2-Locating Sets in a Graph

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Abstract. Let G be an undirected graph with vertex-set V(G) and edge-set E(G), respectively. A set S ⊆ V(G) is a 2-locating set of G if |[(NG(x)\NG(y)) ∩ S] ∪ [(NG(y)\NG(x)) ∩ S]| ≥ 2, for all x, y ∈ V(G)\S with x ≠ y, and for all v ∈ S and w ∈ V(G)\S, (NG(v)\NG(w)) ∩ S ≠ φ or (NG(w)\NG(v)) ∩ S ≠ φ. In this paper, we investigate the concept and study 2-locating sets in graphs resulting from some binary operations. Specifically, we characterize the 2-locating sets in the join, corona, edge corona and lexicographic product of graphs, and determine bounds or exact values of the 2-locating number of each of these graphs.

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1. Introduction

Resolving sets and metric basis are emphasized for their application in computer science, medical sciences and chemistry. The locating set in graphs can be viewed as the set of monitors that can determine the exact location of an intruder. The concept of 2-locating set is obtained from the concept of locating set. Requiring such a set to be 2-locating implies that every pair of vertices where there is no monitor must be connected to at least two monitoring devices that are connected to other monitors. Also, for every vertex and monitoring device there exists at least one monitor that is connected to it. Hence, 2-locating set can be viewed as the set of monitors that can determine the presence of an intruder.

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In 1975, Slater [19] introduced the concept of locating sets and its minimum cardinality as locating number. Harary and Melter also utilized a similar idea, although they referred to the locating set and the locating number, respectively, using the terms resolving set and metric dimension. Resolving sets and locating sets, however, are defined differently in more recent studies. In 2013, Bailey et al. [1] defined a resolving set as a set of vertices \( S \) in a graph \( G \) such that for any two vertices \( u, v \), there exists \( x \in S \) such that the distance \( d(u, x) \neq d(v, x) \). On the other hand, Canoy and Malacas [8] defined a locating set as a set \( S \subseteq V(G) \) of \( G \) such that for every two distinct vertices \( u \) and \( v \) of \( V(G) \setminus S \), \( N_G(u) \cap S \neq N_G(v) \cap S \). Other variations of locating sets are studied in [16], [6], [11], [13], [14], [15] and [7].

In 2021, J. Cabaro and H. Rara [5] studied the idea of the 2-resolving sets in the join and corona of graphs wherein they introduced the idea of 2-locating sets. This work is therefore motivated by the recent studies on these variations of 2-resolving set and 2-metric dimension that utilize the concepts of 2-locating set and 2-locating number. Other studies that deal with the concept of 2-locating sets are located in [6], [9], [10], [12] and [18].

2. Terminology and Notation

In this study, we consider finite, simple, connected, undirected graphs. For basic graph-theoretic concepts, we then refer readers to [3] and [4]. The following concepts are found in [2], [3], [5] and [17] respectively.

The open neighborhood of a vertex \( v \) in a graph \( G \) is defined as the set \( N_G(v) = \{ u \in V(G) : uv \in E(G) \} \), while the closed neighborhood of a vertex \( v \) in \( G \) is defined as \( N_G[v] = N_G(v) \cup \{ v \} \). The open neighborhood of a set \( S \subseteq V(G) \) is defined as \( N_G(S) = \bigcup_{v \in S} N_G(v) \), while its closed neighborhood is \( N_G[S] = N_G(S) \cup S \). A connected graph \( G \) of order \( n \geq 3 \) is point distinguishing if for any two distinct vertices \( u \) and \( v \) of \( G \), \( N_G[u] \neq N_G[v] \). It is totally point determining if for any two distinct vertices \( u \) and \( v \) of \( G \), \( N_G(u) \neq N_G(v) \) and \( N_G[u] \neq N_G[v] \).

For an ordered set of vertices \( W = \{ w_1, w_2, ..., w_k \} \subseteq V(G) \) and a vertex \( v \) in \( G \), we refer to the \( k \)-vector (ordered \( k \)-tuple)

\[
r_G(v/W) = (d_G(v, w_1), d_G(v, w_2), ..., d_G(v, w_k))
\]

as the (metric) representation of \( v \) with respect to \( W \). The set \( W \) is called a resolving set for \( G \) if distinct vertices have distinct representations with respect to \( W \). Hence, if \( W \) is a resolving set of cardinality \( k \) for a graph \( G \) of order \( n \), then the set \( \{ r_G(v/W) : v \in V(G) \} \) consists of \( n \) distinct \( k \)-vectors. A resolving set of minimum cardinality is called a minimum resolving set or a basis, and the cardinality of a basis for \( G \) is the dimension \( \dim(G) \) of \( G \). An ordered set of vertices \( W = \{ w_1, ..., w_k \} \) is a \( k \)-resolving set for \( G \) if, for any distinct vertices \( u, v \in V(G) \), the (metric) representations \( r_G(u/W) \) and \( r_G(v/W) \) of \( u \) and \( v \), respectively, differ in at least \( k \) positions. If \( k = 1 \), then the \( k \)-resolving set is called a resolving set for \( G \). If \( k = 2 \), then the \( k \)-resolving set is called a 2-resolving set for \( G \). If \( G \) has a \( k \)-resolving set, the minimum cardinality \( \dim_k(G) \) of a \( k \)-resolving set is called
the $k$-metric dimension of $G$. If $G$ has a 2-resolving set, we denote the least size of a 2-resolving set by $\dim_2(G)$ is called a 2-metric dimension of $G$. A resolving set of size $\dim_2(G)$ is called a 2-metric basis for $G$.

Let $G$ be any nontrivial connected graph and $S \subseteq V(G)$. A set $S \subseteq V(G)$ is a 2-locating set of $G$ if it satisfies the following conditions:

(i) $|(N_G(x) \setminus N_G(y)) \cap S| \cup |(N_G(y) \setminus N_G(x)) \cap S| \geq 2$, for all $x, y \in V(G) \setminus S$ with $x \neq y$.

(ii) $(N_G(v) \setminus N_G(w)) \cap S \neq \emptyset$ or $(N_G(w) \setminus N_G(v)) \cap S \neq \emptyset$, for all $v \in S$ and for all $w \in V(G) \setminus S$.

The 2-locating number of $G$, denoted by $\ln_2(G)$, is the smallest cardinality of a 2-locating set of $G$. A 2-locating set of $G$ of cardinality $\ln_2(G)$ is referred to as an $\ln_2$-set of $G$.

A set $S \subseteq V(G)$ is a $(2,2)$-locating (or $(2,1)$-locating, respectively) set in $G$ if $S$ is 2-locating and $|N_G(y) \cap S| \leq |S| - 2$ or $|N_G(y) \cap S| \leq |S| - 1$, respectively, for all $y \in V(G)$. The $(2,2)$-locating (or $(2,1)$-locating, respectively) number of $G$, denoted by $\ln_{(2,2)}(G)$ ($\ln_{(2,1)}(G)$, respectively), is the smallest cardinality of a $(2,2)$-locating (or $(2,1)$-locating, respectively) set in $G$. A $(2,2)$-locating $(2,1)$-locating, respectively) set in $G$ of cardinality $\ln_{(2,2)}(G)$ ($\ln_{(2,1)}(G)$, respectively) is referred to as an $\ln_{(2,2)}$-set ($\ln_{(2,1)}$-set, respectively) in $G$.

3. Known Results

The following known results are taken from [5].

**Remark 1.** For any connected nontrivial graph $G$ of order $n \geq 2$, $2 \leq \ln_2(G) \leq n$.

Moreover, $\ln_2(K_n) = n$, for $n \geq 2$.

**Theorem 1.** Let $G$ be a connected nontrivial graph. Then $\ln_2(G) = 2$ if and only if $G \cong P_2$ or $G \cong P_3$.

**Remark 2.** Let $S \subseteq V(G)$ for any pair of vertices $x, y \in S$, $r(x/S)$ and $r(y/S)$ differ in at least 2 positions. Hence, to prove that $S$ is a 2-resolving set in $G$, we only need to show that for every pair of vertices $x, y \in V(G)$ where $x \in S$ and $y \in V(G) \setminus S$ or both $x, y \in V(G) \setminus S$, $r(x/S)$ and $r(y/S)$ differ in at least 2 positions.

**Remark 3.** Every 2-locating set in $G$ is a 2-resolving set in $G$. However, a 2-resolving set in $G$ need not be a 2-locating set in $G$. Thus,

$$\dim_2(G) \leq \ln_2(G).$$

4. Preliminary Results

Every nontrivial connected graph $G$ admits a 2-locating set. Indeed, the vertex-set of $G$ is a 2-locating set.

**Proposition 1.** For any connected graph $G$ of order $n \geq 2$, $2 \leq \ln_2(G) \leq n$. Moreover,
Remark 4. [5] Every 2-locating set of a connected graph \( G \) is 2-resolving. Thus, \( \text{dim}_2(G) \leq \text{ln}_2(G) \).

Theorem 2. Let \( a \) and \( b \) be any positive integers such that \( 2 \leq a \leq b \). Then there exists a connected graph \( G \) such that \( \text{dim}_2(G) = a \) and \( \text{ln}_2(G) = b \).

Proof. Suppose that \( a = b \). Consider the graph \( G = K_a \). Then \( \text{dim}_2(G) = \text{ln}_2(G) = a \). Next, suppose that \( a < b \). Consider the following cases:

Case 1: \( a = 2 \)
Let \( m = b - a \) and consider the graph \( G \) in Figure 2. Let \( S_1 = \{x_1, x_2\} \) and \( S_2 = S_1 \cup \{v_1, v_2, \ldots, v_m\} \). Then \( S_1 \) and \( S_2 \) are \( \text{dim}_2 \)-set and \( \text{ln}_2 \)-set of \( G \), respectively. Hence, \( \text{dim}_2(G) = a \) and \( \text{ln}_2(G) = a + m = b \).

Case 2: \( a \geq 3 \)
Let \( m = b - a \) and consider the graph \( G' \) in Figure 3. Let \( S_1 = \{y_1, y_2, \ldots, y_a\} \) and \( S_2 = S_1 \cup \{u_1, u_2, \ldots, u_m\} \). Then \( S_1 \) and \( S_2 \) are \( \text{dim}_2 \)-set and \( \text{ln}_2 \)-set of \( G' \), respectively. Hence, \( \text{dim}_2(G') = a \) and \( \text{ln}_2(G') = a + m = b \).
Corollary 1. For each positive integer \( n \), there exists a connected graph \( G \) such that \( ln_2(G) - \text{dim}_2(G) = n \), that is, \( ln_2 - \text{dim}_2 \) can be made arbitrarily large.

We now characterize the 2-locating sets in some graphs under some binary operations.

5. Join of Graphs

This section presents the characterizations on the 2-locating sets in the join of graphs.

**Theorem 3.** [6] Let \( G \) and \( H \) be nontrivial connected graphs. A proper subset \( S \) of \( V(G+H) \) is a 2-resolving set in \( G+H \) if and only if \( S_G = V(G) \cap S \) and \( S_H = V(H) \cap S \) are 2-locating sets in \( G \) and \( H \), respectively, where \( S_G \) or \( S_H \) is (2,2)-locating set or \( S_G \) and \( S_H \) are (2,1)-locating sets.

**Theorem 4.** [6] Let \( G \) be a connected graph of order greater than 3 and let \( K_1 = \{v\} \). Then \( S \subseteq V(K_1 + G) \) is a 2-resolving set of \( K_1 + G \) if and only if either \( v \notin S \) and \( S \) is a (2,2)-locating set in \( G \) or \( S = \{v\} \cup T \) is (2,1)-locating set in \( G \).

**Theorem 5.** Let \( G \) and \( H \) be connected graphs. Then \( S \subseteq V(G+H) \) is a 2-locating set in \( G+H \) if and only if \( S \) is a 2-locating set in \( G+H \) where \( S = S_G \cup S_H \), \( S_G \subseteq V(G) \) and \( S_H \subseteq V(H) \).

**Proof.** Suppose \( S \) is a 2-locating set in \( G+H \). Let \( p, q \in V(G+H) \). Consider \( p, q \in V(G+H) \setminus S \) or \( \{p \in V(G+H) \setminus S \text{ or } q \in S\} \). Since \( S \) is a 2-locating set, then \( r_{G+H}(p/S) \) and \( r_{G+H}(q/S) \) differ in at least 2 positions. By Definition of 2-locating set and Remark 2, \( S \) is a 2-locating set.

For the converse, suppose \( S \) is a 2-locating set in \( G+H \). Let \( S = S_G \cup S_H \) where \( S_G \subseteq V(G) \) and \( S_H \subseteq V(H) \). Let \( p, q \in V(G+H) \setminus S \). Consider the following cases.

**Case 1** \( p, q \in V(G) \) \( \setminus S_G \).

Since \( S \) is a 2-locating set, \( r_{G+H}(p/S) \) and \( r_{G+H}(q/S) \) differ in at least 2 positions. By definition of \( G+H \), \( r_{G}(p/S_G) \) and \( r_{G}(q/S_G) \) differ in at least 2 positions. Since \( d_{G+H}(p, u) \) and \( d_{G+H}(q, u) \) is either 0,1 or 2 for each \( u \in V(G+H) \), there exist at least two vertices \( x, y \in S_G \) such that \( x, y \in N_G(p) \setminus N_G(q) \) or \( x, y \in N_G(q) \setminus N_G(p) \) or \( x \in N_G(p) \setminus N_G(q) \) and \( y \in N_G(q) \setminus N_G(p) \). Hence,

\[
\left| (N_{G+H}(p) \setminus N_{G+H}(q)) \cap S \right| \cup \left| (N_{G+H}(q) \setminus N_{G+H}(p)) \cap S \right| \geq 2. \tag{1}
\]

**Case 2** \( p, q \in V(H) \) \( \setminus S_H \)

Proof is similar to Case 1.
Corollary 3. Let \( G \) be any nontrivial connected graph. Then
\[
ln_2(K_1 + G) = \min\{ln_{(2,2)}(G), ln_{(2,1)}(G) + 1\}.
\]

Corollary 4. Let \( G \) and \( H \) be nontrivial connected graphs. A set \( S \subseteq V(G + H) \) is a 2-locating set in \( G + H \) if and only if \( S = S_G \cup S_H \) where \( S_G = V(G) \cap S \) and \( S_H = V(H) \cap S \) are 2-locating sets of \( G \) and \( H \), respectively, where \( S_G \) or \( S_H \) is a \( (2, 2) \)-locating set or \( S_G \) and \( S_H \) are \( (2, 1) \)-locating sets.

Corollary 5. Let \( G \) and \( H \) be nontrivial connected graphs. Then
\[
ln_2(G + H) = \min\{ln_{(2,2)}(G) + ln_2(H), ln_2(G) + ln_{(2,2)}(H), ln_{(2,1)}(G) + ln_{(2,1)}(H)\}.
\]
6. Corona of Graphs

This section presents the characterizations on the 2-locating sets in the corona of graphs.

Theorem 6. Let $G$ and $H$ be nontrivial connected graphs with $\Delta(H) \leq |V(H)| - 3$. A set $S \subseteq V(G \circ H)$ is a 2-locating set of $G \circ H$ if and only if $S = A \cup \bigcup_{v \in V(G)} S_v$ where $A \subseteq V(G)$ and $V(H^v) \cap S \neq \emptyset$ for each $v \in V(G)$ and the following are satisfied

(i) $S_v$ is a 2-locating set of $H^v$ for each $v \in V(G)$ and $S_u$ or $S_v$ is total 2-dominating for $u, v \in V(G) \setminus A$ or otherwise, $S_u$ and $S_v$ are total dominating;

(ii) for each $v \in V(G) \setminus A$, $S_v$ is a $(2,2)$-locating set of $H^v$ with $N_G(v) \cap A = \emptyset$ and $S_v$ is $(2,1)$-locating set, otherwise; and

(iii) for each $v \in A$, $S_v$ is a $(2,1)$-locating set of $H^v$ if $N_G(v) \cap A = \emptyset$.

Proof. Suppose $S \subseteq V(G \circ H)$ is a 2-locating set in $G \circ H$. Let $A = V(G) \cap S$, $S_v = S \cap V(H^v)$ for all $v \in V(G)$. Then $S = A \cup \bigcup_{v \in V(G)} S_v$ where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$. Now, suppose $S_v = \emptyset$ for some $v \in V(G)$. Let $x, y \in V(H^v) \setminus S_v$. Then $|(N_H(x) \setminus N_H(y)) \cap S_v| \cup |(N_H(y) \setminus N_H(x)) \cap S_v| = 0$, for all $x, y \in V(H^v) \setminus S_v$ with $x \neq y$, a contradiction to the assumption of $S_v$. Thus, $S_v \neq \emptyset$ for all $v \in V(G \circ H)$.

To prove (i), let $x, y \in V(H^v)$ where $v \in V(G)$. Then $x, y \in V(G \circ H)$. Since $N_H(x) = N_{G \circ H}(x) \setminus \{v\}$ and $N_H(y) = N_{G \circ H}(y) \setminus \{v\}$, and $S$ is a 2-locating set, this implies that $S_v$ is also 2-locating set in $H^v$. Next, suppose $S_u$ or $S_v$ is not a total dominating, say $S_v$ is not a total dominating set for some $v \in V(G) \setminus A$. Let $x \in V(H^v) \setminus S_u$ and $y \in V(H^v) \setminus S_v$. Since $S$ is a 2-locating set, there exist $w, z \in (N_H(x) \setminus N_H(y)) \cap S_u$ implying that $S_u$ is a total 2-dominating set.

To prove (ii), let $v \in V(G) \setminus A$. Suppose $N_G(v) \cap A = \emptyset$. Since $S_v \subseteq N_{G \circ H}(v)$ and $S$ is 2-locating, there exist at least two vertices $x, y \in S_v \setminus N_{H^v}(p)$ for each $p \in V(H^v)$. Thus, $S_v$ is $(2,2)$-locating set. On the other hand, if $N_G(v) \cap A \neq \emptyset$, there exists at least one vertex $z \in S_v \setminus N_{H^v}(p)$. This implies that $S_v$ is $(2,1)$-locating.

To prove (iii), let $v \in A$ and $N_G(v) \cap A = \emptyset$. Since $S_v$ is a 2-locating set, there exists $r \in S_v \setminus N_{H^v}(p)$ for every $p \in V(H^v)$. Thus, $S_v$ is a $(2,1)$-locating set in $H^v$.

For the converse, suppose $S$ is a set as described and satisfies the given conditions. Let $p, q \in V(G \circ H)$ with $p \neq q$ and let $u, v \in V(G)$ such that $p \in V(u + H^v)$ and $q \in V(v + H^v)$. Suppose $p, q \in V(G \circ H) \setminus S$. Consider the following cases:

Case 1. $u = v$

Subcase 1.1 $p, q \in V(H^v) \setminus S_u$

Since $S_u$ is a 2-locating set of $H^v$, $N_{H^v}(p) = N_{G \circ H}(p)$ and $N_{H^v}(q) = N_{G \circ H}(q)$. Then

$$|[N_{G \circ H}(q) \setminus N_{G \circ H}(p)] \cap S] \cup [(N_{G \circ H}(p) \setminus N_{G \circ H}(q)) \cap S]| \geq 2$$
and for all $r \in S_u$, $(N_{G \circ H}(r) \setminus N_{G \circ H}(q)) \cap S \neq \emptyset$. Thus, $S$ is a 2-locating set.

**Subcase 2.1** $p = v$ and $q \in V(H^u) \setminus S_v$

If $N_G(v) \cap A = \emptyset$, by (ii) $S_v$ is a $(2,1)$-locating set. Hence, there exist at least two distinct vertices $x, y \in V(H^u) \setminus N_{H^u}(q)$. Thus, $x, y \in N_{G \circ H}(p) \setminus N_{G \circ H}(q)$. If $N_G(v) \cap A \neq \emptyset$, then there exists $z \in (N_{G \circ H}(v) \cap A) \setminus N_{G \circ H}(q)$. Since $\gamma(H) \neq 1$, there exists $w \in S_v \setminus N_{H^u}(q)$. Hence, $w, z \in N_{G \circ H}(p) \setminus N_{G \circ H}(q) \cap S$. Thus, $|(N_{G \circ H}(p) \setminus N_{G \circ H}(q) \cap S)| \geq 2$.

**Subcase 1.3** $q = v$ and $p \in V(H^u) \setminus S_u$

The proof is similar to the proof of Subcase 1.2.

**Case 2.** $u \neq v$

**Subcase 2.1** $p \in V(H^u) \setminus S_u$ and $q \in V(H^v) \setminus S_v$

If $u, v \in A$, then we are done. Suppose $u, v \notin A$. Since $S_u$ and $S_v$ are total dominating, there exist $x \in (N_{H^u}(p) \setminus S_u) \setminus N_{H^u}(q)$ and $y \in (N_{H^u}(q) \setminus S_u) \setminus N_{H^u}(p)$.

**Subcase 2.2** $p = u$ and $q \in V(H^v) \setminus S_v$

Since $p \notin A$, $S_u$ is a total dominating set of $H^u$. Hence, $|S_u| \geq 2$. Thus, $|(N_{G \circ H}(p) \setminus N_{G \circ H}(q)) \cap S| \geq 2$.

Suppose $p \in S$ and $q \in V(G \circ H) \setminus S$. Consider the following cases

**Case 1** $u = v$

**Subcase 1.1** $p \in S_v$ and $q \in V(H^v) \setminus S_v$

Since $S_v$ is a 2-locating set, then $(N_{G \circ H}(p) \setminus N_{G \circ H}(q)) \cap S \neq \emptyset$.

**Subcase 1.2** $u = v$ and $q \in V(H^v) \setminus S_v$

If $N_G(p) \cap A \neq \emptyset$, then we are done. Suppose $N_G(p) \cap A = \emptyset$. Then by (iii), $S_v$ is a $(2,1)$-locating set. Thus, $(N_{G \circ H}(p) \setminus N_{G \circ H}(q)) \cap S \neq \emptyset$.

**Case 2** $u \neq v$

**Subcase 2.1** $p \in S_u$ and $q \in V(H^v) \setminus S_v$

If $u \in A$ or $v \in A$, then we are done. If $u, v \notin A$, then by (i) $S_u$ and $S_v$ are total dominating. Hence, there exist $x \in (N_{G \circ H}(p) \setminus S) \setminus N_{G \circ H}(q)$ and $y \in (N_{G \circ H}(q) \cap S) \setminus N_{G \circ H}(p)$.

**Subcase 2.2** $p = u$ and $q \in V(H^v) \setminus S_v$

Since $S_u \neq \emptyset$, $(N_{G \circ H}(p) \cap S) \setminus N_{G \circ H}(q) \neq \emptyset$.

**Subcase 2.3** $p \in S_u$ and $q = v$

Similar to the proof of Subcase 2.2.

Accordingly, $S$ is a 2-locating set of $G \circ H$.

**Corollary 6.** Let $G$ of order $n$ and $H$ be nontrivial connected graphs with $\gamma(H) \neq 1$. Then

(i) $\ln_2(G \circ H) \leq \gamma_1(G) + n \cdot \ln_2(H)$; and

(ii) If $\ln_2(H) = \ln_{(2,1)}(H) = \ln_{(2,2)}(H)$. Then $\ln_2(G \circ H) = n \cdot \ln_2(H)$.

**Proof.** (i) Let $S = V(G \circ H)$ be a 2-locating set in $G \circ H$. Let $A$ be a $\gamma_1$-set of $G$ and $S_v$ be an $\ln_2$-set of $H^v$. Then $S = A \cup \left( \bigcup_{v \in V(G)} S_v \right)$ is a 2-locating set of $G \circ H$. Thus,
\[ \ln_2(G \circ H) \leq |S| \]
\[ = |A| + \sum_{v \in V(G)} |S_v| \]
\[ = \gamma_t(G) + |V(G)|(\ln_2(H)) = \gamma_t(G) + n \cdot \ln_2(H). \]

(ii.) Let \( A = \emptyset \) and \( S_v \) be an \( \ln_2 \)-set of \( H^v \). Then \( S = A \cup \left( \bigcup_{v \in V(G)} S_v \right) \) is a 2-locating set of \( G \circ H \). Thus,
\[ \ln_2(G \circ H) \leq |S| \]
\[ = \sum_{v \in V(G)} |S_v| \]
\[ = |V(G)|\ln_2(H) = n \cdot \ln_2(H). \]

Next, let \( S_0 \) be an \( \ln_2 \)-set in \( G \circ H \). Then by Theorem 6,
\[ S_0 = A_0 \cup \left( \bigcup_{v \in V(G)} S_v \right) \] where \( A_0 \subseteq V(G) \) and \( S_v \) is a 2-locating set of \( H^v \), for all \( v \in V(G) \). Thus,
\[ \ln_2(G \circ H) = |S_0| \]
\[ = |A_0| + \bigcup_{v \in V(G)} S_v | \]
\[ \geq \sum_{v \in V(G)} |S_v| \]
\[ \geq n \cdot \ln_2(H) \]

Thus, equality holds.

7. Edge Corona of Graphs

This section presents characterizations on the 2-locating sets in the edge corona of graphs.

**Theorem 7.** Let \( G \) and \( H \) be nontrivial connected graphs where \( G \neq P_2 \) and \( \Delta(H) \leq |V(H)| - 3 \). A set \( C \subseteq V(G \circ H) \) is a 2-locating set of \( G \circ H \) if and only if \( C \) is a 2-resolving set of \( G \circ H \).

**Proof.** Let \( C \) be a 2-locating set of \( G \circ H \). By Remark 3, \( C \) is a 2-resolving set of \( G \circ H \).
Conversely, suppose $C$ is a 2-resolving set of $G \circ H$. Let $a, b \in V(H^{uw}) \setminus S_{uv}$ where $a \neq b$ or $[a \in S_{uv}$ and $b \notin S_{uv}]$. Since $C$ is a 2-resolving set in $G \circ H$, $r_{G \circ H}(a/C)$ and $r_{G \circ H}(b/C)$ differ in at least 2 positions. Since $N_{G \circ H}(a) = N_{H^{uw}}(a) \cup \{u, v\}$ and $N_{G \circ H}(b) = N_{H^{uw}}(b) \cup \{u, v\}$, $r_{H^{uw}}(a/S_{uv})$ and $r_{H^{uw}}(b/S_{uv})$ must differ in at least 2 positions. By definition of $G \circ H$, there exist at least two vertices say $p, q \in V(H^{uw}) \cap S_{uv}$ such that either $p, q \in N_{H^{uw}}(a) \setminus N_{H^{uw}}(b)$ or $p, q \in N_{H^{uw}}(b) \setminus N_{H^{uw}}(a)$ or $p \in N_{H^{uw}}(a) \setminus N_{H^{uw}}(b)$ and $q \in N_{H^{uw}}(b) \setminus N_{H^{uw}}(a)$. Similarly, if $a \in S_{uv}$ and $b \in V(H^{uw}) \setminus S_{uv}$, then there exists a vertex $s \in V(H^{uw}) \cap S_{uv}$ such that $s \in N_{H^{uw}}(a) \setminus N_{H^{uw}}(b)$ or $s \in N_{H^{uw}}(b) \setminus N_{H^{uw}}(a)$. Thus, it follows that $S_{uv}$ is a 2-locating set of $H^{uw}$.

Accordingly, $C$ is a 2-locating set in $G \circ H$. □

**Theorem 8.** Let $G$ and $H$ be any nontrivial connected graphs where $G \neq P_2$ and $\Delta(H) \leq |V(H)| - 3$. A set $C \subseteq V(G \circ H)$ is a 2-locating set of $G \circ H$ if and only if

$$C = A \cup \left( \bigcup_{uv \in E(G)} S_{uv} \right)$$

where

(i) $A \subseteq V(G)$, $S_{uv} \subseteq V(H^{uw})$ and $V(H^{uw}) \cap C \neq \emptyset$;

(ii) $S_{uv} \subseteq V(H^{uw})$ is a 2-locating set of $H^{uw}$ for all $uv \in E(G)$ or if $uv$ is a pendant edge, then $S_{uv}$ is a $(2,1)$-locating set of $H^{uw}$ whenever $l(\{u, v\}) \subseteq A$ and $S_{uv}$ is a $(2,2)$-locating set of $H^{uw}$ otherwise.

**Proof.** Suppose that $C \subseteq V(G \circ H)$ is a 2-locating set in $G \circ H$. Let $A = V(G) \cap C$ and $S_{uv} = C \cap V(H^{uw})$ for all $uv \in E(G)$. Then $C = A \cup \left( \bigcup_{uv \in E(G)} S_{uv} \right)$ where $A \subseteq V(G)$ and $S_{uv} \subseteq V(H^{uw})$. Now, suppose that $S_{uv} = \emptyset$ for some $uv \in E(G)$. Let $x, y \in V(H^{uw}) \setminus S_{uv}$. Then $|N_{H^{uw}}(x) \setminus N_{H^{uw}}(y)| \cap S_{uv} \cup |N_{H^{uw}}(y) \setminus N_{H^{uw}}(x)| \cap S_{uv}| = 0$, a contradiction to the assumption of $C$. Thus, $S_{uv} \neq \emptyset$ for all $uv \in E(G)$. Next, we claim that $S_{uv}$ is a 2-locating set in $H^{uw}$ for each $uv \in E(G)$. Let $a, b \in V(H^{uw})$ where $uv \in E(G)$. Then $a, b \in V(G \circ H)$. Since $N_{H^{uw}}(a) = N_{G \circ H}(a) \setminus \{u, v\}$ and $N_{H^{uw}}(b) = N_{G \circ H}(b) \setminus \{u, v\}$ and $C$ is a 2-locating set, this implies that $S_{uv}$ is also a 2-locating set in $H^{uw}$. Next, suppose that $uv$ is a pendant edge and suppose $u$ is an end-vertex. Then $u + H^{uw}$ is a subgraph of $G \circ H$. Since $S_{uv} = C \cap V(H^{uw}) \subseteq C$ and $C$ is a 2-locating set, it follows by Corollary 2, $S_{uv}$ is a $(2,1)$-locating set of $H^{uw}$ whenever $u \in C$ and $S_{uv}$ is a $(2,2)$-locating set of $H^{uw}$, otherwise.

Conversely, let $C$ be the set as described and satisfies the given conditions. Let $x, y \in V(G \circ H)$ with $x \neq y$. Then it can be easily verified that $r_{G \circ H}(x/C)$ and $r_{G \circ H}(y/C)$ differ in at least two positions for all $x, y \in V(G)$ or $x \in V(H^{uw})$ and $y \in V(G)$ for all edges $uv \in E(G)$ or $x \in V(H^{ab})$ and $y \in V(H^{ab})$, for some $pq, ab \in E(G)$. Hence, consider only the following cases:
Let $\mathcal{S}$ be an endvertex.

**Case 1:** $x, y \in V(H) \setminus S_{uv}$ or $x \in V(H) \setminus S_{uv}$ and $y \in S_{uv}$ for some edge $uv \in E(G)$. Now, since $S_{uv}$ is 2-locating set, $r_{H}(x/S_{uv})$ and $r_{H}(y/S_{uv})$ differ in at least two positions. Thus, it follows that $r_{G \circ H}(x/C)$ and $r_{G \circ H}(y/C)$ differ in at least two positions.

**Case 2:** $x \in V(H) \setminus S_{uv}$ or $x \in S_{uv}$ and $y = u$ for some pendant edge $uv \in E(G)$ and $u$ is an endvertex.

Since $S_{uv}$ is a $(2,2)$-locating set, there exists $a, b \in S_{uv} \setminus N_{H}(uv)$ such that $a, b \in N_{G \circ H}(y)$. Thus, it follows that $r_{G \circ H}(x/C)$ and $r_{G \circ H}(y/C)$ differ at $a^{th}$ and $b^{th}$ positions. Therefore, $C$ is a 2-resolving set in $G \circ H$. By Theorem 7, $C$ is a 2-locating set in $G$.

**Corollary 7.** Let $G$ and $H$ be any nontrivial connected graphs where $G \neq P_{2}$ with $|E(G)| = m$ and $\Delta(H) \leq |V(H)| - 3$. Then the following statements hold.

(i) If $G$ is a graph with no pendant edges, then $ln_{2}(G \circ H) = m \cdot ln_{2}(H)$.

(ii) If $G$ is a graph with $k \geq 1$ pendant edges, then

\[
ln_{2}(G \circ H) = \min\{ (m - k)ln_{2}(H) + k \cdot ln_{2,1}(H), (m - k)ln_{2}(H) + k \cdot ln_{2,2}(H) \}
\]

and

\[
ln_{2}(G \circ H) = (m - k)ln_{2}(H) + k \cdot ln_{2,2}(H)
\]

whenever $ln_{2,2}(H) = ln_{2,1}(H)$.

**Proof.** (i) Suppose $G$ is a graph with no pendant edges. Now, set $A = \emptyset$ and let $S_{uv}$ be an $ln_{2}$-set of $H_{uv}$ for all $uv \in E(G)$. Then $C = A \cup \left( \bigcup_{uv \in E(G)} S_{uv} \right)$ is a 2-locating set in $G \circ H$ by Theorem 8. Hence,

\[
ln_{2}(G \circ H) \leq |C| = |A| + |E(G)||S_{uv}| = m(ln_{2}(H)).
\]

Next, let $C_{0}$ be an $ln_{2}$-set in $G \circ H$. Then by Theorem 8, $C_{0} = A_{0} \cup \left( \bigcup_{uv \in E(G)} S_{uv} \right)$ where $A_{0} \subseteq V(G)$ and $S_{uv}$ is a 2-locating set of $H_{uv}$ for all $uv \in E(G)$. Thus,

\[
ln_{2}(G \circ H) = |C_{0}|
\]

\[
= |A_{0}| + \big| \bigcup_{uv \in E(G)} S_{uv} \big|
\]

\[
\geq \sum_{uv \in E(G)} |S_{uv}|
\]

\[
\geq m \cdot ln_{2}(H).
\]

Therefore, $ln_{2}(G \circ H) = m \cdot ln_{2}(H)$.

(ii) Let $G$ be a graph with pendant edges and $A \subseteq V(G)$ consists of pendant edges in a graph $G$, that is $|A| = k$. By Theorem 8, $S_{uv}$ is a 2-locating set of $H_{uv}$ for all $uv \in E(G)$ and $S_{uv}$ is a $(2,1)$-locating set of $H_{uv}$ whenever $l(uv) \subseteq A$ and $S_{uv}$ is a $(2,2)$-locating set...
of $H^{uv}$, otherwise. If $S_{uv}$ is a $(2,2)$-locating sets in $H^{uv}$, then

$$(m - k)ln_2(H) + k \cdot ln_{(2,2)}(H) \leq |C| = ln_2(G \circ H).$$

If $S_{uv}$ is a $(2,2)$-locating sets in $H^{uv}$, then

$$(m - k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k \leq |C| = ln_2(G \circ H).$$

Thus,

$$ln_2(G \circ H) \geq \min\{(m - k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k, (m - k)ln_2(H) + k \cdot ln_{(2,2)}(H)\}.$$

Let $(m - k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k \leq (m - k)ln_2(H) + k \cdot ln_{(2,2)}(H)$. Let $S_{uv}$ be the minimum $(2,1)$-locating set in $H^{uv}$ whenever $l(uv) \subseteq A$ and $S_{uv}$ be the minimum $(2,2)$-locating set in $H^{uv}$, otherwise. Then, $C$ is a 2-locating set in $G \circ H$ by Corollary 7. Hence, $ln_2(G \circ H) \leq |C| = (m - k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k$. Similarly, if $(m - k)ln_2(H) + k \cdot ln_{(2,2)}(H) \leq (m - k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k$. Then $ln_2(G \circ H) \leq |C| = (m - k)ln_2(H) + k \cdot ln_{(2,2)}(H)$. Thus,

$$ln_2(G \circ H) \leq \min\{(m - k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k, (m - k)ln_2(H) + k \cdot ln_{(2,2)}(H)\}.$$

Therefore,

$$ln_2(G \circ H) = \min\{(m - k)ln_2(H) + k \cdot ln_{(2,1)}(H) + k, (m - k)ln_2(H) + k \cdot ln_{(2,2)}(H)\}.$$

\[\Box\]

8. Lexicographic Product of Graphs

This section presents characterizations on the 2-locating sets in the lexicographic product of graphs.

**Theorem 9.** [6] Let $G$ and $H$ be nontrivial connected graphs. Then $W = \bigcup_{x \in S} \{x\} \times T_x$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 2-resolving set in $G[H]$ if and only if

(i) $S = V(G)$;

(ii) $T_x$ is a 2-locating set in $H$ for every $x \in V(G)$;

(iii) $T_x$ and $T_y$ are $(2,1)$-locating sets or one of $T_x$ and $T_y$ is a $(2,2)$-locating set in $H$ whenever $x, y \in EQ(G)$; and
(iv) \( T_x \) and \( T_y \) are (2-locating) dominating sets in \( H \) or one of \( T_x \) and \( T_y \) is a 2-dominating set whenever \( x, y \in EQ_2(G) \).

Theorem 10. Let \( G \) and \( H \) be nontrivial connected graphs with \( \Delta(H) \leq |V(H)| - 3 \). Then \( W = \bigcup_{x \in S} \{(x) \times T_x \} \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \), is a 2-locating set in \( G[H] \) if and only if it is a 2-resolving set and it satisfies the following:

(i) \( S = V(G) \);

(ii) \( T_x \) is a 2-locating set in \( H \) for every \( x \in V(G) \);

(iii) \( T_x \) and \( T_y \) is a \((2,1)\)-locating set or one of \( T_x \) and \( T_y \) is a \((2,2)\)-locating set in \( H \) whenever \( x, y \in V(G) \) with \( N_G[x] = N_G[y] \); and

(iv) \( T_x \) and \( T_y \) are \((2-locating)\) dominating sets in \( H \) or one of \( T_x \) and \( T_y \) is a 2-dominating set whenever \( x, y \in V(G) \) with \( d_G(x,y) = 2 \) and \( N_G(x) = N_G(y) \).

Proof. Suppose \( W = \bigcup_{x \in S} \{(x) \times T_x \} \) is a 2-locating set in \( G[H] \). Suppose there exists \( x \in V(G) \setminus S \). Pick \( a, b \in V(H) \), where \( a \neq b \). Then \((x,a),(x,b) \notin W\) and \((x,a) \neq (x,b)\). Since \( x \notin S \), \((x,r) \in V(G[H]) \setminus W \). Note that \((z,c) \in N_{G[H]}(x,a) \cup N_{G[H]}(x,b)\) for all \( z \in N_G(x) \). Thus, \( N_{G[H]}(x,a) \cap N_{G[H]}(x,b) = \emptyset\) and \( N_{G[H]}(x,a) \cap N_{G[H]}(x,b) = \emptyset\). This implies that \( W \) is not a 2-locating set of \( G[H] \), a contradiction to the assumption on \( W \). Therefore, \( S = V(G) \).

To prove (ii), let \( x \in V(G) \) and \( p,q \in V(H) \) where \( p \neq q \). Then \((x,p),(x,q) \notin W\). If \( p,q \notin T_x \) or \([p \in T_x \) and \( q \notin T_x \), then \((x,p),(x,q) \notin W\). Since \( W \) is a 2-locating set in \( G[H] \), by definition of \( G[H] \) there exist at least two vertices \((x,r),(x,s) \in V(H) \cap T_x \) such that either \((x,r),(x,s) \in N_H(x,p) \setminus N_H(x,q)\) or \((x,r),(x,s) \in N_H(x,q) \setminus N_H(x,p)\) or \((x,r) \in N_H((x,p)) \setminus N_H((x,q)) \) and \((x,s) \in N_H((x,q)) \setminus N_H((x,p))\). Similarly, if \((x,p) \in W \) and \((x,q) \notin W \), then there exists a vertex \( t \in V(H) \cap T_x \) such that \((x,t) \in N_H((x,p)) \setminus N_H((x,q)) \) or \((x,t) \in N_H((x,q)) \setminus N_H((x,p))\). Therefore, it follows that \( T_x \) is a 2-locating set of \( H \) for every \( x \in V(G) \). Thus, (ii) follows.

To prove (iii), let \( x,y \in V(G) \) with \( N_G[x] = N_G[y] \). Let \( a,b \in V(H) \), \( a \neq b \). Since \( W \) is a 2-locating set, it is not possible that \( N_H(a) \cap T_x = T_x \) and \( N_H(b) \cap T_y = T_y \). If \( T_x \) or \( T_y \) is \((2,2)\)-locating, then we are done. Otherwise, \( T_x \) and \( T_y \) are \((2,1)\)-locating.

To prove (iv), let \( x,y \in V(G) \) where \( d_G(x,y) = 2 \) and \( N_G(x) = N_G(y) \). Let \( a,b \in V(H) \), \( a \neq b \). Suppose one of \( T_x \) and \( T_y \), say \( T_x \) is not a dominating set in \( H \). Pick \( a \in V(H) \setminus N_H[T_x] \) and let \( b \in V(H) \setminus T_y \). Since \( d_G(H)((x,a),(y,b)) = 2 \), for all \( (y,b) \), it follows that \( |N_H(b) \cap T_y| \geq 2 \), i.e., \( T_y \) is a 2-dominating set.

Conversely, let \( W \) be the set as described and satisfies the given conditions. Let \((x,a),(y,b) \in V(G[H]), (x,a) \neq (y,b)\). Consider the following cases.

Case 1. \( x = y \)

Suppose \((x,a),(y,b) \notin W \). Then \( a \neq b \) and \( a,b \notin T_x = T_y \). By (ii), \( T_x \) is a 2-locating set. On the other hand, if \((x,a),(y,b) \notin W, (y,b) \notin W \), then \( a \in T_x, b \notin T_y \). Since \( T_x \) is a 2-locating set, there exists \((x,s) \in V(H) \cap T_x \) such that \((x,s) \in N_H((x,a)) \setminus N_H((y,b)) \) or \((x,s) \in N_H((y,b)) \setminus N_H((x,a))\). Thus, it follows that \( W \) is a 2-locating set of \( G[H] \).
Case 2. \( x \neq y \).

Subcase 2.1 \( xy \notin E(G) \).

If \( N_G[x] \neq N_G[y] \), then we are done. Suppose \( N_G[x] = N_G[y] \), then by (iii), \( T_x \) and \( T_y \) are \((2,1)\)-locating sets in \( H \) or one of \( T_x \) and \( T_y \) is a \((2,2)\)-locating set in \( H \).

Subcase 2.2 \( xy \notin E(G) \).

If \( d_G(x,y) > 2 \), then we are done. Suppose \( d_G(x,y) = 2 \) and \( N_G(x) = N_G(y) \). Suppose \((x,a),(y,b) \notin W \). Then \( a \notin T_x \) and \( y \notin T_y \). If \( T_x \) and \( T_y \) are both dominating, then there exist at least two vertices \((x,r),(x,s) \in \mathcal{V}(H) \cap T_x \) such that either \((x,r),(x,s) \in N_H((x,p)) \setminus N_H((x,q)) \) or \((x,r),(x,s) \in N_H((x,q)) \setminus N_H((x,p)) \) or \((x,r) \in N_H((x,p)) \setminus N_H((x,q)) \) and \((x,s) \in N_H((x,q)) \setminus N_H((x,p)) \). If one, say \( T_y \), is a 2-dominating set, then there exist at least two vertices \( r,s \in \mathcal{V}(H) \cap T_x \) such that either \((x,r),(x,s) \in N_H((x,p)) \setminus N_H((x,q)) \) or \((x,r),(x,s) \in N_H((x,q)) \setminus N_H((x,p)) \) or \((x,r) \in N_H((x,p)) \setminus N_H((x,q)) \) and \((x,s) \in N_H((x,q)) \setminus N_H((x,p)) \). Similarly, if \((x,a) \in W , (y,b) \notin W \), there exists \((x,s) \in \mathcal{V}(H) \cap T_x \) such that \((x,s) \in N_H((x,a)) \setminus N_H((y,b)) \) or \((x,s) \in N_H((y,b)) \setminus N_H((x,a)) \).

Accordingly, \( W \) is a 2-locating set of \( G[H] \). \( \square \)

Corollary 8. Let \( G \) and \( H \) be nontrivial connected graphs with \( \Delta(H) \leq |V(H)| - 3 \). If \( G \) is a totally point determining graph, then

\[
\ln_2(G[H]) = |V(G)| \cdot \ln_2(H).
\]

Proof. Suppose that \( G \) is totally point determining graph. Let \( S = V(G) \) and let \( T_x \) be an \( \ln_2 \)-set of \( H \) for each \( x \in S \). By Theorem 10, \( W = \bigcup_{x \in S} [\{x\} \times T_x] \) is a 2-locating set of \( G[H] \). It follows that

\[
\ln_2(G[H]) \leq |W| = |V(G)||T_x| = |V(G)| \cdot \ln_2(H).
\]

Now, if \( W_0 = \bigcup_{x \in S_0} [\{x\} \times T_x] \) is an \( \ln_2 \)-set of \( G[H] \), then \( S_0 = V(G) \) and \( T_x \) is a 2-locating set of \( H \) for each \( x \in V(G) \) by Theorem 10. Hence,

\[
\ln_2(G[H]) = |W_0| = |V(G)||T_x| \geq |V(G)| \cdot \ln_2(H).
\]

Therefore, \( \ln_2(G[H]) = |V(G)| \cdot \ln_2(H) \). \( \square \)

9. Conclusion

It is shown that the difference of the 2-metric dimension and 2-locating number can be made arbitrarily large. 2-locating sets in the join, corona, edge corona, and lexicographic product of two graphs have been characterized. From these characterizations, 2-locating numbers have been determined. This new invariant can also be studied for graphs under other binary operations.
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