeness and G_b-convergence are as below:

Definition 3. *[2] Assume (c_n) be a sequence in $(\mathcal{C}, G_b)$. Then the sequence (c_n) is a:*

1. *G_b-Cauchy sequence if $\forall \epsilon > 0$ there is $N \in \mathbb{N}$ such that $\forall n, m, l \geq N$, $G(c_n, c_m, c_l) < \epsilon$;*
2. *G_b-convergent sequence to c if $\forall \epsilon > 0$ there is $N \in \mathbb{N}$ such that $\forall n, m \geq N$, $G(c, c_n, c_m) < \epsilon$;*
3. *G_b-complete if $\forall$ G_b-Cauchy sequence, then G_b is convergent.*

Remark 1. *A sequence (c_n) in $(\mathcal{C}, G_b)$ is G_b-convergent sequence if one of the following conditions is true:*

(1) $G_b(c_n c, c) \to 0$ as $n \to +\infty$;

(2) $G_b(c_n, c_n, c) \to 0$ as $n \to +\infty$.

The concept Ω_b distance mappings (Generalized Ω distance mappings) was introduced by Abodayeh et.al. [1] and they utilized this concept to unify some fixed point results in the literature.

Definition 4. *[1] An Ω_b-distance mappings on $(\mathcal{C}, G_b)$ is a function $\Omega_b : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to [0, +\infty)$ fulfill:*

1. *$\Omega_b(c, c^{'}, c^{''}) \leq b[\Omega_b(c, a, a) + \Omega_b(a, c^{'}, c^{''})]$ for all $c, c^{'}, c^{''}, a \in \mathcal{C}$, $b \in [0, +\infty)$;*
2. *$\forall c, c^{'} \in \mathcal{C}$, $\Omega_b(c, c^{'}, .), \Omega_b(c, ., c^{'}) : \mathcal{C} \to \mathcal{C}$ are lower semi-continuous;*
3. *$\forall \epsilon > 0$ there is an $\alpha > 0$, if $\Omega_b(c, a, a) \leq \alpha$ and $\Omega_b(a, c^{'}, c^{''}) \leq \alpha$, then $G_b(c, c^{'}, c^{''}) \leq \epsilon$, $\forall$ $c, c^{'}, c^{''} \in \mathcal{C}$.*

Definition 5. *If Ω_b distance mappings is equipped with $(\mathcal{C}, G_b)$, then we call $\mathcal{C}$ bounded w.r.t. Ω_b if there exists $L \geq 1$ with $\Omega_b(c_1, c_2, c_3) \leq L$ for all $c_1, c_2, c_3 \in \mathcal{C}$.*

The concept of $\mathcal{H}$-simulation functions which formulated by Bataihah et.al in 2020 is as belows:

Definition 6. *[6] A set of functions $\{h : [1, +\infty) \times [1, +\infty) \to \mathbb{R}\}$ is called $\mathcal{H}$-simulation functions if*

$$h(c, c^{'}) \leq \frac{c^{'}}{c} \ \forall c, c^{'} \in [1, +\infty). \tag{3}$$

Remark 2. *[6] If $h \in \mathcal{H}$ and $(c_n), (c_n^{'})$ are sequences in $[1, +\infty)$ with $1 \leq \lim_{n \to +\infty} c_n^{'} < \lim_{n \to +\infty} c_n$, then*

$$\limsup_{n \to +\infty} h(c_n, c_n^{'}) < 1. \tag{4}$$

Definition 7. *[6, 11] The class of functions: $\{\theta : [0, +\infty) \to [1, +\infty)\}$, θ is continuous and none decreasing functions fulfill the condition:*
$\forall (c_n)$ a sequence in $[0, +\infty)$, $\lim_{n \to +\infty} \theta(c_n) = 1$ if and only if $\lim_{n \to +\infty} c_n = 0$.
Is said to be Θ class

Remark 3. *[11] If $\theta \in \Theta$, then θ^{-1} ($\{1\}$)=0.*

2. Main Results

We start our main results with the following concepts and definitions

Definition 8. *Suppose $(\mathcal{C}, G_b)$ is equipped with Ω_b-distance mappings. A mapping $f : \mathcal{C} \to \mathcal{C}$ is said to be $(\mathcal{H}, \Omega_b)$-interpolative contraction if there are $b \in [1, +\infty)$, $\lambda_i \in (0,1)$ with $i \in \{1,2,3\}$ and $\lambda_2 + \lambda_3 < 1$, $\theta \in \Theta$ and $h \in \mathcal{H}$ such that $\forall$ $c_1, c_2, c_3 \in \mathcal{C}$ we have:*

$$1 \leq h\Bigg(\theta b \Omega_b(fc_1, f^2c_1, fc_2), \theta\lambda_1\Gamma(c_1, c_2, c_3)\Bigg). \tag{5}$$

Where

$$\Gamma(c_1, c_2, c_3) = \max\Bigg\{\Omega_b(c_1, fc_1, c_2), [\Omega_b(c_1, fc_1, fc_1)]^{\lambda_2}[\Omega_b(c_2, fc_2, fc_2)]^{\lambda_3}\Bigg\}.$$

Lemma 1. *Suppose the self function $f : \mathcal{C} \to \mathcal{C}$ fulfills the conditions of $(\mathcal{H}, \Omega_b)$-interpolative contraction. Then*

1. $\Gamma(c_1, c_2, c_3) > 0 \implies \Omega_b(fc_1, f^2c_1, fc_2) \leq \dfrac{\lambda_1}{b}\Gamma(c_1, c_2, c_3)$;
2. $\Gamma(c_1, c_2, c_3) = 0 \implies \Omega_b(fc_1, f^2c_1, fc_2) = 0$.

Proof. (1) If $\Gamma(c_1, c_2, c_3) > 0$, then

$$\begin{aligned} 1 &\leq H(\theta b\Omega_b(fc_1, f^2c_1, fc_2), \theta\lambda_1\Gamma(c_1, c_2, c_3)) \\ &\leq \frac{\theta\lambda_1\Gamma(c_1, c_2, c_3)}{\theta b\Omega_b(fc_1, f^2c_1, fc_2)}. \end{aligned}$$

This implies that, $\theta b\Omega_b(fc_1, f^2c_1, fc_2) \leq \theta\lambda_1\Gamma(c_1, c_2, c_3)$. Due to the fact that the set Θ is a non-decreasing function, we conclude:
$\Omega_b(fc_1, f^2c_1, fc_2) \leq \dfrac{\lambda_1}{b}\Gamma(c_1, c_2, c_3)$. Hence the result.
(2) If $\Gamma(c_1, c_2, c_3) = 0$, then by utilizing condition (1), we have:

$$1 \leq \theta b\Omega_b(fc_1, f^2c_1, fc_2) \leq \theta\lambda\Gamma(c_1, c_2, c_3) = 1.$$

Thus, $\Omega_b(fc_1, f^2c_1, fc_2) = 0$.

Lemma 2. *Suppose the self function $f : \mathcal{C} \to \mathcal{C}$ fulfills the conditions of $(\mathcal{H}, \Omega_b)$-interpolative contraction. Then Λ_f has at most one element.*

Proof. To prove that Λ_f has at most one element, first we claim that, $\Omega_b(\alpha, \alpha, \alpha) = 0$ $\forall \alpha \in \Lambda_f$. Assume $\Omega_b(\alpha, \alpha, \alpha) > 0$ for some $\alpha \in \Lambda_f$, then by employing Lemma 1 we get:

$$\begin{aligned} \Omega_b(f\alpha, f^2\alpha, f\alpha) &\leq \frac{\lambda_1}{b}\Gamma(\alpha, \alpha, \alpha) \\ &= \frac{\lambda_1}{b}\max\{\Omega_b(\alpha, f\alpha, \alpha), [\Omega_b(\alpha, f\alpha, f\alpha)]^{\lambda_2}[\Omega_b(\alpha, f\alpha, f\alpha)]^{\lambda_3}\} \\ &< \Omega_b(\alpha, \alpha, \alpha). \end{aligned}$$

A contradiction. Hence the result.
Now assume that there is $c^*, \alpha \in \Lambda_f$ with $c^* \neq \alpha$, assume that $\Omega_b(c^*, c^*, \alpha) > 0$, so by Lemma 1 we have:

$$\begin{aligned}\Omega_b(c^*, c^*, \alpha) = \Omega_b(fc^*, f^2c^*, f\alpha) \quad &\leq \frac{\lambda_1}{b}\Gamma(c^*, c^*, \alpha) \\ &= \frac{\lambda_1}{b}\max\{\Omega_b(c^*, fc^*, \alpha), [\Omega_b(c^*, c^*, c^*)]^{\lambda_2}[\Omega_b(\alpha, \alpha, \alpha)]^{\lambda_3}\} \\ &< \Omega_b(c^*, c^*, \alpha).\end{aligned}$$

A contradiction. Therefore, $\Omega_b(c^*, c^*, \alpha) = 0$ and by utilizing the definition of of Ω_b (condition (3)) and since $\Omega_b(c^*, c^*, c^*) = 0$, we conclude that $G_b(c^*, c^*, \alpha) = 0$ therefore, $c^* = \alpha$.

For an arbitrary point $c_0 \in \mathcal{C}$ the Picard sequence is defined by iterating $f : \mathcal{C} \to \mathcal{C}$ where $c_{n+1} = f(c_n) = f^n(c_0)$. Henceforth, we mean by the sequence c_n the Picard sequence unless otherwise stated.

Lemma 3. *Suppose the self function $f : \mathcal{C} \to \mathcal{C}$ fulfills the conditions of $(\mathcal{H}, \Omega_b)$-interpolative contraction and suppose that for some $k \in \mathbb{N}$ we have $\Omega_b(c_{k-1}, c_k, c_k) = 0$. Then, $\Lambda_f = \{c_k\}$*

Proof. Note that

$$\Gamma(c_{k-1}, c_k, c_k) = \frac{\lambda_1}{b}\max\left\{\Omega_b(c_{k-1}, c_k, c_k), [\Omega_b(c_{k-1}, c_k, c_k)]^{\lambda_2}[\Omega_b(c_k, c_{k+1}, c_{k+1})]^{\lambda_3}\right\} = 0.$$

So, by Lemma 1, we get that $\Omega_b(c_k, c_{k+1}, c_{k+1}) = \Omega_b(c_{k-1}, c_k, c_k) = 0$. In a similar manner, we can verify that $\Omega_b(c_{k+1}, c_{k+2}, c_{k+2}) = 0$. By utilizing the definition of Ω_b, we conclude that $G_b(c_{k-1}, c_{k+1}, c_{k+1}) = 0$ and so $c_{k-1} = c_{k+1}$. In a typical way, we can prove that $c_k = c_{k+2}$.
Now, by employing the triangle inequality of Ω_b, we get

$$\begin{aligned}\Omega_b(c_k, c_k, c_k) \quad &\leq b[\Omega_b(c_k, c_{k+1}, c_{k+1}) + \Omega_b(c_{k+1}, c_k, c_k)] \\ &= b[\Omega_b(c_k, c_{k+1}, c_{k+1}) + \Omega_b(c_{k+1}, c_{k+2}, c_{k+2})] \\ &= 0.\end{aligned} \tag{6}$$

From inequality (6) and $\Omega_b(c_k, c_{k+1}, c_{k+1}) = 0$, we conclude that $c_k \in \Lambda_f$ and Lemma 2 ensures that c_k is the unique element in Λ_f.

Theorem 1. *Suppose $(\mathcal{C}, G_b)$ is G_b-complete equipped with Ω_b distance mappings with the base $b \in [1, +\infty)$ and $\mathcal{C}$ is bounded w.r.t. Ω_b. Suppose there are $\lambda_i \in (0, 1), i \in \{1, 2, 3\}$ with $\lambda_2 + \lambda_3 < 1$, $\theta \in \Theta$, $h \in \mathcal{H}$ such that the mapping $f : \mathcal{C} \to \mathcal{C}$ is a $(\mathcal{H}, \Omega_b)$-interpolative contraction if one of the following conditions is fulfilled:*
1. The self mapping f is a continuous;
2. For all $c^* \in \mathcal{C}$ if $fc^* \neq c^*$, then $0 < inf\{\Omega_b(c, fc, c^*) : c \in \mathcal{C}\}$,
then Λ_f has only one element.

Proof. Let $c_0 \in \mathcal{C}$ and start by the Picard sequence (c_n). Without lose of generality, we may assume that $\forall n \in \mathbb{N}$, we have $\Omega_b(c_n, c_{n+1}, c_{n+1}) > 0$. So, by Lemma 1, we have

$$\Omega_b(c_n, c_{n+1}, c_{n+1}) \leq \frac{\lambda_1}{b} \max\left\{\Omega_b(c_{n-1}, c_n, c_n), [\Omega_b(c_{n-1}, c_n, c_n)]^{\lambda_2}[\Omega_b(c_n, c_{n+1}, c_{n+1})]^{\lambda_3}\right\}. \tag{7}$$

If $\max\left\{\Omega_b(c_{n-1}, c_n, c_n), [\Omega_b(c_{n-1}, c_n, c_n)]^{\lambda_2}[\Omega_b(c_n, c_{n+1}, c_{n+1})]^{\lambda_3}\right\} = \Omega_b(c_{n-1}, c_n, c_n)$. Therefore, we get

$$\Omega_b(c_n, c_{n+1}, c_{n+1}) \leq \frac{\lambda_1}{b}\Omega_b(c_{n-1}, c_n, c_n); \tag{8}$$

else, we have

$$[\Omega_b(c_n, c_{n+1}, c_{n+1})]^{1-\lambda_3} \leq \frac{\lambda_1}{b}[\Omega_b(c_{n-1}, c_n, c_n)]^{\lambda_2} < \frac{\lambda_1}{b}[\Omega_b(c_{n-1}, c_n, c_n)]^{1-\lambda_3}. \tag{9}$$

From the inequalities (8) and (9), we conclude

$$\begin{aligned} \Omega_b(c_n, c_{n+1}, c_{n+1}) &\leq \frac{\lambda_1}{b}\Omega_b(c_{n-1}, c_n, c_n) \\ &\vdots \\ &\leq (\frac{\lambda_1}{b})^n \Omega_b(c_0, c_1, c_1). \end{aligned} \tag{10}$$

Then there is $L \geq 1$ such that

$$\Omega_b(c_n, c_{n+1}, c_{n+1}) \leq (\frac{\lambda_1}{b})^n L. \tag{11}$$

To show that the iterative sequence (c_n) is G_b-Cauchy, first we prove that $\forall\, m, l \in \mathbb{N}$ with $m \leq l$ we have:

$$\Omega_b(c_{m-1}, c_m, c_l) \leq (\frac{\lambda_1}{b})^{m-1} L. \tag{12}$$

Now,

$$\Omega_b(c_{m-1}, c_m, c_l) \leq \frac{\lambda_1}{b} \max\left\{\Omega_b(c_{m-2}, c_{m-1}, c_{l-1}), [\Omega_b(c_{m-2}, c_{m-1}, c_{m-1})]^{\lambda_2}[\Omega_b(c_{l-1}, c_l, c_l)]^{\lambda_3}\right\}. \tag{13}$$

Assume that $l = m + t$ for some $t \in \mathbb{N}$. Then

$$\begin{aligned} \Omega_b(c_{l-1}, c_l, c_l) &\leq \frac{\lambda_1}{b} \max\left\{\Omega_b(c_{l-2}, c_{l-1}, c_{l-1}), [\Omega_b(c_{l-2}, c_{l-1}, c_{l-1})]^{\lambda_2}[\Omega_b(c_{l-1}, c_l, c_l)]^{\lambda_3}\right\} \\ &= \frac{\lambda_1}{b}\Omega_b(c_{l-2}, c_{l-1}, c_{l-1}) \\ &\leq (\frac{\lambda_1}{b})^t \Omega_b(c_{m-1}, c_m, c_m). \end{aligned} \tag{14}$$

Now,

$$\begin{aligned}\Omega_b(c_{m-1},c_m,c_l) &\leq \frac{\lambda_1}{b}\max\Big\{\Omega_b(c_{m-2},c_{m-1},c_{l-1}),[\Omega_b(c_{m-2},c_{m-1},c_{m-1})]^{\lambda_2+\lambda_3}\Big\}\\ &\leq \frac{\lambda_1}{b}\max\Big\{\frac{\lambda_1}{b}\max\{\Omega_b(c_{m-3},c_{m-2},c_{l-2}),[\Omega_b(c_{m-3},c_{m-2},c_{m-2})]^{\lambda_2+\lambda_3}\},\\ &\quad [\Omega_b(c_{m-2},c_{m-1},c_{m-1})]^{\lambda_2+\lambda_3}\Big\}\\ &\leq (\frac{\lambda_1}{b})^2\Big\{\Omega_b(c_{m-3},c_{m-2},c_{l-2}),[\Omega_b(c_{m-3},c_{m-2},c_{m-2})]^{\lambda_2+\lambda_3}\Big\}\\ &\vdots\\ &\leq (\frac{\lambda_1}{b})^{m-1}\Big\{\Omega_b(c_0,c_1,c_t),[\Omega_b(c_0,c_1,c_1)]^{\lambda_2+\lambda_3}\Big\}\\ &\leq (\frac{\lambda_1}{b})^{m-1}L.\end{aligned} \tag{15}$$

Now, by employing inequalities (11), (12) and condition (1) of the the definition of Ω_b $\forall n < m \leq l$, we get:

$$\begin{aligned}\Omega_b(c_n,c_m,c_l) &\leq b\Omega_b(c_n,c_{n+1},c_{n+1}) + b\Omega_b(c_{n+1},c_m,c_l)\\ &\leq b\Omega_b(c_n,c_{n+1},c_{n+1}) + b^2\Omega_b(c_{n+1},c_{n+2},c_{n+2}) + b^2\Omega_b(c_{n+2},c_m,c_l)\\ &\vdots\\ &\leq b\Omega_b(c_n,c_{n+1},c_{n+1}) + b^2\Omega_b(c_{n+1},c_{n+2},c_{n+2}) + \cdots\\ &\quad + b^{m-n-1}\Omega_b(c_{m-2},c_{m-1},c_{m-1}) + b^{m-n-1}\Omega_b(c_{m-1},c_m,c_l)\\ &\leq b(\frac{\lambda_1}{b})^nL + b^2(\frac{\lambda_1}{b})^{n+1}L + \cdots + b^{m-n-1}(\frac{\lambda_1}{b})^{m-1}L\\ &= bL(\frac{\lambda_1}{b})^n\Big[1+\lambda_1+\lambda_1^2+\cdots+\lambda_1^{m-n-1}\Big]\\ &= bL(\frac{1-\lambda_1^{m-n}}{1-\lambda_1})(\frac{\lambda_1}{b})^n.\end{aligned} \tag{16}$$

By taking the limit as $n \to +\infty$ in above inequality, we find out that (c_n) is a G_b-Cauchy sequence, and since $(\mathcal{C}, G_b)$ is G_b- complete, then there is $c^* \in \mathcal{C}$ s.t. the sequence (c_n) is G_b-convergent to c^*. If f is any continuous mapping, then $fc^* = c^*$. Else, by utilizing the lower semi continuity of Ω_b, we obtain:

$$\Omega_b(c_n,c_m,c^*) \leq \lim_{t\to+\infty}\Omega_b(c_n,c_m,c_t) < \epsilon \ for\ all\ n,m \geq N\ \forall\ \epsilon > 0. \tag{17}$$

Suppose that $m = n+1$. Then $\Omega_b(c_n,c_{n+1},c^*) \leq \lim_{t\to+\infty}\Omega_b(c_n,c_{n+1},c_t) < \epsilon\ \forall n \geq N$.
If $fc^* \neq c^*$, we obtain:

$$0 < \inf\{\Omega_b(c,fc,c^*) : c \in \mathcal{C}\} \leq \inf\{\Omega_b(c_n,c_{n+1},c^*) : n \in \mathbb{N}\} < \epsilon\ \forall\ \epsilon > 0, \tag{18}$$

a contradiction. Hence, $c^* \in \Lambda_f$, the uniqueness follows from Lemma 2. This is complete the proof.

In the next two examples we consider the following:
Define $h : [1,+\infty) \times [1,+\infty) \to [0,+\infty)$, $\theta : [0,+\infty) \to [1,+\infty)$ via $h(c_1, c_2) = \dfrac{c_2}{c_1}$, $\theta(\omega) = e^{\omega}, \forall \omega \in \mathcal{C}$ respectively, then $h \in \mathcal{H}$ *and* $\theta \in \Theta$.
Also, define: $G_b : \mathcal{C}\times\mathcal{C}\times\mathcal{C} \to [0,+\infty)$ by $G_b(c_1, c_2, c_3) = (|c_1 - c_2| + |c_2 - c_3| + |c_1 - c_3|)^2$, then , G_b is a complete with the base $b = 2$.
Moreover, define $\Omega_b : \mathcal{C}\times\mathcal{C}\times\mathcal{C} \to [0,+\infty)$ by $\Omega_b(c_1, c_2, c_3) = (|c_1 - c_2| + |c_1 - c_3|)^2$, Ω_b is a generalized Ω-distance mapping equipped with G_b.

Example 2. *Suppose $\mathcal{C} = \{0, 1, ..., 10\}$, define mapping $f : \mathcal{C} \to \mathcal{C}$ via :*

$$fc = \begin{cases} 0, & c \in \{0, 1, 2\}; \\ 1, & c \in \{3, 4, 5\}; \\ 2, & c \in \{6, 7, ..., 10\}. \end{cases}$$

Then Λ_f has only one element.
To prove this, we need to show that $\forall\ c_1, c_2 \in \mathcal{C}$, we have

$$1 \le h(\theta b\Omega_b(fc_1, f^2c_1, fc_2), \theta\lambda_1\Gamma(c_1, c_2, c_3)).$$

First it is not hard to prove

$$\Omega_b(fc_1, f^2c_1, fc_2) \le 0.45 \max\left\{\Omega_b(c_1, fc_1, c_2), [\Omega_b(c_1, fc_1, fc_1)]^{0.45}[\Omega_b(c_2, fc_2, fc_2)]^{0.45}\right\}.$$

Now,

$$\Omega_b(fc_1, f^2c_1, fc_2) \le \frac{\lambda_1}{b} \max\left\{\Omega_b(c_1, fc_1, c_2), [\Omega_b(c_1, fc_1, fc_1)]^{\lambda_2}[\Omega_b(c_2, fc_2, fc_2)]^{\lambda_3}\right\}$$

$$\iff \theta b\Omega_b(fc_1, f^2c_1, fc_2) \le \theta\lambda_1\Gamma(c_1, c_2, c_3))$$
$$\iff 1 \le H(\theta b\Omega_b(fc_1, f^2c_1, fc_2), \theta\lambda_1\Gamma(c_1, c_2, c_3)).$$

Consequently, f satisfy all conditions of $(\mathcal{H}, \Omega_b)$-interpolative contraction. Theorem 1 confirms that Λ_f has only one element.

Example 3. *Consider the following mapping*

$$f(c) = \frac{1 - c^m}{B + c^m} \text{ where } m \in \mathbb{N} - \{1\} \text{ and } B \ge \sqrt{2}m.$$

Then Λ_f has only one element on $[0,1]$.
To prove this, let $\mathcal{C} = [0,1]$ for all $c_1, c_2, c_3 \in \mathcal{C}$, assume $fc = s$. Then

$$\begin{aligned}
\Omega_b(fc_1, f^2c_1, fc_2) &= \left[\left|\frac{1-c_1^m}{B+c_1^m} - \frac{1-s^m}{B+s^m}\right| + \left|\frac{1-c_1^m}{B+c_1^m} - \frac{1-c_2^m}{B+c_2^m}\right|\right]^2 \\
&= \left[\frac{1}{(B+c_1^m)(B+s^m)}\left|(1-c_1^m)(B+s^m) - (1-s^m)(B+c_1^m)\right|\right. \\
&\quad \left. + \frac{1}{(B+c_1^m)(B+c_2^m)}\left|(1-c_1^m)(B+c_2^m) - (1-c_2^m)(B+c_1^m)\right|\right]^2 \\
&\leq \frac{(B-1)^2}{B^4}\left[|c_1^m - s^m| + |c_1^m - c_2^m|\right]^2 \\
&= \frac{(B-1)^2 m^2}{B^4}\left[|c_1 - s| + |c_1 - c_2|\right]^2 \\
&\leq \frac{(B-1)}{2B^2}\left[|c_1 - fc_1| + |c_1 - c_2|\right]^2 \\
&= \frac{\lambda_1}{b}\Omega_b(c_1, fc_1, c_2).
\end{aligned}$$

Notice that $\lambda_1 = (\dfrac{B-1}{B})^2$ and the base $b = 2$.
Now,

$$\begin{aligned}
b\Omega_b(fc_1, f^2c_1, fc_2) \leq \lambda_1\Omega_b(c_1, fc_1, c_2) &\leq \lambda_1\Gamma(c_1, c_2, c_3) \\
\iff e^{b\Omega_b(fc_1, f^2c_1, fc_2)} &\leq e^{\lambda_1\Gamma(c_1,c_2,c_3)} \\
\iff 1 &\leq \frac{e^{\lambda_1\Gamma(c_1,c_2,c_3)}}{e^{b\Omega_b(fc_1,f^2c_1,fc_2)}} \\
\iff 1 &\leq H(\theta b\Omega_b(fc_1, f^2c_1, fc_2), \theta\lambda_1\Gamma(c_1, c_2, c_3).
\end{aligned}$$

Consequently, f satisfy all conditions of $(\mathcal{H}, \Omega_b)$-interpolative contraction. Theorem 1 confirms that Λ_f has only one element.

Definition 9. *Suppose that $(\mathcal{C}, G_b)$ is equipped with Ω_b-distance mappings and f_1, f_2 are two self mapping on $\mathcal{C}$. We called the pair (f_1, f_2) is a generalized $(\mathcal{H}, \Omega_b)$-interpolative contraction if there exist $b \in [1, +\infty)$, $\lambda_i \in (0,1)$ with $i \in \{1,2,3\}$ and $\lambda_2 + \lambda_3 < 1$, $\theta \in \Theta$ and $h \in \mathcal{H}$ s.t. $\forall$ $c_1, c_2, c_3 \in \mathcal{C}$ we have:*

$$1 \leq h\Big(\theta b\Omega_b(f_1c_1, f_2(f_1c_1), f_2c_2), \theta\lambda_1\Gamma_1(c_1, c_2, c_3)\Big); \tag{19}$$

and

$$1 \leq h\Bigg(\theta b\Omega_b(f_2c_1, f_1(f_2c_1), f_1c_2), \theta\lambda_1\Gamma_2(c_1, c_2, c_3)\Bigg). \tag{20}$$

Where

$$\Gamma_1(c_1, c_2, c_3) = \max\Bigg\{\Omega_b(c_1, f_2c_1, c_2), [\Omega_b(c_1, f_1c_1, f_1c_1)]^{\lambda_2}[\Omega_b(c_2, f_2c_2, f_2c_2)]^{\lambda_3}\Bigg\};$$

and

$$\Gamma_2(c_1, c_2, c_3) = \max\Bigg\{\Omega_b(c_1, f_1c_1, c_2), [\Omega_b(c_1, f_2c_1, f_2c_1)]^{\lambda_2}[\Omega_b(c_2, f_1c_2, f_1c_2)]^{\lambda_3}\Bigg\}.$$

Theorem 2. *Suppose $(\mathcal{C}, G_b)$ is G_b-complete equipped with Ω_b distance mappings with the base $b \in [1, +\infty)$ and $\mathcal{C}$ is bounded w.r.t. Ω_b. Suppose there are $\lambda_i \in (0, 1), i \in \{1, 2, 3\}$ with $\lambda_2 + \lambda_3 < 1$, $\theta \in \Theta$, $h \in \mathcal{H}$ s.t. the pair of self mappings $f_1, f_2 : \mathcal{C} \to \mathcal{C}$ is a generalized $(\mathcal{H}, \Omega_b)$-interpolative contraction if one of the following fulfilled:*
1. If the mappings f_1, f_2 are continuous;
2. If one of the self mappings is continuous and for all $c^* \in \mathcal{C}$ if $f^*c^* \neq c^*$,
then $0 < inf\{\Omega_b(c, f^*c, c^*) : c \in \mathcal{C}\}$, where f^* refers to non-continuous function f_1 or f_2.
then Λ_f has only one element.

Proof. We start our proof our by setting a constructive sequence $(c_n) \in \mathcal{C}$ by iterating $c_{2n+1} = f_1c_{2n}$ *and* $c_{2n+2} = f_2c_{2n+1}$ *for* $n \in \mathbb{N}$ for some arbitrary element $c_0 \in \mathcal{C}$. So we have
$\Omega_b(c_{2n+1}, c_{2n+2}, c_{2n+2}) = \Omega_b(f_1c_{2n}, f_2(f_1c_{2n}), f_2c_{2n+1})$, and so

$$\begin{aligned} 1 &\leq H\Bigg(\theta b\Omega_b(c_{2n+1}, c_{2n+2}, c_{2n+2}), \theta\lambda_1\Gamma(c_{2n}, c_{2n}, c_{2n+1})\Bigg) \\ &\leq \frac{\theta\lambda_1 \max\Bigg\{\Omega_b(c_{2n}, c_{2n+1}, c_{2n+1}), [\Omega_b(c_{2n}, c_{2n+1}, c_{2n+1})]^{\lambda_2}[\Omega_b(c_{2n+1}, c_{2n+2}, c_{2n+2})]^{\lambda_3}\Bigg\}}{\theta b\Omega_b(c_{2n+1}, c_{2n+2}, c_{2n+2})}. \end{aligned} \tag{21}$$

Therefore,

$$\begin{aligned} \Omega_b(c_{2n+1}, c_{2n+2}, c_{2n+2}) \quad &\leq \frac{\lambda_1}{b}\max\Bigg\{\Omega_b(c_{2n}, c_{2n+1}, c_{2n+1}), \\ &\qquad [\Omega_b(c_{2n}, c_{2n+1}, c_{2n+1})]^{\lambda_2}[\Omega_b(c_{2n+1}, c_{2n+2}, c_{2n+2})]^{\lambda_3}\Bigg\}. \end{aligned}$$

By employing the inequalities (8) and (9), we conclude that

$$\Omega_b(c_{2n+1}, c_{2n+2}, c_{2n+2}) \leq \frac{\lambda_1}{b}\Omega_b(c_{2n}, c_{2n+1}, c_{2n+1}). \tag{22}$$

By utilizing typical way, we can easily show that

$$\Omega_b(c_{2n+2}, c_{2n+3}, c_{2n+3}) \leq \frac{\lambda_1}{b}\Omega_b(c_{2n+1}, c_{2n+2}, c_{2n+2}). \tag{23}$$

Hence, we get

$$\Omega_b(c_{n+1}, c_{n+2}, c_{n+2}) \leq \frac{\lambda_1}{b}\Omega_b(c_n, c_{n+1}, c_{n+1}). \tag{24}$$

The completion of the proof of this Theorem is identical to the Theorem 1, and this is complete the proof.

3. Application

Throughout this application, we will emphasize the significant idea that the solution of a fixed point equation (uniqueness and existence) under certain conditions is often comparable to that of other equations.

Consider the following equation:

$$c^{m+1} + c^m + Bc - 1, where\ B \geq \sqrt{2}\ m,\ m \in \mathbb{N} - \{1\}, \tag{25}$$

has a unique solution in the unit interval $[0, 1]$.

To prove this, it is typical to prove that the following self mapping f has a unique fixed point in $[0, 1]$.

$$f(c) = \frac{1 - c^m}{B + c^m}, B \geq \sqrt{2}\ m,\ m \in \mathbb{N} - \{1\}.$$

Example 3 confirms that the self mapping f has a unique fixed point and hence, the Equation (25) has a unique solution.

Next, we discuss an application on Theorem 1. We employ Theorem 1 to prove the uniqueness and existence of a solution for Volterra type integral equation:

$$\eta(t) = \eta_0 + \int_{t_0}^{t} H(r, \eta(r))dr. \tag{26}$$

Suppose that $\|.\|_\infty$ is the superior norm on $C[0, 1]$ which is defined by $\|v\|_\infty = \sup_{t\in[0,1]} v(t)$.

In this application, we consider that $\mathcal{C} = C[0, 1]$ and G_b, Ω_b as follows:

$$G_b(u, v, w) = (\|u-v\|_\infty + \|v-w\|_\infty + \|u-w\|_\infty)^2,\ \Omega_b(u, v, w) = (\|u-v\|_\infty + \|u-w\|_\infty)^2. \tag{27}$$

Next, we have the following theorem:

Theorem 3. *Suppose that $H : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function on $[0, 1] \times \mathbb{R}$ and t_0 is the interior point in $[0, 1]$ and suppose that $\alpha_0 > 0$ such that the function H fulfills the following:*

$$|H(t, u) - H(t, v)| \leq \alpha_0|u - v|\ for\ all\ u, v \in \mathbb{R}\ and\ for\ all\ t \in [0, 1]. \tag{28}$$

Then the integral equation $f\eta(t) = \eta_0 + \int_{t_0}^{t} H(r, \eta(r))dr$ has a unique solution.

Proof. Let $\epsilon > 0$ with $\epsilon < \sqrt{\frac{\lambda_1}{b\alpha_0^2}}$. Define the self mapping $f : C[0,1] \to C[0,1]$ via

$$f\eta(t) = \eta_0 + \int_{t_0}^{t} H(r, \eta(r))dr. \tag{29}$$

Then we show that f satisfies the condition (8) on the interval $C_0 = [t_0, t_0 + \epsilon]$.
It suffices to show that:

$$\Omega_b(fu, f^2u, fv) \leq \frac{\lambda_1}{b}\Omega_b(u, fu, v). \tag{30}$$

Now, for all $u, v \in C[0,1]$, we obtain:

$$\begin{aligned}
\|fu - fv\|_\infty &= \sup_{t\in C_0}|fu(t) - fv(t)| \\
&= \sup_{t\in C_0}|\textstyle\int_{t_0}^{t}(H(r,u(r)) - H(r,v(r)))dr| \\
&\leq \sup_{t\in C_0}\textstyle\int_{t_0}^{t}|(H(r,u(r)) - H(r,v(r)))dr| \\
&\leq \sup_{t\in C_0}\alpha_0|u(t) - v(t)|\textstyle\int_{t_0}^{t} dr \\
&= \alpha_0\|u - v\|_\infty(t - t_0) \\
&= \epsilon\alpha_0\|u - v\|_\infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\|fu - f^2u\|_\infty + \|fu - fv\|_\infty)^2 &= (\sup_{t\in C_0}|fu(t) - f^2u(t)| + \sup_{t\in C_0}|fu(t) - fv(t)|)^2 \\
&= \Big(\sup_{t\in C_0}|\textstyle\int_{t_0}^{t}(H(r,u(r)) - H(r,fu(r))dr| \\
&\quad + \sup_{t\in C_0}|\textstyle\int_{t_0}^{t}(H(r,u(r)) - H(r,v(r))dr|\Big)^2 \\
&\leq (\epsilon\alpha_0)^2(\|u - fu\|_\infty + \|u - v\|_\infty)^2.
\end{aligned}$$

Now, set $\frac{\lambda_1}{b} = (\epsilon\alpha_0)^2$, we get the desire result.

4. Conclusion

In this manuscript, we formulated two significant interpolative contractions namely, $(\mathcal{H}, \Omega_b)$-interpolative contraction for self mapping f and generalized $(\mathcal{H}, \Omega_b)$-interpolative contraction for pair of self mappings (f_1, f_2). By employing these contractions we unify new fixed and common fixed results. We formulated some numerical examples and applications to show the novelty of our results; one of these applications based on the significant idea that the solution of a equation in a certain conditions is typical to solution of fixed point equation. we utilized this idea to prove that this equation not only has solution as the Intermediate value Theorem says but also, this solution is unique. This research can be improved by utilizing the concept of extended G_b-metric spaces.

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