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Some Explicit Formulas of Hurwitz Lerch type Poly-Cauchy Polynomials and Poly-Bernoulli Polynomials

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Abstract. In this paper, the Hurwitz-Lerch poly-Cauchy and poly-Bernoulli polynomials are defined using polylogarithm factorial function. Some properties of these types of polynomials were also established. Specifically, two different forms of explicit formula of Hurwitz-Lerch type poly-Cauchy polynomials were obtained using Stirling numbers of the first and second kind and an explicit formula of Hurwitz-Lerch type poly-Bernoulli polynomials was established using the Stirling numbers of the first kind.

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1. Introduction

It is known that Euler's constant appeared many times in different well-known expressions or formulas such as in exponential integral, the Laplace transform of the natural logarithm, the first of the Laurent series expansion for the Riemann Zeta function, solution of the second kind to Bessel's equation and many more. Surprisingly, the Cauchy numbers have appeared in the formula involving Euler's constant [16]. This fact has attracted several researchers to work further on Cauchy numbers.

The Cauchy numbers [6, 12, 15] of the first and second kind, respectively denoted by c_n and $\hat{c}_n$, are usually defined by its generating functions:

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}, \quad (|t| < 1)$$

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and

$$\frac{t}{(1+t)\ln(1+t)} = \sum_{n=0}^{\infty} \hat{c}_n \frac{t^n}{n!}, \quad (|t| < 1).$$

The Bernoulli numbers [1] denoted by B_n are defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad (|t| < 2\pi).$$

One of its combinatorial relations is given by

$$B_n = (-1)^n \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array} \right] \frac{(-1)^m m!}{m+1}.$$

The Cauchy numbers appear in the Laplace Summation Formula [15] as a coefficient and are also called the Cauchy numbers of the first kind. In this formula, the Cauchy numbers are expressed in terms of Stirling numbers as follows,

$$\int f(t)dt = \Delta^{-1} \sum_{k=0}^{\infty} \frac{c_k}{k!} \Delta^k$$

where Δ is the forward difference operator. This is analogous to Euler McLaurin Summation Formula where Bernoulli numbers are expressed in terms of Stirling numbers, however, differentiation is being used instead of the difference operators as shown below:

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x)dx + \sum_{v=1}^{n} \frac{v!}{B_v} f^{(v-1)}(x)\Big|_a^b - R_n[f].$$

A variation of Cauchy numbers of the first kind was introduced by Komatsu [12] inspired by the polylogarithm factorial functions

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}.$$

These numbers are called poly-Cauchy numbers of the first and second kind, denoted by $c_n^{(k)}$ and $\hat{c}_n^{(k)}$, respectively. More precisely, these numbers are defined by means of integrals as follows:

$$c_n^{(k)} = n! \int_0^1 \cdots \int_0^1 \binom{t_1 t_2 \cdots t_k}{n} dt_1 dt_2 \cdots dt_k$$

and

$$\hat{c}_n^{(k)} = n! \int_0^1 \cdots \int_0^1 \binom{-t_1 t_2 \cdots t_k}{n} dt_1 dt_2 \cdots dt_k.$$

These numbers have combinatorial relations with Stirling numbers of the first and second kind as follows

$$\sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\} c_m^{(k)} = \frac{1}{(n+1)^k},$$

$$\sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\} \hat{c}_m^{(k)} = \frac{(-1)^n}{(n+1)^k}$$

and explicit formulas

$$c_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array} \right] \frac{(-1)^m}{(m+1)^k},$$

$$\hat{c}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array} \right] \frac{1}{(m+1)^k}$$

where $\left[{n \atop m}\right]$ and $\left\{{n \atop m}\right\}$ are the Stirling numbers of the first and second kind, respectively, with generating functions:

$$\frac{[\ln(1+t)]^m}{m!} = \sum_{n=m}^{\infty} (-1)^{n-m} \left[{n \atop m}\right] \frac{t^n}{n!}, \quad (|t| < 1)$$

and

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{{n \atop m}\right\} \frac{t^n}{n!}, \quad (|t| < 1)$$

such that $\left[{n \atop m}\right] = 0$ and $\left\{{n \atop m}\right\} = 0$ for $n < m$.

Parallel to this, Kaneko [11] defined certain variation of Bernoulli numbers in terms of polylogarithm function

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}, \quad (|z| < 1)$$

which are called poly-Bernoulli numbers denoted by $B_n^{(k)}$. These types of numbers are defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}.$$

Certain generalization of poly-Cauchy numbers of the first and second kind was introduced by Cenkci and Young [4]. This generalization was motivated by the concept of Hurwitz-Lerch factorial zeta function defined by

$$\Phi f(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{n!(n+a)^s}$$

for $s \in \mathbb{C}$ when $|z| < 1$, Re $s > 1$ when $|z| = 1$ and $a \notin \{0, -1, -2, \cdots\}$. These numbers were called Hurwitz type poly-Cauchy numbers of the first and second kind, denoted by

$c_n^{(k)}(a)$ and $\hat{c}_n^{(k)}(a)$, which are respectively defined by

$$\Phi f(\log(1+t),k,a)=\sum_{n=0}^{\infty} c_n^{(k)}(a)\frac{t^n}{n!}$$

and

$$\Phi f(-\log(1+t),k,a)=\sum_{n=0}^{\infty} \hat{c}_n^{(k)}(a)\frac{t^n}{n!}.$$

These numbers possessed the following properties which are analogous to those of poly-Cauchy numbers: explicit formulas

$$\begin{aligned} c_n^{(k)}(a) &= (-1)^n \sum_{m=0}^{n} \frac{(-1)^m S_1(n,m)}{(m+a)^k}, \\ \hat{c}_n^{(k)}(a) &= (-1)^n \sum_{m=0}^{n} \frac{S_1(n,m)}{(m+a)^k}, \end{aligned}$$

relations with Stirling numbers of the second kind

$$\sum_{m=0}^{n} S_2(n,m)c_m^{(k)}(a)=\frac{1}{(n+a)^k},$$
$$\sum_{m=0}^{n} S_2(n,m)\hat{c}_m^{(k)}(a)=\frac{(-1)^n}{(n+a)^k},$$

and expressions of Hurwitz type poly-Bernoulli numbers in terms of Hurwitz type poly-Cauchy numbers

$$\begin{aligned} B_n^{(k)}(a) &= \sum_{l=0}^{n}\sum_{m=0}^{n}(-1)^{m+n}m!S_2(n,m)S_2(m,l)c_l^{(k)}(a), \\ B_n^{(k)}(a) &= \sum_{l=0}^{n}\sum_{m=0}^{n}(-1)^{m}m!S_2(n,m)S_2(m,l)\hat{c}_l^{(k)}(a), \\ c_n^{(k)}(a) &= \sum_{l=0}^{n}\sum_{m=0}^{n}\frac{(-1)^{m+n}}{m!}S_1(n,m)S_1(m,l)B_l^{(k)}(a), \\ \hat{c}_n^{(k)}(a) &= \sum_{l=0}^{n}\sum_{m=0}^{n}\frac{(-1)^{n}}{m!}S_1(n,m)S_1(m,l)B_l^{(k)}(a). \end{aligned}$$

Recently, several generalizations of these numbers have been introduced relating to some well-known special numbers. For instance, the poly-Cauchy polynomials are expressed in terms of polylogarithm factorial function and multi poly-Cauchy polynomials, multi poly-Bernoulli and multi poly-Euler numbers and polynomials are expressed in terms of multiple polylogarithm factorial function [7, 8, 10]. Moreover, other well known families of polynomials such as the Apell-type classical polynomials and Apostol-type polynomials have attracted research attention due to their important applications in the areas of applied mathematics, physics and engineering [3, 5].

This present study aims to establish other variation of generalizing Cauchy and Bernoulli polynomials that can be related to the well-known Hurwitz-Lerch factorial zeta function. The generalization may contribute to the development of numerous applications in number theory, numerical analysis and difference-differential equations.

2. Hurwitz-Lerch type Poly-Cauchy and Poly-Bernoulli Polynomials

Kamano and Komatsu [13] defined the poly-Cauchy polynomials of the first and second kind, $c_n^{(k)}(x)$ and $\hat{c}_n^{(k)}(x)$, respectively, as follows:

$$c_n^{(k)}(x) = n! \int_0^1 \cdots \int_0^1 \binom{t_1 t_2 \cdots t_k + x}{n} dt_1 dt_2 \cdots dt_k, \quad (k \geq 1)$$

and

$$\hat{c}_n^{(k)}(x) = n! \int_0^1 \cdots \int_0^1 \binom{-t_1 t_2 \cdots t_k - x}{n} dt_1 dt_2 \cdots dt_k, \quad (k \geq 1)$$

with generating functions

$$(1+t)^x \mathrm{Lif}_k(\ln(1+t)) = \sum_{n=0}^{\infty} c_n^{(k)}(x) \frac{t^n}{n!} \tag{1}$$

$$\frac{\mathrm{Lif}_k(-\ln(1+t))}{(1+t)^x} = \sum_{n=0}^{\infty} \hat{c}_n^{(k)}(x) \frac{t^n}{n!} \tag{2}$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}. \tag{3}$$

Observe that from (3), we get

$$\Phi f(\ln(1+t), k, a) = \sum_{n=0}^{\infty} \frac{(\ln(1+t))^n}{n!(n+a)^k} = \mathrm{Lif}_k(\ln(1+t))(a).$$

If $a = 1$, we get $\Phi f(z, k, 1) = \mathrm{Lif}_k(z)$. Comparing this with the left hand side of (1) and (2), it would be logical to define the Hurwitz-Lerch poly-Cauchy polynomials of the first and seconds kind as follows:

Definition 1. *The Hurwitz-Lerch type poly-Cauchy polynomials of the first kind denoted by* $c_{n,a}^{(k)}(x)$ *are defined by*

$$(1+t)^x \Phi f(\ln(1+t), k, a) = \sum_{n=0}^{\infty} c_{n,a}^{(k)}(x) \frac{t^n}{n!}.$$

Definition 2. *The Hurwitz-Lerch type poly-Cauchy polynomials of the second kind denoted by* $\hat{c}_{n,a}^{(k)}(x)$ *are defined by*

$$\frac{\Phi f(-\ln(1+t),k,a)}{(1+t)^x}=\sum_{n=0}^{\infty}\hat{c}_{n,a}^{(k)}(x)\frac{t^n}{n!}.$$

These polynomials have an explicit formula involving the Stirling numbers of the first and second kind.

Theorem 1. For $k\in\mathbb{Z}, n\geq 0$ we have

$$c_{n,a}^{(k)}(x) \quad = \quad \sum_{s=0}^{n}\frac{x!}{(x-n+s)!}(-1)^{s-m}\binom{n}{s}\sum_{m=0}^{s}\frac{\left[{s \atop m}\right]}{(m+a)^k}.$$

Proof.

$$\sum_{n=0}^{\infty}c_{n,a}^{(k)}(x)\frac{t^n}{n!}=(1+t)^x\Phi f(\ln(1+t),k,a).$$

Working on the right hand side, we have

$$\begin{aligned}
(1+t)^x\Phi f(\ln(1+t),k,a) &= \left(\sum_{s=0}^{\infty}\binom{x}{s}t^s\right)\left(\sum_{m=0}^{\infty}\frac{(\ln(1+t))^m}{m!(m+a)^k}\right)\\
&= \left(\sum_{s=0}^{\infty}\binom{x}{s}t^s\right)\left(\sum_{m=0}^{\infty}\left(\sum_{n=m}^{\infty}\frac{(-1)^{n-m}\left[{n \atop m}\right]}{(m+a)^k}\right)\right)\\
&= \sum_{s=0}^{\infty}\sum_{n=0}^{s}\left\{\binom{x}{s-n}(-1)^{n-m}t^{s-n}\left(\sum_{m=0}^{n}\left(\frac{\left[{n \atop m}\right]}{(m+a)^k}\frac{t^n}{n!}\right)\right)\right\}\\
&= \sum_{s=0}^{\infty}\left\{\sum_{n=0}^{s}\frac{x!(-1)^{n-m}}{(s-n)!(x-s+n)!}\left(\sum_{m=0}^{n}\left(\frac{\left[{n \atop m}\right]}{(m+a)^k}\frac{t^s s!}{n!s!}\right)\right)\right\}\\
&= \sum_{s=0}^{\infty}\left\{\sum_{n=0}^{s}\frac{x!}{(x+s-n)!}(-1)^{n-m}\left(\sum_{m=0}^{n}\left(\frac{\left[{n \atop m}\right]}{(m+a)^k}\right)\frac{s!}{n!(s-n)!}\frac{t^s}{s!}\right)\right\}\\
&= \sum_{s=0}^{\infty}\left\{\sum_{n=0}^{s}\frac{x!}{(x-s+n)!}(-1)^{n-m}\binom{s}{n}\left(\sum_{m=0}^{n}\left(\frac{\left[{n \atop m}\right]}{(m+a)^k}\right)\right)\right\}\frac{t^s}{s!}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{s=0}^{n}\frac{x!}{(x-n+s)!}(-1)^{s-m}\binom{n}{s}\left(\sum_{m=0}^{s}\left(\frac{\left[{s \atop m}\right]}{(m+a)^k}\right)\right)\right\}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients completes the proof. ■

Theorem 2. For $k\in\mathbb{Z}, n\geq 0$ we have

$$\hat{c}_{n,a}^{(k)}(x)=\sum_{s=0}^{n}\frac{(x+n-s-1)!}{(x-1)!}(-1)^n\binom{n}{s}\sum_{m=0}^{s}\frac{\left[{s \atop m}\right]}{(m+a)^k}.$$

Proof.

$$\sum_{n=0}^{\infty} \hat{c}_{n,a}^{(k)}(x)\frac{t^n}{n!} = \frac{1}{(1+t)^x}\Phi f(-\ln(1+t), k, a).$$

Working on the right hand side, we have

$$\begin{aligned}
\frac{1}{(1+t)^x}\Phi f(-\ln(1+t), k, a) &= \left(\sum_{s=0}^{\infty}\binom{x+s-1}{s}(-1)^s t^s\right)\left(\sum_{m=0}^{\infty}\frac{(-\ln(1+t))^m}{m!(m+a)^k}\right)\\
&= \left(\sum_{s=0}^{\infty}\binom{x+s-1}{s}(-1)^s t^s\right)\left(\sum_{m=0}^{\infty}\left(\frac{(\ln(1+t))^m(-1)^m}{m!}\frac{1}{(m+a)^k}\right)\right)\\
&= \left(\sum_{s=0}^{\infty}\binom{x+s-1}{s}(-1)^s t^s\right)\left(\sum_{m=0}^{\infty}\left(\sum_{n=m}^{\infty}\frac{(-1)^n\left[{n \atop m}\right]}{(m+a)^k}\right)\frac{t^n}{n!}\right)\\
&= \sum_{s=0}^{\infty}\sum_{n=0}^{s}\left\{\binom{x+s-n-1}{s-n}(-1)^s t^{s-n}\left(\sum_{m=0}^{n}\left(\frac{\left[{n \atop m}\right]}{(m+a)^k}\right)\frac{t^n}{n!}\right)\right\}\\
&= \sum_{s=0}^{\infty}\left\{\sum_{n=0}^{s}\frac{(x+s-n-1)!}{(s-n)!(x-1)!}(-1)^s\left(\sum_{m=0}^{n}\left(\frac{\left[{n \atop m}\right]}{(m+a)^k}\right)\frac{t^s s!}{n!s!}\right)\right\}\\
&= \sum_{s=0}^{\infty}\left\{\sum_{n=0}^{s}\frac{(x+s-n-1)!}{(x-1)!}(-1)^s\left(\sum_{m=0}^{n}\left(\frac{\left[{n \atop m}\right]}{(m+a)^k}\right)\frac{s!}{n!(s-n)!}\frac{t^s}{s!}\right)\right\}\\
&= \sum_{s=0}^{\infty}\left\{\sum_{n=0}^{s}\frac{(x+s-n-1)!}{(x-1)!}(-1)^s\binom{s}{n}\left(\sum_{m=0}^{n}\left(\frac{\left[{n \atop m}\right]}{(m+a)^k}\right)\right)\right\}\frac{t^s}{s!}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{s=0}^{n}\frac{(x+n-s-1)!}{(x-1)!}(-1)^n\binom{n}{s}\left(\sum_{m=0}^{s}\left(\frac{\left[{s \atop m}\right]}{(m+a)^k}\right)\right)\right\}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients yields the result. ■

The Hurwitz-Lerch type poly-Bernoulli Polynomials can also be defined by means of Hurwitz-Lerch zeta function $\Phi(z, s, a)$.

Definition 3. *The Hurwitz-Lerch type poly-Bernoulli polynomials denoted by $B_{n,a}^{(k)}(x)$ are defined by*

$$\Phi(1-e^{-t}, k, a)e^{tx} = \sum_{n=0}^{\infty} B_{n,a}^{(k)}(x)\frac{t^n}{n!}.$$

where

$$e^{tx} = \sum_{n=0}^{\infty}\frac{(tx)^n}{n!}.$$

Bayad and Hamahata [2] established an explicit formula for poly-Bernoulli polynomials. Analogous to this, the Hurwitz-Lerch type poly-Bernoulli polynomials have explicit explicit formula involving Stirling numbers.

Theorem 3. *For $k \in \mathbb{Z}, n \geq 0$ we have*

$$B_{n,a}^{(k)}(x) = \sum_{s=0}^{n} x^{n-s}(-1)^{s-m}\binom{n}{s}\sum_{m=0}^{s}\frac{\left\{ {s \atop m} \right\} m!}{(m+a)^k}.$$

Proof.

$$\sum_{n=0}^{\infty} B_{n,a}^{(k)}(x)\frac{t^n}{n!} = \Phi(1-e^{-t}, k, a)e^{xt}.$$

Working on the right hand side, we have

$$\begin{aligned}
e^{xt}\Phi(1-e^{-t},k,a) &= \left(\sum_{s=0}^{\infty}\frac{(xt)^s}{s!}\right)\left(\sum_{m=0}^{\infty}\frac{(1-e^{-t})^{m_r}}{(m+a)^k}\right) \\
&= \left(\sum_{s=0}^{\infty}\frac{(xt)^s}{s!}\right)\left(\sum_{m=0}^{\infty}\frac{(e^{-t}-1))^m(-1)^m}{m!}\frac{m!}{(m+a)^k}\right) \\
&= \left(\sum_{s=0}^{\infty}\frac{(xt)^s}{s!}\right)\left(\sum_{m=0}^{\infty}\left(\sum_{n=m}^{\infty}\frac{(-1)^{n-m}\left\{ {n \atop m} \right\} m!}{(m+a)^k}\right)\frac{t^n}{n!}\right) \\
&= \sum_{s=0}^{\infty}\sum_{n=0}^{s}\left\{\frac{x^{s-n}}{(s-n)!}(-1)^{n-m}t^{s-n}\left(\sum_{m=0}^{n}\left(\frac{\left\{ {n \atop m} \right\} m!}{(m+a)^k}\right)\frac{t^n}{n!}\right)\right\} \\
&= \sum_{s=0}^{\infty}\sum_{n=0}^{s}\left\{x^{s-n}(-1)^{n-m}\left(\sum_{m=0}^{n}\left(\frac{\left\{ {n \atop m} \right\} m!}{(m+a)^k}\right)\frac{s!}{(s-n)!n!}\frac{t^s}{s!}\right)\right\} \\
&= \sum_{s=0}^{\infty}\left\{\sum_{n=0}^{s}x^{s-n}(-1)^{n-m}\binom{s}{n}\left(\sum_{m=0}^{n}\left(\frac{\left\{ {n \atop m} \right\} m!}{(m+a)^k}\right)\right)\right\}\frac{t^s}{s!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{s=0}^{n}x^{n-s}(-1)^{s-m}\binom{n}{s}\left(\sum_{m=0}^{s}\left(\frac{\left\{ {s \atop m} \right\} m!}{(m+a)^k}\right)\right)\right\}\frac{t^n}{n!}
\end{aligned}$$

Comparing the coefficients gives the result. ∎

3. Hurwitz-Lerch type Multi-Poly-Cauchy and Multi-Poly-Bernoulli Polynomials

Further generalization of poly-Cauchy numbers of the first and second kind was defined by Lacpao et al. [14] in polynomial form by means of multiple polylogarithm function

$$\mathrm{Li}_{k_1,k_2,\cdots,k_r}(z) = \sum_{0<m_1<m_2<\cdots<m_r}\frac{z^{m_1}}{m_1^{k_1}\cdots m_r^{k_r}}.$$

and Hurwitz-Lerch multi-factorial zeta functions

$$\Phi\mathrm{f}(z,(k_1,k_2,\cdots,k_r),a) = \sum_{0\leq m_1<m_2<\cdots<m_r}\left(z^{m_r}\prod_{i=1}^{r}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\right), \qquad (4)$$

respectively.

When $r=1$, (4) gives

$$\Phi \mathrm{f}(z,k_1,a)=\sum_{0\le m_1}\frac{z^{m_1}}{m_1!(m_1+a)^{k_1}},$$

the Hurwitz-Lerch factorial zeta function. These polynomials, denoted by $c_n^{(k_1,k_2,\cdots,k_r)}(x)$ and $\hat{c}_n^{(k_1,k_2,\cdots,k_r)}(x)$, are respectively defined by

$$\begin{aligned}(1+t)^x\mathrm{Lif}_{k_1,k_2,\cdots,k_r}(\ln(1+t)) &= \sum_{n=0}^{\infty}c_n^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!},\\ \frac{\mathrm{Lif}_{k_1,k_2,\cdots,k_r}(-\ln(1+t))}{(1+t)^x} &= \sum_{n=0}^{\infty}\hat{c}_n^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}.\end{aligned}$$

We define the Hurwitz-Lerch type multi-poly-Cauchy polynomials of the first and second kind analogous to the definitions of Hurwitz-Lerch type poly-Cauchy polynomials as follows:

Definition 4. *The Hurwitz-Lerch type multi-poly-Cauchy polynomials of the first kind are defined by*

$$(1+t)^x\Phi f(\ln(1+t),(k_1,k_2,\cdots,k_r),a)=\sum_{n=0}^{\infty}c_n^{(k_1,k_2,\cdots,k_r)}(x,a)\frac{t^n}{n!}.$$

Definition 5. *The Hurwitz-Lerch type multi-poly-Cauchy polynomials of the second kind are defined by*

$$\frac{\Phi f(-\ln(1+t),(k_1,k_2,\cdots,k_r),a)}{(1+t)^x}=\sum_{n=0}^{\infty}\hat{c}_n^{(k_1,k_2,\cdots,k_r)}(x,a)\frac{t^n}{n!}.$$

Similarly, these polynomials have explicit formula involving Stirling numbers.

Theorem 4. *For $k_1,k_2,\cdots,k_r\in\mathbb{Z},n\ge 0$, we have*

$$\begin{aligned}&c_n(k_1,k_2,\cdots,k_r)(x,a)\\ &=\sum_{k=0}^{n}\frac{x!}{(x-k)!}(-1)^{n-k}\binom{n}{k}\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{(-1)^{m_r}{n-k \brack m_r}}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i)!(m_i+a-r+i)^{k_i}}\right).\end{aligned}\tag{5}$$

Proof.

$$\sum_{n=0}^{\infty}c_n(k_1,k_2,\cdots,k_r)(x,a)\frac{t^n}{n!}=(1+t)^x\Phi f(\ln(1+t),(k_1,k_2,\cdots,k_r),a)$$

Working on the right hand side, we have

$$
\begin{aligned}
& (1+t)^x \Phi f(\ln(1+t),(k_1,k_2,\cdots,k_r),a) \\
= & \sum_{k=0}^{\infty}\binom{x}{k}t^k \\
& \sum_{0\le m_1\le m_2\le\cdots\le m_r}\frac{(\ln(1+t))^{m_r}}{m_1!(m_1+a-r+1)^{k_1}m_2!(m_2+a-r+2)^{k_2}\cdots m_r!(m_r+a)^{k_r}} \\
= & \sum_{k=0}^{\infty}\binom{x}{k}t^k \sum_{0\le m_1\le m_2\le\cdots\le m_r}\left(\frac{(\ln(1+t))^{m_r}}{m_r!}\frac{1}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i)!(m_i+a-r+i)^{k_i}}\right) \\
= & \sum_{k=0}^{\infty}\binom{x}{k}t^k \sum_{0\le m_1\le m_2\le\cdots\le m_r} \\
& \left(\sum_{n=m_r}^{\infty}(-1)^{n-m_r}\begin{bmatrix} n \\ m_r\end{bmatrix}\frac{t^n}{n!}\frac{1}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\right) \\
= & \sum_{k=0}^{\infty}\binom{x}{k}t^k \sum_{n=0}^{\infty}\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n}\left(\frac{(-1)^{n-m_r}\left[{n \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i)!(m_i+a-r+i)^{k_i}}\frac{t^n}{n!}\right) \\
= & \sum_{k=0}^{\infty}\binom{x}{k}t^k \sum_{n=0}^{\infty}(-1)^n\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n}\left(\frac{(-1)^{m_r}\left[{n \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i)!(m_i+a-r+i)^{k_i}}\right)\frac{t^n}{n!} \\
= & \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{x!}{(x-k)!}(-1)^n \\
& \sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n}\left(\frac{(-1)^{m_r}\left[{n \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i)!(m_i+a-r+i)^{k_i}}\right)\frac{t^{n+k}(n+k)!}{k!n!(n+k)!} \\
= & \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{x!}{(x-k)!}(-1)^{n-k} \\
& \sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{(-1)^{m_r}\left[{n-k \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i)!(m_i+a-r+i)^{k_i}}\right)\frac{t^n n!}{k!(n-k)!n!} \\
= & \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{x!}{(x-k)!}(-1)^{n-k}\binom{n}{k}\right. \\
& \left.\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{(-1)^{m_r}\left[{n-k \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i)!(m_i+a-r+i)^{k_i}}\right)\right\}\frac{t^n}{n!}.
\end{aligned}
$$

It follows that

$$
\sum_{n=0}^{\infty}c_n(k_1,k_2,\cdots,k_r)(x,a)\frac{t^n}{n!}
$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{x!}{(x-k)!}(-1)^{n-k}\binom{n}{k}\right.$$
$$\left.\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{(-1)^{m_r}\left[{n-k \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i)!(m_i+a-r+i)^{k_i}}\right)\right\}\frac{t^n}{n!}.$$

Equating the coefficients gives the result. ■

Theorem 5. *For $k_1, k_2, \cdots, k_r \in \mathbb{Z}, n \geq 0,$ we have*

$$\hat{c}_n(k_1, k_2, \cdots, k_r)(x, a)$$
$$= \sum_{k=0}^{n}\frac{(x+k-1)!}{(x-k)!}(-1)^n\binom{n}{k}\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{\left[{n-k \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\right).$$

Proof.

$$\sum_{n=0}^{\infty}c_n(k_1, k_2, \cdots, k_r)(x, a)\frac{t^n}{n!} = \frac{1}{(1+t)^x}\Phi f(-\ln(1+t), (k_1, k_2, \cdots, k_r), a)$$

Working on the right hand side, we have

$$\frac{1}{(1+t)^x}\Phi f(-\ln(1+t), (k_1, k_2, \cdots, k_r), a)$$
$$= \sum_{k=0}^{\infty}\binom{x+k-1}{k}(-1)^k t^k$$
$$\sum_{0\le m_1\le m_2\le\cdots\le m_r}\frac{(-\ln(1+t))^{m_r}}{m_1!(m_1+a-r+1)^{k_1}m_2!(m_2+a-r+2)^{k_2}\cdots m_r!(m_r+a)^{k_r}}$$
$$= \sum_{k=0}^{\infty}\binom{x+k-1}{k}(-1)^k t^k$$
$$\sum_{0\le m_1\le m_2\le\cdots\le m_r}\left(\frac{(\ln(1+t))^{m_r}(-1)^{m_r}}{m_r!}\frac{1}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\right)$$
$$= \sum_{k=0}^{\infty}\binom{x+k-1}{k}(-1)^k t^k$$
$$\sum_{0\le m_1\le m_2\le\cdots\le m_r}\left(\sum_{n=m_r}^{\infty}(-1)^{n-m_r}\begin{bmatrix} n \\ m_r\end{bmatrix}(-1)^{m_r}\frac{t^n}{n!}\frac{1}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\right)$$
$$= \sum_{k=0}^{\infty}\binom{x+k-1}{k}(-1)^k t^k$$

$$\sum_{n=0}^{\infty}\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n}\left(\frac{(-1)^n\left[{n \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\frac{t^n}{n!}\right)$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty}\binom{x+k-1}{k}(-1)^k t^k \\
&\quad \sum_{n=0}^{\infty}(-1)^n\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n}\left(\frac{\left[{n \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\right)\frac{t^n}{n!}\\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(x+k-1)!}{(x-1)!}(-1)^{n+k}\\
&\quad \sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n}\left(\frac{\left[{n \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\right)\frac{t^{n+k}(n+k)!}{k!n!(n+k)!}\\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(x+k-1)!}{(x-1)!}(-1)^{n}\\
&\quad \sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{\left[{n-k \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\right)\frac{t^{n}n!}{k!(n-k)!n!}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{(x+k-1)!}{(x-1)!}(-1)^{n}\binom{n}{k}\right.\\
&\quad \left.\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{\left[{n-k \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\right)\right\}\frac{t^{n}}{n!}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\hat{c}_n(k_1,k_2,\cdots,k_r)(x,a)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{(x+k-1)!}{(x-1)!}(-1)^{n}\binom{n}{k}\right.\\
&\quad \left.\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{\left[{n-k \atop m_r}\right]}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{m_i!(m_i+a-r+i)^{k_i}}\right)\right\}\frac{t^{n}}{n!}.
\end{aligned}$$

The result follows by comparing the coefficients. ■

Corcino et. al. defined the Hurwitz-Lerch multi-poly Bernoulli polynomials as follows: The Hurwitz-Lerch type multi-poly-Bernoulli numbers $B_n^{(k_1,k_2,\cdots,k_r)}(a)$ are defined by the generating function

$$\Phi((1-e^{-t}),(k_1,k_2,\cdots,k_r),a)=\sum_{n=0}^{\infty}B_n^{(k_1,k_2,\cdots,k_r)}(a)\frac{t^n}{n!}. \tag{6}$$

They established an explicit formula of Hurwitz-Lerch multi-poly-Bernoulli polynomials in [9, Theorem 3.1]. Parallel to this, we obtained an explicit formula of Hurwitz-Lerch multi-poly-Bernoulli-polynomials in terms of Stirling numbers of the second kind as follows:

Theorem 6. *For $k_1, k_2, \cdots, k_r \in \mathbb{Z}, n \geq 0$, we have*

$$\begin{aligned} B_n^{(k_1,k_2,\cdots,k_r)}(x,a) &= \sum_{k=0}^{n}(rx)^k(-1)^{n-k}\binom{n}{k} \\ &\quad \sum_{0\leq m_1\leq m_2\leq\cdots\leq m_r}^{n-k}\left(\frac{(-1)^{m_r}m_r!\left\{ {n-k \atop m_r} \right\}}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i+a-r+i)^{k_i}}\right) \end{aligned}$$

Proof.

$$\sum_{n=0}^{\infty} B_n^{(k_1,k_2,\cdots,k_r)}(x,a)\frac{t^n}{n!} = e^{rxt}\Phi(1-e^{-t},(k_1,k_2,\cdots,k_r),a)$$

Working on the right hand side, we get

$$\begin{aligned} & e^{rxt}\Phi(1-e^{-t},(k_1,k_2,\cdots,k_r),a) \\ = & \sum_{k=0}^{\infty}\frac{(rxt)^k}{k!}\sum_{0\leq m_1\leq m_2\leq\cdots\leq m_r}\frac{(1-e^{-t})^{m_r}}{(m_1+a-r+1)^{k_1}(m_2+a-r+2)^{k_2}\cdots(m_r+a)^{k_r}} \\ = & \sum_{k=0}^{\infty}\frac{(rxt)^k}{k!} \\ & \sum_{0\leq m_1\leq m_2\leq\cdots\leq m_r}\frac{(-1)^{m_r}m_r!}{(m_1+a-r+1)^{k_1}(m_2+a-r+2)^{k_2}\cdots(m_r+a)^{k_r}}\frac{(e^{-t}-1)^{m_r}}{m_r!} \\ = & \sum_{k=0}^{\infty}\frac{(rxt)^k}{k!} \\ & \sum_{0\leq m_1\leq m_2\leq\cdots\leq m_r}\left(\sum_{n=m_r}^{\infty}(-1)^{m_r}m_r!\left\{ {n \atop m_r} \right\}\frac{(-t)^n}{n!}\frac{1}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i+a-r+i)^{k_i}}\right) \\ = & \sum_{k=0}^{\infty}\frac{(rxt)^k}{k!} \\ & \sum_{0\leq m_1\leq m_2\leq\cdots\leq m_r}\left(\sum_{n=m_r}^{\infty}(-1)^{n+m_r}m_r!\left\{ {n \atop m_r} \right\}\frac{t^n}{n!}\frac{1}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i+a-r+i)^{k_i}}\right) \\ = & \sum_{k=0}^{\infty}\frac{(rxt)^k}{k!}\sum_{n=0}^{\infty}\sum_{0\leq m_1\leq m_2\leq\cdots\leq m_r}^{n}\left(\frac{(-1)^{n+m_r}m_r!\left\{ {n \atop m_r} \right\}}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i+a-r+i)^{k_i}}\right)\frac{t^n}{n!} \\ = & \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(rx)^k(-1)^n \end{aligned}$$

$$
\begin{aligned}
&\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n}\left(\frac{(-1)^{m_r}m_r!\left\{{n\atop m_r}\right\}}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i+a-r+i)^{k_i}}\right)\frac{t^{n+k}(n+k)!}{k!n!(n+k)!}\\
=\;&\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}(rx)^k(-1)^{n-k}\\
&\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{(-1)^{m_r}m_r!\left\{{n-k\atop m_r}\right\}}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i+a-r+i)^{k_i}}\right)\frac{t^n n!}{k!(n-k)!n!}\\
=\;&\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}(rx)^k(-1)^{n-k}\binom{n}{k}\right.\\
&\left.\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{(-1)^{m_r}m_r!\left\{{n-k\atop m_r}\right\}}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i+a-r+i)^{k_i}}\right)\right\}\frac{t^n}{n!}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{n=0}^{\infty}B_n^{(k_1,k_2,\cdots,k_r)}(x,a)\frac{t^n}{n!}\;=\;&\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}(rx)^k(-1)^{n-k}\binom{n}{k}\right.\\
&\left.\sum_{0\le m_1\le m_2\le\cdots\le m_r}^{n-k}\left(\frac{(-1)^{m_r}m_r!\left\{{n-k\atop m_r}\right\}}{(m_r+a)^{k_r}}\prod_{i=1}^{r-1}\frac{1}{(m_i+a-r+i)^{k_i}}\right)\right\}\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients proves the theorem. ■

4. Conclusions

This paper defined variations of poly-Cauchy and poly-Bernoulli polynomials called the Hurwitz-Lerch poly-Cauchy and poly-Bernoulli polynomials and Hurwitz-Lerch multi poly-Cauchy polynomials using polylogarithm factorial and multiple polylogarithm factorial functions, respectively. Moreover, explicit formulas parallel to those of poly-Cauchy and poly-Bernoulli polynomials were explored and obtained. These types of polynomials could have significant applications in the field of numerical analysis, analytic number theory and difference-differential equations. The generalizations of these polynomials yield symmetries for Stirling number series and lead to a unified investigation of algebraic properties for these polynomials. Furthermore, future researches may include properties of these polynomials with multiple parameters and relations to other family of polynomials.

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