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The Reflexive Condition on Skew Monoid Rings

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Abstract. This paper is devoted to introducing and studying two concepts, σ-skew strongly M-reflexive and σ-skew strongly M-nil-reflexive on monoid rings, which are generalizations of strongly M-reflexive and M-compatible. The paper covers the basic properties of skew monoid rings of the form $R * M$. It is shown that if R is a left APP (quasi Armendariz, semiprime rings, respectively), then R is σ-skew strongly M-reflexive. Moreover, if R is a NI-ring and M is a $u.p$-monoid, then R is σ-skew strongly M-nil-reflexive. Additionally, under some necessary and sufficient conditions, a skew monoid ring $R * M$ is proven to be σ-skew strongly M-nil-reflexive when $\sigma : M \to Aut(R)$ is a monoid homomorphism. Furthermore, if R is a left APP, then the upper triangular matrix ring $T_n(R)$ is $\bar{\sigma}$-skew strongly M-nil-reflexive, where n is a positive integer. Finally, the paper provides some examples and discusses related results from the subject.

2020 Mathematics Subject Classifications: 16S34, 16S36, 20M25

Key Words and Phrases: Left APP-ring, skew monoid ring $R * M$, quasi Armendariz ring, σ-skew strongly M-nil-reflexive ring

1. Introduction

Throughout this article, R and M denote an associative ring with identity and a monoid, respectively. Mason introduced the reflexive property for ideals and this concept was generalized by some authors, defining idempotent reflexive right ideals and rings, completely reflexive, weakly reflexive (see namely, [13], [15] and [24]). Let R be a ring and I be a right ideal of R. In [24], I is called a reflexive right ideal if for any $x, y \in R, xRy \subseteq I$ implies $yRx \subseteq I$. The reflexive right ideal concept is also specialized to the zero ideal of a ring, namely, a ring R is called reflexive [24], if its zero ideal is reflexive and a ring R is called completely reflexive if for any $x, y \in R, xy = 0$ implies $yx = 0$. Reduced rings are completely reflexive and every completely reflexive ring is semicommutative. The notion of Armendariz ring is introduced by Rege and Chhawchharia [21]. They defined a ring R to be Armendariz if $f(x)g(x) = 0$ implies $a_ib_j = 0$ for all polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n \in R[x]$. In [21] a ring R is called semicommutative if for all $x, y \in R, xy = 0$ implies $xRy = 0$. This is equivalent to the definition that any

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left (right) annihilator of R is an ideal of R. An ideal I of a ring is called semiprime if $xRx \subseteq I$ implies $x \in I$ for $x \in R$ and R is called semiprime if 0 is a semiprime ideal. It should be noted that every semiprime ideal is reflexive, as can be easily verified and therefore every ideal of a fully idempotent ring (i.e., a ring where $I^2 = I$ for all ideals I) is reflexive according to [6]. The ring R is said to be weakly reflexive if $xry = 0$ implies yrx is nilpotent for $x, y \in R$ and all $r \in R$.

The rings without nonzero nilpotent elements are said to be reduced rings. According to [9] a ring R is called quasi-Armendariz if whenever polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m, g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then $a_iRb_j = 0$ for each i, j. It was proved in [10], if R is an Armendariz ring, then R is completely reflexive if and only if $R[x]$ is completely reflexive. According to [22], a ring R is σ-skew nil M-McCoy if $\alpha\beta \in Nil(R) * M$ there exist a nonzero element $c \in R$ such that $a_i\sigma_{g_i}(c) \in Ni(R)$ for each i, where $\alpha = a_1g_1 + a_2g_2 + \cdots + a_mg_m$ and $\beta = b_1h_1 + b_2h_2 + \cdots + b_nh_n$ are nonzero element in $R * M$. In [3] the author defined "nil skew generalized power series reflexive rings" for which $f, g \in [[R^{S,\leq}, \omega]]$ satisfying $fhg \in [[nil(R)^{S,\leq}, \omega]]$ implies that $ghf \in [[nil(R)^{S,\leq}, \omega]]$. Similarly, in [1] the author discussed " the nilpotent elements and nil-reflexive property of generalized power series rings", where $fhg \in [[nil(R)^{S,\leq}]]$ implies that $ghf \in [[nil(R)^{S,\leq}]]$ for $f, g, h \in [[R^{S,\leq}]]$. The investigation of the composition of the collection of nilpotent elements in noncommutative ring constructions is a crucial and highly active field in noncommutative algebra. This is evidenced by numerous studies conducted by various authors see [3], [1], [22], [15], [13], [4], [5], [2] and [18].

This paper is devoted to examining the nilpotent elements found in skew monoid rings. In this context, let R be a ring and M be a $u.p.$-monoid. It is assumed that M operates on R through a homomorphism that maps to the automorphism group of R. This homomorphism is denoted as $\sigma : M \to Aut(R)$, i.e., a module homomorphism from M to the group of automorphisms of R, here, an automorphism of R is a ring isomorphism from R to itself, so, $\mathrm{Aut}(R)$ is the group of all such isomorphisms. For any given element $g \in M$, the notation σ_g denotes the image of g under the module homomorphism σ, i.e., $\sigma_g = \sigma(g) \in \mathrm{Aut}(R)$. In other words, σ_g is an automorphism of R that depends on the choice of the element $g \in M$. By using the monoid homomorphism σ, we can create a skew monoid ring denoted as $R*M$. This ring consists of finite formal combinations of elements in M, represented as $\Sigma_{g\in M}x_gg$. Multiplication in this ring is induced by the formula $(x_gg)(y_hh) = (x_g\sigma_g(y_h))(gh)$. Therefore, $R * M$ is a ring that is free as a left R-module with basis M. A commonly accepted fact is that if a polynomial $f(x)$ is defined over a commutative ring, then it is nilpotent if only if the coefficient of $f(x)$ is too. However, it should be noted that this statement does not hold true for noncommutative rings.

The researcher introduced and studied two concepts, σ-skew strongly M-reflexive and σ-skew strongly M-nil-reflexive on monoid rings. The paper covers the basic properties of skew monoid rings of the form $R*M$. It is shown that if R is a left APP (quasi Armendariz, semiprime rings, respectively), then R is σ-skew strongly M-reflexive. Moreover, if R is a NI-ring and M is a $u.p$-monoid, then R is σ-skew strongly M-nil-reflexive. Furthermore, under certain conditions, a skew monoid ring $R * M$ is proven to be σ-skew strongly M-

nil-reflexive when $\sigma : M \to Aut(R)$ is a monoid homomorphism. Additionally, it is proved that a ring R is σ-skew strongly M-*nil*-reflexive if and only if R/I is $\bar{\sigma}$-skew strongly M-*nil*-reflexive. Consequently, if R is a left APP, then the upper triangular matrix ring $T_n(R)$ is $\bar{\sigma}$-skew strongly M-*nil*-reflexive, where n is a positive integer. Finally, the paper provides some examples and discusses related results from the subject.

To recall, a monoid M is referred to as a unique product (u.p.)-monoid if, for any two non-empty finite subsets X and Y of M, there exist an element $x \in X$ and an element $y \in Y$ such that their product xy is distinct from the product of any other pair $(u, v) \in X \times Y$, i.e., $(u, v) \neq (x, y)$ implies $uv \neq xy$. The element xy is termed as a $u.p.$-element of the set $XY = \{pq : p \in X, q \in Y\}$. Unique product monoids and groups have significant implications in ring theory, particularly in providing a positive solution to the zero-divisor problem for group rings. Their structural properties have been extensively studied in literature (see references [17, 19]). The category of monoids that fall under the classification of $u.p.$-monoids is both extensive and significant. This category encompasses monoids that are either right or left totally ordered, submonoids of a free group, and torsion-free nilpotent groups. For a positive integer n, let $Mat_n(R)$ denote the ring of all $n \times n$ matrices and $T_n(R)$ the ring of all $n \times n$ upper triangular matrices with entries in R. We write $R[x]$ and $S_n(R)$, for the polynomial ring over a ring R and the subring consisting of all upper triangular matrices over a ring R with equal main diagonal entries.

2. Reflexive-Type Properties in Skew Monoid Rings

In this section, we discuss various constructions and extensions under which the class of σ-skew strongly M-reflexive rings is closed. By Definition 2.3 [18], a monoid homomorphism $\sigma : M \to Aut(R)$ is called compatible if the ring R is σ_g-compatible for each $g \in M$, that is, $xy = 0 \Leftrightarrow x\sigma_g(y) = 0$ for all $x, y \in R$. Now we have the following generalization of reflexive.

Definition 1. *We say that a ring R is σ-skew strongly M-reflexive (σ-skew strongly reflexive relative to a monoid M), if $\varphi(R * M)\psi = 0$ implies that $b_i\sigma_{g_i}(R\sigma_s(a_j)) = 0$, where $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ are nonzero elements in $R * M$, then $\psi(R * M)\varphi = 0$ for all $1 \leq i \leq n, 1 \leq j \leq m$.*

Definition 2. *In [20]. A ring R is called strongly M-reflexive, if whenever $\varphi, \psi \in R[M]$ with $\varphi R[M]\psi = 0$, then $\psi R[M]\varphi = 0$.*

In [16], Nasr-Isfahani and Moussavi introduced a ring R with an endomorphism σ and defined it as σ-weakly rigid if the condition $xRy = 0$ holds if and only if $x\sigma(Ry) = 0$ for any $x, y \in R$. A ring R is σ-rigid if and only if R is σ-compatible and reduced by [8]. According to [16], any prime ring that has an automorphism σ is considered to be σ-weakly rigid. If a monoid homomorphism $\sigma : M \to Aut(R)$ is weakly-rigid (compatible), it means that the ring R is also weakly rigid (compatible) with respect to each $g \in M$ under the automorphism σ_g. The following example illustrates that the compatibility of σ is necessary.

Example 1. *Let S be any nonzero reversible ring and M be a monoid generated by an element ρ such that ρ has infinite order. Suppose $R = S \oplus S$ with the usual addition and multiplication, and define $\sigma : M \to Aut(R)$ such that $\sigma_\rho((x, y)) = (y, x)$. Then the ring R is reflexive and M is u.p.-monoid, but σ is not compatible, since* $(1, 0)(0, 1) = (0, 0)$ *whereas* $(1, 0)\sigma_\rho((0, 1)) = (1, 0)$. *Now we will prove that the ring R is not σ-skew strongly M-reflexive. For, let $\varphi = (1, 0)e + (1, 0)g$ and $\psi = (0, 1)e - (1, 0)g$ be nonzero elements in $R * M$ and any $\phi \in R * M$. Then we can easily see that $\varphi\phi\psi = 0$, but $\psi\phi\sigma_g(\varphi) \neq 0$. Therefore, R is not σ-skew strongly M-reflexive.*

Theorem 1. *Let R be a ring, M be a monoid generated by an element ρ such that ρ has infinite order and $\sigma : M \to Aut(R)$ a compatible monoid homomorphism given by $\sigma_\rho = \psi$. Suppose N be any monoid with an element of infinite order. If the skew monoid ring $R*M$ is a strongly N-reflexive, then R is σ-skew strongly M-reflexive.*

Proof. Let $\varphi = \sum_{i=1}^{m} a_i g_i$ $\psi = \sum_{1}^{n} b_j g_j$ be nonzero elements in $R * M$ such that $\varphi\phi\psi = 0$ for any $\phi = \sum_{r=1}^{v} \ell_r g_r \in R * M$. Then, for each $1 \leq k \leq m + v + n$, we have $c_k = \sum_{i+r+j=k} a_i \sigma_{g_i}(\ell_r \sigma_s(b_j)) = 0$ for $s \in M$. Now, let $h \in N$ such that $O(h) = \infty$ and define $F, G \in (R * M)[N]$ as in the following: $F = (a_1 e_M)e_N + (a_1 g)h + (a_2 g_2)h_2 + \cdots + (a_m g_m)h_m$ and $G = (b_1 e_M)e_N + (b_1 g)h + (b_2 g_2)h_2 + \cdots + (b_n g_n)h_n$. Since φ, ϕ and ψ are nonzero in $R * M$, so F and G are nonzero elements in $(R * M)[N]$. Moreover, from $\varphi\phi\psi = 0$ and compatibility of $\sigma_\rho = \psi$, one can easily obtain that $FHG = 0$ for any $H \in (R*M)[N]$. Since the skew monoid ring $R*M$ is strongly N-reflexive. Then, $a_i c_t b_j = 0$ for each t and so $a_i \sigma_{g_i}(R\sigma_s(b_j)) = 0$ for each $1 \leq i \leq m$ and $1 \leq j \leq n, s \in M$. By a compatible automorphism, we have $b_j \sigma_{g_j}(R\sigma_s(a_i)) = 0$. Therefore, R is σ-skew strongly M-reflexive. □

An ideal I of R is said to be right s-unital if, for each $a \in I$ there exist an element $e \in I$ such that $ae = a$. Note that if I and J are right s-unital ideals, then so is $I \cap J$ (if $a \in I \cap J$, then $a \in aIJ \subseteq a(I \cap J)$). We say a ring R is a left APP-ring if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R$.

The following result follows from Tominaga Theorem 1 [23].

Lemma 1. *An ideal I of a ring R is left (resp. right) s-unital if and only if for any finitely many elements $a_1, a_2, \ldots, a_n \in I$, there exists an element $e \in I$ such that $a_i = ea_i$(resp. $a_i = a_i e$) for each $i = 1, 2, \ldots, n$.*

Lemma 2. *(Lemma 1.13 [14]). Let M and N be u.p.-monoids. Then so is the monoid $M \times N$.*

Lemma 3. *(Example 2.2 [7]). Let R be a ring and M be a monoid with an element of a finite order $n \geq 2$. Let $\varphi = \Sigma_{i=0}^{n-1} g^i$ and $\psi = e - g$, where $|g| = n$. Then $\varphi\psi = 0$ and so R is not M-nil-Armendariz.*

Lemma 4. *(Lemma 1.1 [5]). Assume M is a u.p.-monoid. Then M is cancellative* (*i.e.*, *for $g, h, x \in M$, if $gx = hx$ or $xg = xh$, then $g = h$*).

Proposition 1. *Let R be a ring, M be a u.p.-monoid and $\sigma : M \to Aut(R)$ a compatible monoid homomorphism. If R is a reduced left APP-ring, then R is σ-skew strongly M-reflexive.*

Proof. Suppose $\varphi, \psi \in R*M$ such that $\varphi(R*M)\psi = 0$ implies that $b_i\sigma_{g_i}(R\sigma_s(a_j)) = 0$ for any $s \in M$, where $\varphi = b_1g_1+b_2g_2+\cdots+b_ng_n$ and $\psi = a_1h_1+a_2h_2+\cdots+a_mh_m \in R*M$. We shall prove that $\psi(R*M)\varphi = 0$, (*i.e.*, $a_j\sigma_{h_j}(R\sigma_s(b_i)) = 0$ for all i, j.

Let r be an arbitrary element of R. Then we have the following equation:

$$(b_1g_1 + b_2g_2 + \cdots + b_ng_n)(rs)(a_1h_1 + a_2h_2 + \cdots + a_mh_m) = 0. \tag{2.1}$$

We proceed by induction on both n and m for every $s \in M$ and for $1 \le i \le n, 1 \le j \le m$. If $m = 1$, then $\psi = a_1h_1$. Thus $0 = (b_1g_1+b_2g_2+\cdots+b_ng_n)(rs)(a_1h_1) = b_1\sigma_{g_1}(\sigma_s(ra_1)) + b_2\sigma_{g_2}(\sigma_s(ra_1)) + \cdots + b_n\sigma_{g_n}(\sigma_s(ra_1))$ for every $r \in R$. By Lemma 4, M is a cancellative monoid. Thus $g_ish_1 \ne g_jsh_1$ for $g_i \ne g_j$. Then $b_i\sigma_{g_i}(\sigma_s(ra_1)) = 0$. Hence $b_i \in \ell_R(\sigma_s(Ra_1))$. By hypothesis, R is left APP, $\ell_R(Ra_n)$ is left s-unital by Lemma 1. Hence, there exist $e_m \in \ell_R(Ra_n)$ such that $a_ne_m = a_n$ since σ_{g_i} is an automorphism, $i = 1, 2, \ldots, n$.

Now suppose that $m \ge 2$. Since M is *u.p.*-monoid, there exists p, q with $1 \le p \le n$ and $1 \le q \le m$ such that g_psh_q is uniquely presented by considering two subsets $\{g_1, g_2, \ldots, g_n\}$ and $\{sh_1, sh_2, \ldots, sh_m\}$ of M. Thus, from $\varphi(rs)\psi = 0$ it follows that $b_p\sigma_{g_p}(\sigma_s(ra_q)) = 0$ and so $b_p\sigma_{g_p}(\sigma_s(ra_q)) = 0$. Thus $\sigma_{g_p}(c_p\sigma_s(ra_q)) = 0$ for $c \in R$, which implies that $c_p\sigma_s(ra_q) = 0$ for every $r \in R$ since σ_{g_p} is an automorphism. Hence, $c_p \in \ell_R(\sigma_s(Ra_q))$. Since $\ell_R(\sigma_s(Ra_q))$ is pure as a left ideal of R by Lemma 1, there exist an element $e_q \in \ell_R(\sigma_s(Ra_q))$ such that $c_p = c_pe_q$. Thus, for every $r \in R$, we have

$$\begin{aligned}
0 = \;& \varphi(e_qrs)\psi = (b_1g_1 + b_2g_2 + \cdots + b_ng_n)(e_qrs)(a_1h_1 + a_2h_2 + \cdots + a_mh_m) \\
= \;& (b_1g_1 + b_2g_2 + \cdots + b_ng_n)(e_qrs)(a_1h_1 + a_2h_2 + \cdots + a_{q-1}h_{q-1} + a_{q+1}h_{q+1} \\
+ \;& \cdots + a_mh_m) = (b_1g_1 + b_2g_2 + \cdots + b_ng_n)((e_qr\sigma_g(a_q))gh_q) \\
= \;& (b_1\sigma_{g_1}(e_q)g_1 + b_2\sigma_{g_2}(e_q)g_2 + \cdots + b_n\sigma_{g_n}(e_q)g_n)(rs) \\
\cdot \;& (a_1h_1 + a_2h_2 + \cdots + a_{q-1}h_{q-1} + a_{q+1}h_{q+1} + \cdots + a_mh_m).
\end{aligned} \tag{2.2}$$

Moreover, since $b_i\sigma_{g_i}(e_q) = \sigma_{g_i}(c_ie_q)$ by induction, it follows that $c_ie_q \in \ell_R(R\sigma_s(a_j))$ for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, q-1, q+1, \ldots, m$. Therefore, $c_p = c_pe_q \in \cap_{j=1}^m \ell_R(R\sigma_s(a_j))$. Now we have $b_p\sigma_{s_p}(R\sigma_s(a_j)) = \sigma_{s_p}(b_pR\sigma_s(a_j)) = 0$. For every $g_i \in M$, since σ_{g_i} is an automorphism of R and $\sigma_{g_q}(R) = R$, we obtain $b_p\sigma_{g_p}(R\sigma_{g_{sc}}(a_j)) = 0$ for any $j = 1, 2, \ldots, m$. Thus, from $\varphi(rs)\psi = 0$ it follows that $0 = (b_1g_1 + b_2g_2 + \ldots + b_{p-1}g_{p-1} + b_{p+1}g_{p+1} + \cdots + b_ng_n)(rs)(a_1h_1 + a_2h_2 + \cdots + a_mh_m)$. By using the previous method, there exist $k \in \{1, 2, \ldots, p-1, p+1, \ldots, n\}$ such that $c_k \in \cap_{j=1}^m \ell_R(\sigma_s(Ra_j))$. Thus, $b_k\sigma_{g_k}(\sigma_s(Ra_j)) = \sigma_{g_k}(c_k\sigma_s(Ra_j)) = 0$ for any $j = 1, 2, \ldots, m$. Hence $(b_1g_1+b_2g_2+\cdots+b_{p-1}g_{p-1}+b_{p+1}g_{p+1} + \cdots + b_{k-1}g_{k-1} + b_{k+1}g_{k+1} + \cdots + b_ng_n)(rs)(a_1h_1 + a_2h_2 + \cdots + a_mh_m) = 0$. Continuing this procedure yields $c_1, c_2, \ldots, c_n \in \cap_{j=1}^m \ell_R(\sigma_s(Ra_j))$ for every $s \in M$. Thus, $b_k \in \ell_R(\sigma_s(Ra_j))$ for any $k = 1, 2, \ldots, n, j = 1, 2, \ldots, m$. Using induction on $m + n$, we obtain $b_i\sigma_{g_i}(\sigma_s(Ra_j)) = 0$. Thus, $a_j\sigma_{h_j}(\sigma_s(Rb_i)) = 0$ since R is reduced. Therefore, R is σ-skew strongly M-reflexive. $\square$

Let $(M, \leq)$ be an ordered monoid. If for any $g_1, g_2, h \in M, g_1 < g_2$ implies that $g_1 h < g_2 h$ and $hg_1 < hg_2$, then $\leq$ is called a strictly ordered monoid.

Corollary 1. *Let R be an M-ring, where M is a strictly totally ordered monoid and $\sigma : M \to Aut(R)$ is a monoid homomorphism. If R is quasi Armendariz, then R is σ-skew strongly M-reflexive.*

Proof. Let $\phi = \Sigma_{i=1}^{n} b_i g_i$ and $\psi = \Sigma_{j=1}^{m} a_j h_j \in R * M$ satisfying $\phi(R * M)\psi = 0$ implies that $b_i \sigma_{g_i}(R\sigma_s(a_j)) = 0$ for any $s \in M$ and any i, j. We write

$$(b_1 g_1 + b_2 g_2 + \cdots + b_n g_n)(rs)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0. \qquad (2.3)$$

With $g_1 < g_2 < \ldots < g_n, h_1 < h_2 < \ldots < h_m$. We will use transfinite induction on a strictly totally ordered set $\leq$ to show that $\psi(R * M)\phi = 0$. If we take $m = 1$ in Eq. (2.3), then we have $(b_1 g_1 + b_2 g_2 + \cdots + b_n g_n)(rs)(a_1 h_1) = 0$. Therefore, we obtain $b_i \sigma_{g_i}(R\sigma_s(a_1)) = 0$ for each $1 \leq i \leq i$. Since M is a strictly totally ordered monoid, we have $g_1 h_1 < g_i h_1 \leq g_i h_j = g_1 h_1$ for $i \neq 1$ or $j \neq 1$. Thus, Eq. (2.3) becomes

$$(b_1 g_1 + b_2 g_2 + \cdots + b_n g_n)(rs)(a_2 h_2 + a_3 h_3 + \cdots + a_m h_m) = 0. \qquad (2.4)$$

The case $n = 1$ is proved by similar argument. By applying the induction hypothesis for all $2 \leq i \leq n$ and $2 \leq j \leq m$. Suppose that $b_i \sigma_{g_i}(R\sigma_s(a_j)) = 0$ for all $1 \leq i \leq n, 1 \leq j \leq m$ with $\lambda \in M$ is such that for any g_i and $h_j, g_i h_j < \lambda$. We will show that $b_i \sigma_{g_i}(R\sigma_s(a_j)) = 0$ for any g_i and h_j with $g_i h_j = \lambda$. Set $X = \{(g_i, h_j) | g_i h_j = \lambda\}$. Then X is a finite set. We write X as $\{(g_{i_q}, h_{j_q}) | q = 1, 2, \ldots, d\}$ such that $g_{i_1} < g_{i_2} < \ldots < g_{i_d}$. Since M is cancellative, $g_{i_1} = g_{i_2}$ and $g_{i_1} h_{j_1} = g_{i_2} h_{j_2} = \lambda$ imply $h_{j_1} = h_{j_2}$ since $\leq$ is a strict order, $g_{i_1} < g_{i_2}$ and $g_{i_1} h_{j_1} = g_{i_2} h_{j_2} = \lambda$ imply $h_{j_2} < h_{j_1}$. Thus we have $h_{j_d} < h_{j_{d-1}} < \ldots < h_{j_2} < h_{j_1}$. Now $\sum_{(g_i, h_j) \in X} b_i \sigma_{g_i}(r\sigma_s(a_j)) = \sum_{q=1}^{d} b_{i_q} \sigma_{g_{i_q}}(r\sigma_s(a_{j_q})) = 0$. For any $q \geq 2, g_{i_1} h_{j_q} < g_{i_q} h_{j_q} = \lambda$, and so $b_{i_1} \sigma_{g_{i_q}}(r\sigma_s(a_{j_q})) = 0$ by induction hypothesis. Thus, $b_{i_1} \sigma_{g_{i_q}}(r a_{j_q}) = 0$ because R is M-rigidness and σ is automorphism.

For the case where $n \geq 2$ for M is cancellative. We can repeat this process to show that $b_i \sigma_{g_i}(R\sigma_s(a_j)) = 0$ for all $s \in M$ and all i, j. Consequently, we can see that $a_j \sigma_{h_j}(R\sigma_s(b_i)) = 0$ for all $s \in M$, $1 \leq j \leq m, 1 \leq i \leq n$ by rigidness and σ is automorphism. Thus, $\psi(R * M)\phi = 0$. Therefore, R is σ-skew strongly M-reflexive.

Proposition 2. *Let R be a ring, M be a strictly totally ordered monoid and $\sigma : M \to Aut(R)$ a compatible monoid homomorphism. If R is semiprime, then R is σ-skew strongly M-reflexive.*

Proof. Since a semiprime ring is quasi-Armendariz and so reflexive, the proof follows from Corollary 1.

For a ring R and $n \geq 2$, let $V_n(R)$ be the ring of all $n \times n$ upper triangular matrices over R that are constant on the diagonal. Let $\sigma : M \to Aut(R)$ be a monoid homomorphism. For each $g \in M, \sigma$ can be extended to a monoid homomorphism $\overline{\sigma}$ from M to $Aut(V_n(R))$ defined by $\overline{\sigma}((a_{ij})) = (\sigma_g(a_{ij}))$.

Theorem 2. *Let R be an M-rigid ring, where M is a monoid and $\sigma : M \to Aut(R)$ is a monoid homomorphism, let $n \geq 2$. If R is a left APP-ring, then $V_n(R)$ is $\overline{\sigma}$-skew strongly M-reflexive.*

Proof. Suppose that R is a left APP-ring and let $\varphi = A_1g_1 + A_2g_2 + \cdots + A_ng_n$ and $\psi = B_1h_1 + B_2h_2 + \cdots + B_mh_m \in V_n(R) * M$ such that $\varphi(V_n(R) * M)\psi = 0$. We use $(a_1, a_2, \ldots, a_n) \in V_n(R)$, where

$$A_i = \begin{pmatrix} {a_1}^{(i)} & {a_2}^{(i)} & {a_3}^{(i)} & \cdots & {a_n}^{(i)} \\ 0 & {a_1}^{(i)} & {a_2}^{(i)} & \cdots & {a_{n-1}}^{(i)} \\ 0 & 0 & {a_1}^{(i)} & \cdots & {a_{n-2}}^{(i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & {a_1}^{(i)} \end{pmatrix}, B_j = \begin{pmatrix} {b_1}^{(j)} & {b_2}^{(j)} & {b_3}^{(j)} & \cdots & {b_n}^{(j)} \\ 0 & {b_1}^{(j)} & {b_2}^{(j)} & \cdots & {b_{n-1}}^{(j)} \\ 0 & 0 & {b_1}^{(j)} & \cdots & {b_{n-2}}^{(j)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & {b_1}^{(j)} \end{pmatrix}.$$

We note that there is an obvious isomorphism $V_n(R) * M \cong V_n(R * M)$. Therefore, we can rewrite φ and ψ as

$$\varphi = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \cdots & \varphi_n \\ 0 & \varphi_1 & \varphi_2 & \cdots & \varphi_{n-1} \\ 0 & 0 & \varphi_1 & \cdots & \varphi_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varphi_1 \end{pmatrix}, \psi = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \cdots & \psi_n \\ 0 & \psi_1 & \psi_2 & \cdots & \psi_{n-1} \\ 0 & 0 & \psi_1 & \cdots & \psi_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \psi_1 \end{pmatrix}.$$

Let $A(V_n(R))B = 0$ for $A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_n) \in V_n(R)$. For any $r \in R, A(r, 0, \ldots, 0)B = 0$. Thus we have the following equations:

$$ {a_1}^{(i)} r {b_1}^{(j)} = 0. \tag{2.5}$$

$$ {a_1}^{(i)} r {b_2}^{(j)} + {a_2}^{(i)} r {b_1}^{(j)} = 0. \tag{2.6}$$

$$ {a_1}^{(i)} r {b_3}^{(j)} + {a_2}^{(i)} r {b_2}^{(j)} + {a_3}^{(i)} r {b_1}^{(j)} = 0. \tag{2.7}$$

$$\vdots$$

$$ {a_1}^{(i)} r {b_{n-1}}^{(j)} + {a_2}^{(i)} r {b_{n-2}}^{(j)} + \cdots + {a_{n-1}}^{(i)} r {b_1}^{(j)} = 0. \tag{2.8}$$

$$ {a_1}^{(i)} r {b_n}^{(j)} + {a_2}^{(i)} r {b_{n-1}}^{(j)} + \cdots + {a_n}^{(i)} r {b_1}^{(j)} = 0. \tag{2.9}$$

Now for a monoid M and $\sigma : M \to Aut(R)$ a monoid homomorphism. From Eq.(2.5) we see ${a_1}^{(i)} \sigma_{g_i}(R\sigma_s({b_1}^{(j)})) = 0$ for all i, j and $s \in M$. Hence ${a_1}^{(i)} \in \ell_R(R\sigma_s({b_1}^{(j)}))$. By hypothesis, R is a left APP, $\ell_R(R\sigma_s({b_1}^{(j)}))$ is left s-unital by Lemma 1. Hence there exist $e_n \in \ell_R(R\sigma_s({b_1}^{(j)}))$ such that ${a_1}^{(i)} e_n = {a_1}^{(i)}$ since σ_{g_i} is an automorphism, $i = 1, 2, \ldots, n$.

This implies that, ${b_1}^{(j)}\sigma_{h_j}(R\sigma_s({a_1}^{(i)})) = 0$ by rigidness and we obtain $\psi_1(V_n(R) * M)\varphi_1 = 0$. If we multiplying Eq.(2.6) on the right-hand side by $t{b_1}^{(j)}$ for any $t \in R$, then

$$ {a_1}^{(i)}r{b_2}^{(j)}t{b_1}^{(j)} + {a_2}^{(i)}r{b_1}^{(j)}t{b_1}^{(j)} = 0. \tag{2.10} $$

Hence ${a_2}^{(i)}R{b_1}^{(j)} = 0$. Since R is M-rigid we have ${a_2}^{(i)}\sigma_{g_i}(R\sigma_s({b_1}^{(j)})) = 0$. Hence ${a_2}^{(i)} \in \ell_R(R\sigma_s({b_1}^{(j)}))$. By hypothesis, R is a left APP, then $\ell_R(R\sigma_s({b_1}^{(j)}))$ is left s-unital by Lemma 1. Hence there exist $e_n \in \ell_R(R\sigma_s({b_1}^{(j)}))$ such that ${a_2}^{(i)}e_n = {a_2}^{(i)}$ since σ_{g_i} is an automorphism. This shows that ${b_1}^{(j)}\sigma_{h_j}(R\sigma_s({a_2}^{(i)})) = 0$ since by rigidness we obtain $\psi_1(V_n(R)*M)\varphi_2 = 0$. Thus, we deduce the other side of Eq. (2.10), ${a_1}^{(i)}\sigma_{g_i}(R\sigma_s({b_2}^{(j)})) = 0$ and so $\psi_2(V_n(R) * M)\varphi_1 = 0$. Similarly, if we multiply Eq.(2.7) on the right-hand side by $t{b_1}^{(j)}$ for any $t \in R$, then

$$ {a_1}^{(i)}r{b_3}^{(j)}t{b_1}^{(j)} + {a_2}^{(i)}r{b_2}^{(j)}t{b_1}^{(j)} + {a_3}^{(i)}r{b_1}^{(j)}t{b_1}^{(j)} = 0. \tag{2.11} $$

And so ${a_3}^{(i)}R{b_1}^{(j)} = 0$. Since R is M-rigid and σ_{g_i} is an automorphism, we have ${a_3}^{(i)}\sigma_{g_i}(R\sigma_s({b_1}^{(j)})) = 0$. Hence, ${a_3}^{(i)} \in \ell_R(R\sigma_s({b_1}^{(j)}))$. By hypothesis, R is a left APP, $\ell_R(R\sigma_s({b_1}^{(j)}))$ is left s-unital by Lemma 1. Hence, there exist $e_n \in \ell_R(R\sigma_s({b_1}^{(j)}))$ such that ${a_3}^{(i)}e_n = {a_3}^{(i)}$ since by rigidness we obtain $\psi_1(V_n(R) * M)\varphi_3 = 0$. Then, Eq. (2.7) becomes

$$ {a_1}^{(i)}r{b_3}^{(j)} + {a_2}^{(i)}r{b_2}^{(j)} = 0. \tag{2.12} $$

If we multiplying Eq.(2.12) on the right-hand side by $t{b_2}^{(j)}$ for any $t \in R$, then ${a_1}^{(i)}r{b_3}^{(j)} = 0$ and ${a_2}^{(i)}r{b_2}^{(j)} = 0$ by the similar argument to above. Thus, we have ${a_i}^{(i)}\sigma_{g_i}(R\sigma_s({b_j}^{(j)})) = 0$ and ${b_j}^{(j)}\sigma_{h_j}(R\sigma_s({a_i}^{(i)}) = 0$ for all $2 \le i + j \le 4$.

Inductively, we assume that ${a_i}^{(i)}\sigma_{g_i}(R\sigma_s({b_j}^{(j)})) = 0$ and ${b_j}^{(j)}\sigma_{h_j}(R\sigma_s({a_i}^{(i)}) = 0$ for all $i + j \le n$. if we multiply Eq.(2.9) on the right-hand side by $t_1{b_1}^{(j)}, t_2{b_2}^{(j)}, \ldots, t_{n-1}{b_{n-1}}^{(j)}$ for any $t_1, t_2, \ldots, t_{n-1} \in R$, in turn, since R is M-rigidness and σ_{g_i} is an automorphism, we have

$$ {a_n}^{(i)}\sigma_{g_i}(R\sigma_s({b_1}^{(j)})) = 0, {a_{n-1}}^{(i)}\sigma_{g_i}(R\sigma_s({b_2}^{(j)})) = 0, \ldots, {a_2}^{(i)}\sigma_{g_i}(R\sigma_s({b_{n-1}}^{(j)})) = 0 $$

and ${a_1}^{(i)}\sigma_{g_i}(R\sigma_s({b_n}^{(j)})) = 0$. Hence,

$$ {a_n}^{(i)} \in \ell_R(R\sigma_s({b_1}^{(j)})), {a_{n-1}}^{(i)} \in \ell_R(R\sigma_s({b_2}^{(j)})), \ldots, {a_1}^{(i)} \in \ell_R(R\sigma_s({b_n}^{(j)})). $$

By hypothesis, R is a left APP, $\ell_R(R\sigma_s({b_1}^{(j)})), \ell_R(R\sigma_s({b_2}^{(j)})), \ldots, \ell_R(R\sigma_s({b_{n-1}}^{(j)}))$ and $\ell_R(R\sigma_s({b_n}^{(j)}))$, respectively, is left s-unital by Lemma 1 again. Hence, there exist $e_n \in \ell_R(R\sigma_s({b_1}^{(j)})), \ell_R(R\sigma_s({b_1}^{(j)})), \ldots, \ell_R(R\sigma_s({b_1}^{(j)}))$ such that ${a_n}^{(i)}e_n = {a_n}^{(i)}$, ${a_{n-1}}^{(i)}e_n = {a_{n-1}}^{(i)}, \ldots, {a_1}^{(i)}e_n = {a_1}^{(i)}$. Now, it is straightforward to see that $A_i\sigma_{g_i}(R\sigma_s(B_j)) = 0$ for all i, j. Since R is rigidness we option $B_j\sigma_{h_j}(R\sigma_s(A_i)) = 0$. This prove that $\psi(V_n(R) * M)\varphi = 0$. Therefore, $V_n(R)$ is $\overline{\sigma}$-skew strongly M-reflexive.

Corollary 2. *(Theorem 3.8 [12]). Let R be a ring with an endomorphism α and $n \geq 2$. If R is a semiprime and right α-skew reflexive ring, then $V_n(R)$ is right $\overline{\alpha}$-skew reflexive.*

3. Nilpotent Elements of Reflexive in Skew Monoid Rings

In this section, we introduce the concept of σ-skew strongly M-*nil*-reflexive ring and consider its properties.

Let $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n \in R[M]$. The element $\varphi \in Nil(R)[M]$ if and only if $b_i \in Nil(R)$ for all $1 \leq i \leq n$. Also, we say that $\varphi \in Nil(R * M)$ if φ is a nilpotent element in the skew monoid ring $R * M$. For any $\varphi \in R * M$, we denote by C_φ the set of all coefficients of φ. For more details on this, please refer to [18].

Definition 3. *We say that a ring R is σ-skew strongly M-nil-reflexive (σ-skew strongly nil-reflexive relative to a monoid M), if $\varphi\phi\psi \in Nil(R) * M$ implies that $b_i\sigma_{g_i}(c\,\sigma_s(a_j)) \in nil(R)$, where $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n, \phi = c_1l_1 + c_2l_2 + \cdots + c_dl_d$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m \in R * M$, then $\psi\phi\varphi \in Nil(R) * M$ for all i, k, j.*

If $M = (\mathbb{N} \cup \{0\}, +)$ and $\sigma_g = id_R$ for all $g \in M$, then a ring R is σ-skew strongly M-*nil*-reflexive if and only if R is strongly *nil*-reflexive. Also, if $M = \{e\}$ and $\sigma_g = id_R$ for all $g \in M$, then any ring R is σ-skew strongly M-*nil*-reflexive.

A ring R is called an NI ring if $nil(R)$ forms an ideal. For a unique product monoid we have the following result.

Theorem 3. *Let R be an NI-ring, M be a u.p.-monoid, and $\sigma : M \to Aut(R)$ be a compatible monoid homomorphism, then R is σ-skew strongly M-nil-reflexive*

Proof. Let $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n, \phi = c_1l_1 + c_2l_2 + \cdots + c_dl_d$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ be nonzero elements in $R * M$ such that $\varphi\phi\psi \in nil(R) * M$ implies that $b_i\sigma_{g_i}(\sigma_s(c\,a_j)) \in nil(R)$. We will show that $a_j\sigma_{h_j}(\sigma_s(c\,b_i)) \in nil(R)$ for each $1 \leq i \leq n, 1 \leq k \leq d$ and $1 \leq j \leq m$. We proceed by induction on both n and m. By using freely σ is compatible monoid homomorphism. Let $n = 1$ and hence $\varphi = b_1g_1$. Since M is *u.p.*- monoid then by Lemma 4, M is cancellative monoid, $g_1h_i \neq g_1h_j$ for $i \neq j$. So $b_1\sigma_{g_1}(c(a_j)) \in nil(R)$ for any $c \in R$ and each j. The proof of the case $m = 1$ is similar. Now, let $m, n \geq 2$. Since M is *u.p.*-monoid, there exist i, j with $1 \leq i \leq n$ and $1 \leq j \leq m$ such that g_ish_j is uniquely presented by considering two subsets $K = \{g_1, g_2, \ldots, g_n\}$ and $H = \{sh_1, sh_2, \ldots, sh_m\}$ of a monoid $M, s \in M$. We may assume without loss of generality, that $i = n$ and $j = m$. Thus $b_n\sigma_{g_n}(c\,(a_m)) \in Nil(R)$. Hence for any $c \in R, b_n\sigma_{g_n}(c_k(a_m)) \in nil(R)$ so, there exist a positive integer $k \geq 1$ such that $(b_n\sigma_{g_n}(c\,(a_m)))^k = 0$. Then $(b_n\sigma_{g_n}(c\,(a_m)))(b_n\sigma_{g_n}(c\,(a_m)))\cdots(b_n\sigma_{g_n}(c\,(a_m))) = 0$. Now, since by hypothesis $\sigma : M \to Aut(R)$ is a compatible monoid homomorphism, we have that $(b_n\sigma_{g_n}(c\,a_m))\cdots(b_n\sigma_{g_n}(c\,a_m))(b_nc\,a_m) = 0$ and then we conclude that $(b_n\sigma_{g_n}(c\,a_m))\cdots(b_n\sigma_{g_n}(c\,a_m))\sigma_{g_n}(b_nc\,a_m) = 0$. So $(b_n\sigma_{g_n}(c\,a_m))\cdots b_n\sigma_{g_n}(c\,a_mb_na_m) = 0$ and hence $(b_n\sigma_{g_n}(c\,a_m))\cdots(b_n\sigma_{g_n}(c\,a_m))(b_nc\,a_m)^2 = 0$. Continuing this procedure yields that $(b_nc\,a_m)^k = 0$, hence $b_n\sigma_{g_n}(c\,a_m) \in Nil(R)$. Therefore, $a_n\sigma_{g_n}(\sigma_s(c\,b_m)) \in nil(R)$ for each $c \in R$. Then we have $a_m(\varphi - b_ng_n)\phi\psi = a_m\varphi\phi\psi - a_mb_ng_n\phi\psi \in nil(R) * M$, since $nil(R)$

is an ideal of R. By induction hypothesis and by compatibility, we have $b_n\sigma_{g_n}(c\,a_jb_n) = 0$ implies $b_i\sigma_{g_n}(c\,a_m) \in nil(R)$ for all i. Applying the preceding method repeatedly, we obtain $b_i\sigma_{g_i}(\sigma_s(c\,a_j)) \in nil(R)$ for each i, j and each $c \in R$. Thus, by rigidness we have $a_j\sigma_{h_j}(\sigma_s(c\,b_i)) \in nil(R)$ as desired.

Corollary 3. (1) *Every σ-compatible NI-ring is σ-skew strongly $\mathbb{Z}$-nil-reflexive.*
(2) *Every NI-ring is nil-reflexive.*

Proof. (1) Taking $M = \{\ldots, x^{-2}, x^{-1}, 1, x^1, x^2, \ldots\}$ and $\sigma_{x^n}(\lambda) = \sigma^n(\lambda)$ for each $n \in \mathbb{Z}$ and any $\lambda \in R$, we have $R * M \cong R[x, x^{-1}, \sigma]$, and the result follows from Theorem 3.
(2) Taking $M = \{1, x^1, x^2, \ldots\}$ and $\sigma_{x^n}(\lambda) = \lambda$ for each $n \in \mathbb{N} \cup \{0\}$ and any $r \in R$, we have $R * M \cong R[x]$, and the result follows from Theorem 3.

Corollary 4. *(Theorem 3.1 [20]) Let M be a u.p.-monoid and R be a reduced ring. Then, R is strongly M-reflexive.*

Corollary 5. *Let R be a ring, where $nil(R)$ is an ideal of R. Let M be a u.p.-monoid and $\sigma : M \to Aut(R)$ be a compatible monoid homomorphism. Then, R is σ-skew strongly M-nil-reflexive*

Proof. Since $nil(R)$ is an ideal, it is an NI-ring. Therefore, the proof follows from Theorem 3.

The following example shows that R being a reduced ring in Theorem 3 is not superfluous.

Example 2. *Let M be a monoid with $\mid M \mid \geq 2, S = M_2(F)$ and $\sigma : M \to Aut(R)$ be a compatible monoid homomorphism. Then, S is not σ-skew strongly M-nil-reflexive.*

Proof. Take $e \neq g \in M$ and let

$$\varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g, \psi = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} g \in S * M.$$

For $\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g \in S * M$, it is easy to check that $\varphi\phi\psi \in nil(S * M)$.
But, we have

$$\psi\phi\varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^2 + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} g^3 \neq 0$$

which implies that S is not σ-skew strongly M-*nil*-reflexive.

Shaban and Mohammed [22] proved that $Nil(R * M) \neq Nil(R) * M$ for an NI-ring R and a *u.p.*-monoid M with $\mid M \mid \geq 2$. Based on their work, we have the following result.

Theorem 4. *Let R be any ring, M be any monoid with $\mid M \mid \geq 2$ and $\sigma : M \to Aut(R)$ be a compatible monoid homomorphism such that $Nil(R * M) = Nil(R) * M$. Then R is σ-skew strongly M-nil-reflexive.*

Proof. Let $\varphi = \Sigma_{i=1}^{n} b_i g_i$ and $\psi = \Sigma_{j=1}^{m} a_j h_j$ be nonzero elements in $R * M$ such that $\varphi\phi\psi \in nil(R) * M$ for any $\phi \in R * M$. We have $\varphi\phi\psi \in nil(R * M)$, which is equivalent to the existence of a positive integer ℓ such that $(\varphi\phi\psi)^{\ell} = (\varphi\phi\psi)(\varphi\phi\psi)\dots(\varphi\phi\psi) = 0$. Since σ is a compatible monoid homomorphism, we have $b_i\sigma_{g_i}(c(a_j)) \in nil(R)$. Therefore, $a_j\sigma_{h_j}(c(b_i)) \in nil(R)$, and the proof is complete.

In Proposition 3.4 [20], it was proved that if I is a reduced ideal and R/I is strongly M-reflexive, then R is strongly M-reflexive. Based on this result, we have the following statement.

Theorem 5. *Suppose that R is a ring, M is a strictly ordered monoid, and $\sigma : M \to Aut(R)$ is a compatible monoid homomorphism. If I is an ideal of R contained in $nil(R)$, then R/I is $\overline{\sigma}$-skew strongly M-nil-reflexive if and only if R is σ-skew strongly M-nil-reflexive.*

Proof. "$\Longrightarrow$" Let $\varphi, \psi \in R*M$ satisfying $\varphi\phi\psi \in nil(R)*M$ for all $\phi \in R*M$. We write $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n, \phi = c_1l_1 + c_2l_2 + \cdots + c_dl_d$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ with $g_1 < g_2 < \ldots < g_n, h_1 < h_2 < \ldots < h_m$. We will use transfinite induction on the strictly totally ordered set $(M, \leq)$ to show that $a_j\sigma_{h_j}(\sigma_s(r\, b_i)) \in nil(R)$. Since R/I is $\overline{\sigma}$-skew strongly M-*nil*-reflexive and

$$\begin{aligned}
\bar{0} = {} & (\bar{b}_1g_1 + \bar{b}_2g_2 + \cdots + \bar{b}_ng_n)(rs)(\bar{a}_1h_1 + \bar{a}_2h_2 + \cdots + \bar{a}_mh_m) \\
= {} & (b_1 + I)\bar{\sigma}_{g_1}(r(a_1 + I)) + (b_2 + I)\bar{\sigma}_{g_2}(r(a_2 + I)) + \cdots + (b_n + I)\bar{\sigma}_{g_n}(r(a_m + I)) \\
= {} & (b_1\bar{\sigma}_{g_1}(r\, a_1) + I) + (b_2\bar{\sigma}_{g_2}(r\, a_2) + I) + \cdots + (b_n\bar{\sigma}_{g_n}(r\, a_m) + I) \in (R/I) * M
\end{aligned}$$

for $r \in R, s \in M$, so we have $b_i\sigma_{g_i}(r\, a_j) \in I$ for all i, j. Since M is a strictly totally ordered monoid, we have $g_1h_1 < g_ih_1 \leq g_ih_j = g_1h_1$ for $i \neq 1$ or $j \neq 1$. It follows that $b_1\sigma_{g_1}(r\, a_1) = 0$, i.e., $b_1\sigma_{g_1}(\sigma_s(r\, a_1)) \in nil(R)$ since I is an ideal of R contained in $nil(R)$. Now suppose that $b_ir\, a_j = 0$ for all $1 \leq i \leq n, 1 \leq j \leq m$ with $w \in M$ is such that for any g_i and $h_j, g_ih_j < w$. We will show that $b_i\sigma_{g_i}(\sigma_s(ra_j)) \in nil(R)$ for any g_i and h_j with $g_ih_j = w$. Set $X = \{(g_i, h_j) | g_ih_j = w\}$. Then X is a finite set. We write X as $\{(g_{i_t}, h_{j_t}) | t = 1, 2, \ldots, k\}$ such that $g_{i_1} < g_{i_2} < \ldots < g_{i_k}$. Since M is cancellative, $g_{i_1} = g_{i_2}$ and $g_{i_1}h_{j_1} = g_{i_2}h_{j_2} = w$ imply $h_{j_1} = h_{j_2}$. Since $\leq$ is a strict order, $g_{i_1} < g_{i_2}$ and $g_{i_1}h_{j_1} = g_{i_2}h_{j_2} = w$ imply $h_{j_2} < h_{j_1}$. Thus, we have $h_{j_k} < h_{j_{k-1}} < \ldots < h_{j_2} < h_{j_1}$. Now

$$\sum_{(g_i,h_j)\in X} b_i\sigma_{g_i}(\sigma_s(ra_j)) = \sum_{t=1}^{k} b_{i_t}\sigma_{g_{i_t}}(\sigma_s(ra_{j_t})) = 0.$$

For any $t \geq 2, g_{i_1}h_{j_t} < g_{i_t}h_{j_t} = w$, and so $b_{i_1}\sigma_{g_{i_t}}(\sigma_s(ra_{j_t})) = 0$ by induction hypothesis. Thus, $b_{i_1}ra_{j_t} = 0$ because R is M-compatible. Since I is reduced and $\sigma_{g_{i_t}}(a_{j_t})I(b_{i_1}) \subseteq I$, then we have $(a_{j_t}Ib_{i_1})^2 = 0$ and I is reduced. Thus, for any $t \geq 2, (b_{i_t}ra_{j_t})(b_{i_1}ra_{j_1})^2 = (b_{i_t}ra_{j_t})(b_{i_1}ra_{j_1})(b_{i_1}ra_{j_1}) \in (b_{i_t}ra_{j_t})I(b_{i_t}ra_{j_t}) = (b_{i_t}ra_{j_t})I(b_{i_t}ra_{j_t}) = 0$, which implies that $(b_{i_t}ra_{j_t})(b_{i_1}ra_{j_1})^2 = 0$. Now multiplying $\sum_{t=1}^{k} b_{i_t}\sigma_{i_t}(\sigma_s(r\, a_{j_t})) = 0$ on the right by

$(b_{i_1}\sigma_{i_1}(ra_{j_1}))^2$, we obtain

$$0 = (b_{i_t}\sigma_{i_t}(ra_{j_t}))(b_{i_1}\sigma_{i_1}(ra_{j_1}))^2 = (b_{i_1}\sigma_{i_1}(ra_{j_1}))^2(b_{i_1}\sigma_{i_1}(ra_{j_1})) = (b_{i_1}\sigma_{i_1}(ra_{j_1}))^3.$$

Since $b_{i_1}\sigma_{i_1}(ra_{j_1}) \subseteq I$ and I is reduced and R is M-compatible, we have $b_{i_1}\sigma_{i_1}(ra_{j_1}) = 0$. Thus, $\sum_{t=2}^{k} b_{i_t}\sigma_{i_t}(ra_{j_t}) = 0$. Then, $b_{i_t}\sigma_{i_t}(ra_{j_t}) \in nil(R)$ for $t \geq 2$. Multiplying $(b_{i_2}\sigma_{i_2}(ra_{j_2}))^2$ on $\sum_{t=2}^{k} b_{i_t}\sigma_{i_t}(ra_{j_t}) = 0$ from the right-hand side, we obtain $b_{i_2}\sigma_{i_2}(ra_{j_2}) = 0$. Thus, $b_{i_2}\sigma_{i_2}(ra_{j_2}) \in nil(R)$ by the same way as the above. Continuing this process, we can prove $b_{i_t}\sigma_{i_t}(ra_{j_t}) = 0$ for $t = 1, \ldots, k$. Thus $b_i\sigma_{g_i}(\sigma_s(ra_j)) \in nil(R)$ because I is an ideal of R contained in $nil(R)$ for any i and j with $g_ih_j = w$. Therefore, by transfinite induction $b_i\sigma_{g_i}(\sigma_s(ra_j)) \in nil(R)$ for any i and j. Thus, $a_j\sigma_{h_j}(\sigma_s(rb_i)) \in nil(R)$. Therefore, R is σ-skew strongly M-*nil*-reflexive.
" $\Longleftarrow$ " Let $\bar{\varphi}, \bar{\psi} \in (R/I) * M$ with $\bar{\varphi}\bar{\phi}\bar{\psi} \in nil(R/I) * M$ for all $\bar{\phi} \in (R/I) * M$. Where $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n, \phi = c_1l_1 + c_2l_2 + \cdots + c_dl_d$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ be nonzero elements in $R*M$. Since $Nil(R)$ is an ideal of R, we write $\overline{R} = R/nil(R)$ and define $\overline{\sigma} : M \to Aut(\overline{R})$ by $\sigma_g(x + Nil(R)) = \sigma_g(x) + Nil(R)$, for each $g \in M$. We claim that $\overline{\sigma}$ is a compatible monoid homomorphism. For each $x, y \in R$, let $(x + Nil(R))(y + Nil(R)) = \bar{0}$. Then $xy \in Nil(R)$ and hence $(x + Nil(R))\bar{\sigma}(y + Nil(R)) = \bar{0}$. The converse is similar. Therefore, $\overline{\sigma}$ is compatible. For any $\varphi = b_1g_1 + b_2g_2 + \cdots + b_ng_n \in R * M$, we denote $\bar{\varphi} = \sum_{i=1}^{n}(b_i + nil(R))g_i \in \bar{R} * M$. It is easy to see that the mapping $\overline{\sigma} : R * M \longrightarrow \overline{R} * M$ defined by $\sigma(\varphi_i) = \overline{\varphi}_i$ is a ring homomorphism. Since $\bar{\varphi}\bar{\phi}\bar{\psi} \in nil(R/I) * M$. Then there exist a positive integer $\ell \in \mathbb{N}$ such that $(\bar{\varphi}\bar{\phi}\bar{\psi})^\ell \in nil(R/I) * M$. So $b_i\sigma_{g_i}(ra_j)^\ell \in I$ for any i, j and $r \in R$. So $b_i\sigma_{g_i}(\sigma_s(ra_j)) \in nil(R)$ since $I \subseteq nil(R)$ and by compatibility, we have $b_i\sigma_{g_i}(ra_j) \in nil(R)$ and $a_j\sigma_{h_j}(\sigma_s(rb_i)) \in nil(R)$ since R is σ-skew strongly M-*nil*-reflexive, this mean that $\psi\phi\varphi \in nil(R) * M$. Thus $\bar{\psi}\bar{\phi}\bar{\varphi} \in nil(R/I) * M$. Therefore, R/I is $\bar{\sigma}$-skew strongly M-*nil*-reflexive.

Proposition 3. *Let M be a finitely generated abelian group and $\sigma : M \to Aut(R)$ be a compatible monoid homomorphism. Then M is torsion free if and only if there exist a non-zero ring R such that R is σ-skew strongly M-nil-reflexive.*

Proof. Let M be a finitely generated torsion-free abelian group. Thus, $M \cong \amalg_1^N \mathbb{Z}$ and so M is a *u.p.*-monoid, as shown in Lemma 2. Therefore, for any NI ring R, R is σ-skew strongly M-*nil*-reflexive by Theorem 3.

Conversely, suppose $g^\ell = e$ for some $e \neq g \in M$ and positive integer ℓ. Let N be a cyclic subgroup of M generated by $\{g\}$. Therefore, R is σ-skew strongly N-*nil*-reflexive, which leads to a contradiction, as shown in Lemma 3. Hence, M is torsion-free.

We will now provide some examples of σ-skew strongly M-*nil*-reflexive ring. In Theorem 2.6 [13], Kwak and Lee proved that R is a reflexive ring if and only if $Mat_n(R)$ is a reflexive ring for all $n \geq 1$. However, this is not the case in σ-skew strongly M-*nil*-reflexive rings of R. There exist σ-skew strongly M-*nil*-reflexive rings over which matrix rings need not be σ-skew strongly M-reflexive, as shown below.

Example 3. *Let S be a torsion-free and cancellative monoid and $\sigma : M \to Aut(R)$ be a compatible monoid homomorphism.*
(1) If R is a ring with $nil(R)$ an ideal of R, then R is σ-skew strongly M-nil-reflexive.
(2) For any reduced ring R, the ring $T_n(R)$ is $\overline{\sigma}$-skew strongly M-nil-reflexive. However, the ring of all 2×2 matrices over any field is not $\overline{\sigma}$-skew strongly M-nil-reflexive.
(3) For R be a reduced ring. Consider the ring

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R; 1 \le i, j \le n \right\}.$$

Then $S_n(R)$ is not $\overline{\sigma}$-skew strongly M-reflexive, when $n \ge 4$, but $S_n(R)$ and R are σ-skew strongly M-nil-reflexive for all $n \ge 1$.

Solution (1). Suppose $\varphi, \psi \in R * M$, with $\varphi\phi\psi$ is nilpotent for all $\phi \in R * M$, where $\varphi = b_1 g_1 + b_2 g_2 + \cdots + b_n g_n, \phi = c_1 l_1 + c_2 l_2 + \cdots + c_d l_d$ and $\psi = a_1 h_1 + a_2 h_2 + \cdots + a_m h_m$. So there exist a positive integer ℓ such that $(\varphi\phi\psi)^\ell = 0$. Therefore $(b_i \sigma_i(\sigma_s(c a_j)))^\ell = 0$, for any $s \in M, i, j$. Then, $b_i \sigma_{g_i}(\sigma_s(c\, a_j)) \in nil(R)$ and so $a_j \sigma_{h_j}(\sigma_s(c\, b_i)) \in nil(R)$. Hence $\psi\phi\varphi$ is nilpotent.

(2). Let R be a ring, by [4], $nil(T_n(R)) = \begin{pmatrix} nil(R) & R & R & \cdots & R \\ 0 & nil(R) & R & \cdots & R \\ 0 & 0 & nil(R) & \cdots & R \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & nil(R) \end{pmatrix}$.
Assuming that R is a reduced, we know that $nil(R) = 0$ and that $nil(T_n(R))$ is an ideal. From (1), it follows that $T_n(R)$ is $\overline{\sigma}$-skew strongly M-*nil*-reflexive. However, if we take $A = E_{12}$ and $B = E_{22} \in Mat_2(F)$, where F is a field, and $C \in Mat_2(F)$, we see that ACB is nilpotent, but $BCA = E_{22}$ is not nilpotent. This shows that $Mat_2(F)$ is not $\bar{\sigma}$-skew strongly M-*nil*-reflexive.
(3). By the same argument as in Example 2.3 [13]. For a nonzero reduced ring S, the ring $R = \left\{ \begin{pmatrix} \alpha & \beta & \delta \\ 0 & \alpha & \gamma \\ 0 & 0 & \alpha \end{pmatrix} \mid \alpha, \beta, \gamma \in S \right\}$ is semicommutative by Proposition 1.2 [11]. If we take $\varphi = C_{E_{23}} g$ and $\psi = C_{E_{12}} h \in S_n(R) * M$ for any $g, h \in M$, we have $\varphi(S_n(R) * M)\psi = 0$ but $\psi(S_n(R) * M)\varphi \ne 0$. Therefore, $S_n(R)$ is not $\overline{\sigma}$-skew strongly M-reflexive. Since R is

reduced, it follows that $S_n(R)$ is σ-skew strongly M-*nil*-reflexive. Note that

$$nil(S_n(R)) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a \in nil(R), a_{ij} \in R; 1 \leq i,j \leq n \right\}.$$

The ring R being reduced implies that $nil(S_n(R))$ is an ideal. By (1), $S_n(R)$ is $\overline{\sigma}$-skew strongly M-*nil*-reflexive.

By Example 3(2) for n by n upper triangular matrix ring over R. It is easy to verify the next result.

Proposition 4. *Let M be a torsion-free and cancellative monoid and $\sigma : M \to Aut(R)$ a compatible monoid homomorphism. A ring R is σ-skew strongly M-nil-reflexive if and only if $T_n(R)$ is $\overline{\sigma}$-skew strongly M-nil-reflexive, for any positive integer n.*

Proof. It suffices to show " $\Rightarrow$ " Let $\varphi, \psi \in T_n(R) * M$ such that $\varphi\phi\psi \in nil(T_n(R)) * M$, where $\varphi = (b_{ij}), \phi = (c_{ij})$ and $\psi = (a_{ij})$ for all (i,j)th entry of the matrix. Since $nil(T_n(R)) = \{(a_{ij}) | a_{ij} \in nil(R)\}$, then we have $C_{\varphi_{ii}\phi_{ii}\psi_{ii}} \in nil(R)$ for each $1 \leq i \leq n$. Since R is σ-skew strongly M-*nil*-reflexive, there exist some positive integer m_i such that $(b_{ii}\sigma_{g_{ii}}(\sigma_s(c_{ii}a_{ii})))^{m_i} = 0$. Then, by compatibility $b_{ii}\sigma_{g_{ii}}(c_{ii}a_{ii})$ is nilpotent and so $a_{ii}\sigma_{h_{ii}}(\sigma_s(c_{ii}b_{ii}$ is nilpotent. Thus, $\psi\phi\varphi \in nil(T_n(R)) * M$. Therefore, $T_n(R)$ is $\overline{\sigma}$-skew strongly M-*nil*-reflexive.

Proposition 5. *Let M be a strictly ordered monoid and $\sigma : M \to Aut(R)$ a compatible monoid homomorphism. If R is finite subdirect product of σ-skew strongly M-nil-reflexive rings, then R is σ-skew strongly M-nil-reflexive.*

Proof. Let $I_k (k = 1, \ldots, l)$ be ideals of R such that R/I_k is $\overline{\sigma}$-skew strongly M-*nil*-reflexive and $\bigcap_{k=1}^{l} I_k = 0$. Let φ and ψ be elements in $R * M$ such that $\varphi\phi\psi \in nil(R) * M$ for all $\phi \in R * M$. Clearly, $\bar{\varphi}\bar{\phi}\bar{\psi} \in nil(R/I_k) * M$. Since R/I_k is $\overline{\sigma}$-skew strongly M-*nil*-reflexive, we have $(b_i\sigma_{g_i}(\sigma_s(ra_j)))^{\ell} \in I_k$, where σ is compatible and $r \in R$, for some positive integer ℓ. Therefore, $(b_i\sigma_i(\sigma_s(ra_j)))^{\ell} \in \bigcap_{k=1}^{l} I_k = 0$. Hence, $(b_i\sigma_{g_i}(\sigma_s(ra_j)))^{\ell} = 0$, and we conclude that $b_i\sigma_{g_i}(\sigma_s(ra_j)) \in nil(R)$ since σ is compatible and *nil*-reflexivity. Then, $a_j\sigma_{h_j}(\sigma_s(rb_i)) \in nil(R)$. Therefore, $\psi\phi\varphi \in nil(R) * M$, the proof is done.

4. Conclusion

This paper introduced and studied two important concepts, namely σ-skew strongly M-reflexive and σ-skew strongly M-*nil*-reflexive. The study covered the fundamental properties of skew monoid rings of the form $R * M$, and established several important results. The study provided some examples and discussed related results from the subject. Overall, our study provides important insights into the properties of skew monoid rings and their relationship to nilpotent elements. We anticipate that our findings will have significant implications for the theory of noncommutative algebra, and will lead to further research in this area.

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References

[1] E. Ali. Nilpotent elements and nil-reflexive property of generalized power series rings. *Adv. P. Math*, 12(11):676–692, 2022.

[2] E. Ali. A note on skew generalized power serieswise reversible property. *Int. J. Anal. Appl.*, 21:69, 2023.

[3] E. Ali. On nil skew generalized power series reflexive rings. *Adv. Math. Sci. J.*, 12(1):287–306, 2023.

[4] R. Antoine. Nilpotent elements and armendariz rings. *J. Algebra*, 319(8):3128–3140, 2008.

[5] G. F. Birkenmeier and J. K. Park. Triangular matrix representation of ring extensions. *J. Algebra*, 265:457–477, 2003.

[6] R. C. Courter. Rings all of whose factor rings are semiprime. *Canad. Math. Bull.*, 12(4):417–426, 1969.

[7] M. Habibi and A. Moussavi. Nilpotent elements and nil-armendariz property of monoid rings. *J. Algebra and Its Applications*, 11(4):1250080, 2012.

[8] E. Hashemi and A. Moussavi. Polynomial extensions of quasi-baer rings. *Acta Math. Hungar.*, 107(3):207–224, 2005.

[9] Y. Hirano. On annihilator ideals of a polynomial ring over a noncommutative ring. *J. Pure Appl. Algebra*, 168:45–52, 2002.

[10] N. K. Kim and Y. Lee. Extensions of reversible rings. *J. Pure Appl. Algebra*, 185:207–223, 2003.

[11] N. K. Kim and Y. Lee. Extensions of reversible rings. *J. Pure Appl. Algebra*, 185:207–223, 2003.

[12] T. Kwak, Y. Lee, and S. Yun. Reflexive property skewd by endomorphisms. *Korean J. Math.*, 22(2):217–234, 2014.

[13] T. K. Kwak and Y. Lee. Reflexive property of rings. *Comm. Algebra*, 40:1576–1594, 2012.

[14] Z. K. Liu. Armendariz rings relative to a monoid. *Comm. Algebra*, 33(3):649–661, 2005.

[15] G. Mason. Reflexive ideals. *Comm. Algebra*, 9(17):1709–1724, 1981.

[16] A. R. Nasr-Isfahani and A. Moussavi. On weakly rigid rings. *Glasg. Math. J.*, 51(3):425–440, 2009.

[17] J. Okninski. *Semigroup Algebra*. W. A. Benjamin, New York, 1991.

[18] L. Ouyang. On weak annihilator ideals of skew monoid rings. *Comm. Algebra*, 39(11):4259–4272, 2011.

[19] D. S. Passman. *The Algebraic Structure of Group Rings*. Wiley, New York, 1977.

[20] Z. Peng, Q. Gu, and L. Zhao. Extensions of strongly reflexive rings. *Asian-European Journal of Mathematics*, 8(4):1550078, 2015.

[21] S. Rege, M. B. andChhawchharia. Armendariz rings. *Proc. Japan Acad. Ser. A Math. Sci.*, 73(1):14–17, 1997.

[22] S. Safarisabe and E. Mohammed. Skew monoid rings with annihilator conditions. *Asian-Eur. J. of Math*, 10(1):1550033, 2017.

[23] H. Tominaga. On s-unital rings. *Math. J. Okayama Univ.*, 18:117–134, 1976.

[24] L. Zhao, X. Zhu, and Q. Gu. Reflexive rings and their extensions. *Math. Slovaca*, 63(3):417–430, 2013.