Nuclearity of a class of vector-valued sequence spaces

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Abstract. In this note, we deal with a perfect sequence space λ and a convex bornological space E to introduce and study the space λ(E) of all totally λ-summable sequences from E. We prove that λ(E) is complete if and only if λ and E are complete, nuclear if and only if λ and E are nuclear, and we make use of a result of Ronald C. Rosier [10] to give a similar characterization of the nuclearity of the space λ{E} of all absolutely λ-summable sequences in a locally convex E.

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Introduction

In connection with the nuclearity of a locally convex space E, A. Pietsch in [9] introduced the spaces ℓ_p(E) and ℓ_p{E} respectively of weakly ℓ_p-summable and absolutely ℓ_p-summables sequences in E. In [8], he used these spaces to study the absolutely p-summing operators. Later, he introduced and studied also the space λ{E} of λ-summable sequences in E, for a perfect sequence space λ in the sense of Köthe endowed with its normal topology. Many other authors were interested in the study of these spaces. Ronald C. Rosier in [10] considered a general polar topology on λ{E} and got a precise description of the topological dual and its equicontinuous subsets. M. Florencio and P. J. Paúl [3], considering general polar topologies, obtained many interesting results such as barreledness conditions. In [1] and [2], they studied the space λ(E) of weakly λ-summables sequences in E and represented this space as the completion of the injective tensor product λ⊗E. In [6] and [7], L. Oubbī and M. A. Ould Sidaty reconsidered the space λ(E) and obtained some of its properties. They mainly described the continuous dual space of λ(E). While in [11] and [13], characterizations of the reflexivity of λ(E) in terms of that of λ and E and the AK-property are given. A characterization of the nuclearity of the space of weakly λ-summable sequences is given in [12].

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In this note, we are concerned with the nuclearity of the convex bornological space $\lambda(E)$ of all totally $\lambda$-summable sequences in $E$, in the sense of [3], where $E$ is a convex bornological space.

In sections 1 and 2, we endow this space with a structure of b-space, and study some of its properties.

The section 3 is devoted to the nuclearity of $\lambda(E)$. We prove mainly that $\lambda(E)$ possesses this property if and only if both of $\lambda$ and $E$ have.

In Section 4, we provide an application of the results of Section 3 on the nuclearity of the space $\lambda\{E\}$ of absolutely $\lambda$-summable sequences in a locally convex space $E$.

1. Preliminaries

For a linear space $E$, we mean by a convex bornology on $E$, a collection of subsets of $E$ covering $E$, hereditary for the inclusion, and closed for the finite unions, the addition, the scalar multiplication and the formation of absolutely convex hulls. We say then that $E$ is a convex bornological space or simply a b-space. The elements of the bornology of $E$ are called bounded sets of $E$.

A collection $B$ of bounded sets of $E$ is a basis for its bornology if every bounded set in $E$ is contained in an element of $B$. In the sequel, we assume that the members of $B$ are absolutely convex.

A b-space $E$ is said to be Hausdorff if the only bounded linear subspace of $E$ is $\{0\}$. We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ converges to $x \in E$, or that $x$ is a limit of $\{x_n\}_{n=1}^{\infty}$ in $E$ if there exists an element $B \in B$ such that $\{x_n - x\}_{n=1}^{\infty}$ is contained and convergent to 0 in the normed space $(E_B, \| \cdot \|_B)$, where $E_B$ is the subspace of $E$ generated by $B$ and $\| \cdot \|_B$ is the gauge of $B$.

A subset of a b-space $E$ will be said to be closed if it contains the limits of all its sequences.

A Banach disk in a b-space $E$ is an element $B \in B$ for which the normed space $E_B$ is complete. $E$ is said to be b-complete or simply complete if every bounded set in $E$ is contained in a Banach disk in $E$.

A linear mapping between two b-spaces $E$ and $F$ is said to be bounded if it transforms bounded sets of $E$ to bounded sets of $F$. A bounded linear mapping transforms convergent sequences to convergent ones. A bornological isomorphism is a bounded linear bijection whose inverse is also bounded.

The Köthe dual of a sequence space $\lambda$ is defined as

$$\lambda^\times = \left\{ (\beta_n) \subset \mathbb{C} : \sum_{n=1}^{\infty} |a_n \beta_n| \text{ converges for all } (a_n) \in \lambda \right\}.$$ 

We see that $\lambda \subset \lambda^{\times \times} = (\lambda^{\times})^{\times}$; we say that $\lambda$ is perfect if the equality holds.

The normal cover of a subset $S$ of $\lambda$ is the subset of $\lambda$ formed by the sequences of the
form $(\varepsilon_n\alpha_n)_n$ where $(\alpha_n)_n \in S$ and $(\varepsilon_n)_n \subset \mathbb{C}$ with $|\varepsilon_n| \leq 1$, for all $n$. We see that $S$ is contained in its normal cover. $S$ is said to be normal or solid if it coincides with its normal cover.

For the general theory of locally convex spaces and Köthe sequence spaces, we refer the reader to [5].

Throughout this paper, $\lambda$ will be a perfect (and then a normal) sequence space endowed with a normal bornology, that is a convex bornology having a basis $S$ of solid sets, and for which the standard coordinate projections from $\lambda$ to $\mathbb{C}$ are bounded.

Following the terminology of [3], a sequence $(x_n)_n \subset E$ is said to be totally $\lambda$-summable in $E$ if there exists an absolutely convex element $B \in \mathbb{B}$ such that $(x_n)_n \subset E_B$ and $(\|x_n\|_B)_n \in \lambda$. In other words, $(x_n)_n = (\alpha_n b_n)_n$, with $(\alpha_n)_n \in \lambda$ and \{b_n\}_n=1^\infty \subset B.

Starting from this definition, we introduce the vector valued sequence space

$$
\lambda(E) = \left\{ (x_n) \in E : \exists B \in \mathbb{B}, (x_n)_n \subset E_B \text{ and } (\|x_n\|_B)_n \in \lambda \right\}.
$$

Due to the properties of $\mathbb{B}$, the triangle inequality of the norms $\| \cdot \|_B$ and the fact that $\lambda$ is normal, we see that $\lambda(E)$ is a linear space. For $S \in S$ and $B \in \mathbb{B}$, we define

$$
S(B) = \left\{ (x_n)_n \subset E_B, (\|x_n\|_B)_n \in S \right\}.
$$

2. Properties of $\lambda(E)$

In the sequel, the b-spaces $E$ equipped with the convex bornology with basis $\mathbb{B}$ and $\lambda$ with the normal bornology with basis $S$, will be supposed to be Hausdorff spaces.

Starting from this setting, one can define, in a natural way, a convex bornology on $\lambda(E)$ with basis $S(\mathbb{B})$ by setting

$$
S(\mathbb{B}) = \left\{ H \subset \lambda(E) : \exists S \in S, B \in \mathbb{B} \text{ such that } H = S(B) \right\}.
$$

In view of the hypothesis made on $S$ and $\mathbb{B}$, $S(\mathbb{B})$ is indeed a basis for a convex bornology on $\lambda(E)$ for which $\lambda(E)$ is a Hausdorff space.

**Lemma 1.** For a fixed $k \in \mathbb{N}$, denote by $\pi_k$ the projection from $\lambda(E)$ on $E$ defined by

$$
\pi_k(x) = x_k, \text{ for all } x = (x_n) \in \lambda(E).
$$

Then, $\pi_k$ is a bounded linear map.

**Proof.** Let $B \in \mathbb{B}$ and $S \in S$ and fix $k \in \mathbb{N}$. Since the bornology of $\lambda$ is normal, the set $\{\alpha_k : (\alpha_n)_n \in S\}$ is bounded in $\mathbb{C}$, and then so is $\{\|x_k\| : (x_n)_n \in S(B)\}$. This means that $\{x_k : (x_n)_n \in S(B)\}$ is bounded in $E_B$. Thus, $\pi_k$ is bounded. ■
Proposition 1. The spaces $\lambda$ and $E$ can be identified with closed subspaces of $\lambda(E)$.

Proof. Let $I : E \rightarrow \lambda(E), t \rightarrow te_1$, where $t$ is at the first component. It is clear that $I$ is linear and one to one. Let $B \in \mathbb{B}$, and $S \in \mathbb{S}$ such that $e_1 \in S$, then $I(B) \subset S(B)$ and $I$ is bounded. Inversely, $I^{-1} : I(E) = Ee_1 \rightarrow E$ is the restriction of $\pi_1$ to the subspace $I(E)$, and then it is bounded by Lemma 1. It remains to show that $I(E)$ is closed in $\lambda(E)$. We have $I(E) = \bigcap_{k \neq 1} \pi_k^{-1}\{\{0\}\}$. Since $E$ is supposed to be a Hausdorff space, then $\{0\}$ is closed and so is $I(E)$.

Now, fix $0 \neq x_0 \in E$ and let $g : \lambda \rightarrow \lambda(E)$, $\alpha = (\alpha_n)_n \rightarrow (\alpha_n x_0)_n = \alpha x_0$. It is clear that $g$ is bounded. Inversely, if $\lambda \in \mathbb{S}$ and $B \in \mathbb{B}$ with $x_0 \in B$. Then, $g(S) \subset S(B)$; so $g$ is bounded. Inversely, if $S \in \mathbb{S}$ and $B \in \mathbb{B}$, then $g^{-1}(S(B) \cap \lambda x_0) = \frac{1}{\|x_0\|_B}S$, and then $g^{-1} : g(E) = \lambda x_0 \rightarrow \lambda$ is bounded. It remains to show that $g(\lambda)$ is closed in $\lambda(E)$. Let $(\alpha^{(k)}_n x_0 = (\alpha^{(k)}_n, x_0)_n)_{k=1}^\infty$ be a sequence in $\lambda x_0$ which converges to $x = (x_n)_n \in \lambda(E)$. By Lemma 1, $(\alpha^{(k)}_n x_0)_{k=1}^\infty$ converges to $x_n$ in $E$, for every $n$. As, the subspace $\mathbb{C} x_0$ of $E$ is closed in $E$, $x_n$ must belong to $\mathbb{C} x_0$. Then, there is $\alpha = (\alpha_n)$ such that $x = (x_n)_n = \alpha x_0$. It is easy to see that $\alpha \in \lambda$. We conclude that $\lambda x_0$ is closed in $\lambda(E)$.

Proposition 2. $\lambda(E)$ is complete if and only if $\lambda$ and $E$ are complete.

Proof. If $\lambda(E)$ is complete, then so are $\lambda$ and $E$ by Proposition 1. Inversely, suppose $\lambda$ and $E$ are complete. We only show that if $B$ and $S$ are Banach disks in $E$ and $\lambda$ respectively, then $S(B)$ is a Banach disk in $\lambda(E)$. To simplify the notations, we set $F = \lambda(E)$, $H = S(B)$ and $\pi$ the gauge of $H$.

Let $((x^i)_n)_{n=1}^\infty$ be a Cauchy sequence in $(F, \pi)$. We have

$$\left\|\left\|x^i_n\right\|_B\right\|_S - \left\|\left\|x^j_n\right\|_B\right\|_S \leq \left\|\left\|x^i_n\right\|_B - \left\|x^j_n\right\|_B\right\|_S \leq \left\|\left\|x^i_n\right\|_B - \left\|x^j_n\right\|_B\right\|_S$$

This means that $\{\left\|x^i\right\|_B\}_{i=1}^\infty$ is a Cauchy sequence in the complete space $(\lambda_S, \|\cdot\|_S)$; let $\alpha = (\alpha_n)$ be its limit in $\lambda_S$. Fix $n \in \mathbb{N}$. Due to the boundedness of the projections, $\{\left\|x^i_n\right\|_B\}_{i=1}^\infty$ converges to $\alpha_n$ and $\{x^i_n\}_{i=1}^\infty$ is a Cauchy sequence in the complete space $E_B$; denote by $x_n$ its limit. Thus, $\left\|x_n\right\|_B = \alpha_n$, and $x = (x_n)_n \in \lambda(E)$. It remains to prove the convergence of $\{x^i_n\}_{i=1}^\infty$ to $x$. This derives from the fact that $\{\left\|x^i - x\right\|_B\}_{i=1}^\infty$ is a Cauchy sequence in $(\lambda_S, \|\cdot\|_S)$ and its limit is nothing but the zero sequence in $\lambda$.

3. Nuclearity of $\lambda(E)$

A linear mapping $f : E \rightarrow F$ between complete normed spaces is said to be nuclear if there exist $(\varepsilon_n)_n \in \ell_1$, a bounded sequence $(a_n)_n$ in the continuous dual $E'$ of $E$ and a
bounded sequence \((y_n)_n \subset F\) such that

\[
f(x) = \sum_{n=1}^{\infty} \varepsilon_n a_n(x)y_n, \text{ for all } x \in E.
\]

A b-space \(E\) is said to be nuclear (a Schwartz space) if for every Banach disk \(A\) in \(E\) there is a Banach disk \(B \supset A\) in \(E\) such that the inclusion mapping \(E_A \to E_B\) is nuclear (compact).

**Proposition 3.** The tensor product \(\lambda \otimes E\) is identifiable with a subspace of \(\lambda(E)\).

**Proof.** We see that for all \(\alpha = (\alpha_n)_n \in \lambda\) and \(x \in E\), \((\alpha_n x)_n \in \lambda(E)\). Define the bilinear mapping \(\varphi : \lambda \times E \to \lambda(E)\), such that \(\varphi(\alpha, x) = (\alpha_n x)_n\). There exists a linear mapping \(\ell : \lambda \otimes E \to \lambda(E)\), with \(\ell(\alpha \otimes x) = (\alpha_n x)_n\). Let us show that \(\ell\) is one to one. Suppose that \(z \in \lambda \otimes E\) such that \(\ell(z) = 0\). We can write \(z = \sum_{i=1}^{k}(\alpha^i_n)_n \otimes x_i\), for which \((\alpha^i_n)_{i=1}^{k}\) and \((x_i)_{i=1}^{k}\) are linearly independent. But,

\[
\ell(z) = \sum_{i=1}^{k} \ell(\alpha^i \otimes x_i) = \sum_{i=1}^{k} (\alpha^i_n x_i)_n = \left(\sum_{i=1}^{k} \alpha^i_n x_i\right)_n.
\]

Since \(\ell(z) = 0\) then \(\left(\sum_{i=1}^{k} \alpha^i_n x_i\right)_n = 0\) and \(\sum_{i=1}^{k} \alpha^i_n x_i = 0\), for every \(n\). But, as \((x_i)_{i=1}^{k}\) is linearly independent, \(\alpha^i_n = 0\), for all \(1 \leq i \leq k\) and \(n \in \mathbb{N}\). Thus, \(z = \sum_{i=1}^{k}(\alpha^i_n)_n \otimes x_i = 0\), and \(\ell\) is one to one. \(\blacksquare\)

**Lemma 2.** Let \(S\) and \(B\) be Banach disks in \(\lambda\) and \(E\) respectively, \(N(x) = \left\|\left(\|x_n\|_B\right)_n\right\|_S\) for all \(x = (x_n)_n \in \lambda_S(E_B)\) and \(N_1(z) = N(\ell(z))\) for all \(z \in \lambda_S \otimes E_B\). Then,

1. \(N_1\) is a cross-norm on \(\lambda_S \otimes E_B\), that is \(N(\alpha \otimes x) = \left\|\alpha\right\|_S \|x\|_B\), for every \(\alpha \in \lambda_S\) and \(x \in E_B\).

2. The mapping \(\ell : \lambda_S \otimes E_B \to \lambda_S(E_B)\) is isometric and can be extended to a unique linear mapping \(\ell : \lambda_S \otimes N_1 E_B \to \lambda_S(E_B)\), where \(\lambda_S \otimes N_1 E_B\) the completion of the normed space \((\lambda_S \otimes N_1 E_B, N_1)\).

**Proof.** Since \(N\) is a solid norm and \(\ell\) is a one to one linear mapping, \(N_1\) is a norm. It is clear that \(N_1(\alpha \otimes x) = \|\alpha\|_S \|x\|_B\), and \(J.\) holds. By the definition of \(N_1\), we see that \(\ell\) is isometric from \(\lambda_S \otimes E_B\) to the complete space \(\lambda_S(E_B)\), and then it has an extension to the completion \(\lambda_S \otimes N_1 E_B\) of \(\lambda_S \otimes N_1 E_B\). This gives the second item. \(\blacksquare\)

We will make use of the following result to represent \(\lambda(E)\) as a bornological tensor product.

**Proposition 4.** [4, Ch VIII, Prop. 4]

1. There is a convex bornology \(b\) on \(\lambda \otimes E\) (the finest one) making bounded the inclusion mappings \(\lambda_S \otimes N_1 E_B \to \lambda(E)\). Moreover, \(\lambda \otimes_b E = \lim \lambda_S \otimes N_1 E_B\).

2. \(b\) is located between the projective bornology \(\pi\) and the injective bornology \(\varepsilon\).

3. If \(\lambda\) or \(E\) is nuclear, then \(\pi = b = \varepsilon\).

4. If \(\lambda\) and \(E\) are nuclear, the bornological completion \(\lambda \otimes_b E\) of \(\lambda \otimes_b E\) is the inductive limit of the Banach spaces \(\lambda_S \otimes N_1 E_B\).
Now, we prove

**Theorem 1.** If \( \lambda \) and \( E \) are nuclear, the equality \( \lambda(E) = \lambda \otimes_b E \) holds algebraically and bornologically.

**Proof.** Consider the linear mapping \( \ell : \lambda \otimes_b E \to \lambda(E) \) defined in the proof of Proposition 3.

According to the definition of the norms \( N \) and \( N_1 \), we see that \( \ell \) is bounded, and since \( \lambda(E) \) is complete, \( \ell \) can be extended to a bounded linear mapping \( \hat{\ell} \) from the bornological completion \( \lambda \otimes_b E \) of \( \lambda \otimes_b E \) to \( \lambda(E) \).

We will prove that \( \tilde{\ell} \) makes \( \lambda \otimes_b E \) and \( \lambda(E) \) bornologically isomorphic.

Let \( z \in \lambda \otimes_b E \) be such that \( \tilde{\ell}(z) = 0 \). By [4, Ch VIII, Prop. 2], a sequence \( \{z_k\}_{k=1}^{\infty} \) of elements of \( \lambda \otimes_b E \) converges to \( z \). Then \( \{z_k - z\}_{k=1}^{\infty} \) is a null sequence in some subspace \( \lambda S \otimes_b E_B \).

Thus,
\[
\tilde{\ell}(z) = \tilde{\ell}(\lim k \ell(z_k)) = \lim k(\ell \circ \iota)(z_k) = \lim k(\ell(z_k)) = \lim k(\ell(z_k)) = \tilde{\ell}(z) = 0.
\]

Here \( \iota \) is the canonical injection from \( \lambda \otimes_b E \) to its completion \( \lambda \otimes_b E \).

By Lemma 2, \( \ell \) is isometric and then it is one to one, then \( z = 0 \), and \( \tilde{\ell} \) is one to one.

We will prove that \( \tilde{\ell} \) is onto as follows. Let \( A \in B \) be a Banach disk; since \( E \) is nuclear we can select a Banach disk \( B \in \mathbb{B} \) containing \( A \) such that the inclusion \( E_A \to E_B \) is nuclear.

There are \( (\epsilon_k)_k \in \ell_1 \), a bounded sequence \( (a_k)_k \) in the continuous dual \( (E_A)' \) of \( E_A \) and a bounded sequence \( (y_k)_k \subset E_B \) such that
\[
x = \sum_{k=1}^{\infty} \epsilon_k a_k(x) y_k, \text{ for all } x \in E_A.
\]

Let \( x = (x_n)_n \in \lambda S(E_A) \), and \( \alpha^k = (\alpha^k_n)_n = (a_k(x_n))_n \). We have
\[
|\alpha^k_n| = |a_k(x_n)| \leq \|a_k\| \|x_n\|_A \leq \left( \sup_p \|a_p\| \right) \|x_n\|_A, \text{ for all } k, n.
\]

The sequence \( (a_k)_k \) being bounded in \( (E_A)' \), \( \sup_p \|a_p\| \) is finite, \( \alpha^k = (\alpha^k_n)_n \in \lambda S(E_A) \), for all \( k \), and, by (2), \( \|\alpha^k\|_S \leq (\sup_p \|a_p\|) (\|x_n\|_A)_n \|S \) and then \( \sup_k \|\alpha^k\|_S \) is finite. Then,
\[
\sum_{k=1}^{r} N_1(\epsilon_k \alpha^k \otimes y_k) = \sum_{k=1}^{r} |\epsilon_k| \|\alpha^k\|_S \|y_k\|_B \leq (\sup_p \|a_p\|)(\sup_p \|y_p\|) N(x) \sum_{k=1}^{r} \epsilon_k.
\]

As, \( \lambda S(E_B) \) is a complete normed spaces, the series \( \sum_{k=1}^{\infty} \epsilon_k \alpha^k \otimes y_k \) converges in \( \lambda S(E_B) \) to a limit \( g(x) \). Moreover,
\[
\tilde{\ell}(g(x)) = x.
\]

Indeed, if \( z = (z_n)_n \in \lambda S(E_B) \) is such that \( z = \tilde{\ell}(g(x)) \), then
\[
z = (z_n)_n = \tilde{\ell} \left( \sum_{k=1}^{\infty} \epsilon_k (a_k(x_n))_n \otimes y_k \right) = \sum_{k=1}^{\infty} \epsilon_k \tilde{\ell}((a_k(x_n))_n \otimes y_k)
\]
\[ \sum_{k=1}^{\infty} \varepsilon_k \ell((a_k(x_n))_n \otimes y_k) = \sum_{k=1}^{\infty} \varepsilon_k (a_k(x_n)y_k)_n. \]

But the projections are bounded by Lemma 1, then
\[ z_n = \sum_{k=1}^{\infty} \varepsilon_k a_k(x_n)y_k, \text{ for all } n. \]

By (1), \( z_n = x_n \), for all \( n \), and \( \tilde{\ell}(g(x)) = x \). This means that \( \tilde{\ell} \) is onto. In the other hand, if \( K \) is bounded in \( \lambda(E) \), then \( K \) is contained and bounded in some \( \lambda_S(E_B) \), and \( \tilde{\ell}(g(K)) = K \), from what, we conclude that the inverse of \( \tilde{\ell} \) is bounded.

We are now ready to prove the main result of this section.

**Theorem 2.** Let \( E \) be a complete b-space and \( \lambda \) be a normal sequence space. Then \( \lambda(E) \) is nuclear if and only if \( \lambda \) and \( E \) are nuclear.

**Proof.** If \( \lambda(E) \) is nuclear then, by Proposition 1, \( E \) and \( \lambda \) are closed subspaces of \( \lambda(E) \) and then they are nuclear also.

Inversely, suppose that \( E \) and \( \lambda \) are nuclear. By Proposition 4, \( \lambda \tilde{\otimes}_b E \) is nuclear. So by Theorem 1, \( \lambda(E) \) is nuclear. ■

**Theorem 3.** Let \( E \) be a complete b-space and \( \lambda \) be a normal sequence space.

(i) If \( \lambda \) is nuclear then, \( \lambda(E) \) is a Schwartz space if and only if \( E \) is a Schwartz space.

(ii) If \( E \) is nuclear then, \( \lambda(E) \) is a Schwartz space if and only if \( \lambda \) is a Schwartz space.

**Proof.** Suppose that \( E \) is nuclear. If \( \lambda(E) \) is a Schwartz space, then \( \lambda \), being a closed subspace of \( \lambda(E) \) by Proposition 1, is a Schwartz space. Inversely, suppose that \( E \) is nuclear and \( \lambda \) is a Schwartz space. Let \( A \in \mathcal{B} \) and \( S \in \mathcal{S} \) be a Banach disks in \( E \) and \( \lambda \) respectively. Since \( E \) is nuclear we can select a Banach disk \( B \in \mathcal{B} \) containing \( A \) such that the inclusion \( E_A \rightarrow E_B \) is nuclear. So, there are \( (\varepsilon_k)_k \in \ell_1 \), a bounded sequence \( (a_k)_k \) in the continuous dual \( (E_A)' \) of \( E_A \) and a bounded sequence \( (y_k)_k \subset E_B \) such that
\[ x = \sum_{k=1}^{\infty} \varepsilon_k a_k(x)y_k, \text{ for all } x \in E_A. \]  

(5)

Since \( \lambda \) is a Schwartz space, there is a Banach disk \( T \) in \( \lambda \) such that the injection \( \lambda_S \rightarrow \lambda_T \) is compact. We will show that the injection \( \lambda_S(E_A) \rightarrow \lambda_T(E_B) \) is compact. Let
\[ \{x^i = (x^i_n)_n\}_{i=1}^{\infty} \]

(6)

\[ x^i = \sum_{n=1}^{\infty} \varepsilon_n a_n^i(x_n)y_n, \text{ for all } x^i \in E_A. \]

(7)
be a sequence in $S(A)$. By (5), we have
\[ x_n^i = \sum_{k=1}^{\infty} \varepsilon_k a_k(x_n^i)y_k, \text{ for all } n, i. \] (7)

The sequence $(a_k)_k$ being bounded in $(E_A)'$, there is a constant $c > 0$ such that
\[ |a_k(x_n^i)| \leq c \|x_n^i\|_A \text{ for all } i, k, n. \]

This means that $\{(a_k(x_n^i))_k\}_{i=1}^{\infty} \subset \lambda_S$ and that
\[ \{(a_k(x_n^i))_k\}_{i=1}^{\infty} \subset cS. \] (8)

A subsequence $\{(a_k(x_n^i))_k\}_{j=1}^{\infty}$ of $\{(a_k(x_n^i))_k\}_{i=1}^{\infty}$ should converge in $\lambda_T$ to $\alpha^k = (\alpha^k_n)_n$.

In the other hand, the equation (8) shows that the sequence $\{(a_k(x_n^i))_k\}_{i=1}^{\infty}$ is bounded in $\lambda_S$. For every $n \in \mathbb{N}$, there $c_n > 0$ such that for all $j, k$
\[ |a_k(x_n^i)| \leq c_n \text{ and then } |\alpha^k_n| \leq c_n. \] (9)

For every $n \in \mathbb{N}$, since $\{(\alpha^k_n)_k\}_{k=1}^{\infty}$ is bounded in the complete normed space $E_B$, the series $\sum_{k} \varepsilon_k \alpha^k_n y_k$ converges to a limit $x_n \in E_B$. Let $x = (x_n)_n$. Since $\{(\alpha^k_n)_k\}_{k=1}^{\infty}$ is bounded in $\lambda_S$ and $\{y_k\}_{k=1}^{\infty}$ is bounded in $E_B$, the sequence $\{(\alpha^k_n y_k)_{k=1}^{\infty} \subset \lambda_T (E_B)$ and then in $\lambda_T (E_B)$. Thus, the series $\sum_k \varepsilon_k a_k^0 y_k_n$ converges in $\lambda_T (E_B)$ to $z = (z_n)_n$. Since the projections are bounded by Lemma 1, one has $z_n = \sum_k \varepsilon_k \alpha^k_n y_k$ for all $n$, and then $x = z \in \lambda_T (E_B)$.

It remains to prove that $\{x^j\}_{i=1}^{\infty}$ converges in $(\lambda_T (E_B), N)$ to $x$. We have,
\[ x^j - x = \sum_{k} \varepsilon_k (\alpha_n^j(x_n^i) - \alpha_n^0) y_k \]

and
\[ N(x^j - x) \leq \sum_{k} |\varepsilon_k| ||(\alpha_n^j(x_n^i) - \alpha_n^0)| |_S ||y_k||_B \] (10)

For $j, k$, let
\[ \beta^j_k = ||a_k(x_n^i) - \alpha^k_n||_T \text{ and } \gamma_k = ||y_k||_B. \] (11)

Then, $(\gamma_k)_k \subset c_0$ and $\{ (\varepsilon_k \beta^j_k)_k \}_{j=1}^{\infty}$ is a sequence in $\ell_1$ which is $\sigma(\ell_1, c_0)$--bounded, then it has a convergent subsequence say,
\[ \{(\varepsilon_k \beta^j_k)_k\}_{j=1}^{\infty}. \] (12)

As, $\lim_{r \to \infty} \varepsilon_k \beta^j_k = 0$, for all $k$, then the sequence in (12) converges to 0 in $(\ell_1, \sigma(\ell_1, c_0))$. By (11) and (10), we have
\[ N(x^r - x) \leq \sum_{k} |\varepsilon_k \beta^j_k| \gamma_k, \text{ for all } r \in \mathbb{N}. \]

Thus, $\{x^r - x\}_{r=1}^{\infty}$ converges to 0 in $\lambda_T (E_B)$, and (6) has a convergent subsequence. This finishes the proof of (i). The proof of (ii) is similar by interchanging the roles of $E$ and $\lambda$ in the proof. ■
4. Nuclearity of $\lambda\{E\}$

Notice that a locally convex space is said to be nuclear (resp. a Schwartz space) if the convex bornology of equicontinuous subsets of its topological dual is nuclear (resp. of Schwartz).

Let $\lambda$ be a perfect sequence space and $E$ a locally convex space whose topology is defined by a family $\mathcal{M}$ of absolutely convex equicontinuous subsets of its topological dual $E'$. Define

$$\lambda\{E\} = \{(x_n)_n \subset E : (P_M(x_n)) \subset \lambda\}, \text{ where } P_M(x_n) = \sup_{a \in M} |a(x_n)|.$$

If a topology on $\lambda$ is defined by family $S$ of normal, absolutely convex and $\sigma(\lambda^\times, \lambda)$–bounded subsets of $\lambda^\times$, then a locally convex topology can be defined on $\lambda\{E\}$ by the family of semi-norms $(\pi_{S,M})_{S \in S, M \in M}$, such that, if $x = (x_n)_n \subset \lambda\{E\}$ then

$$\pi_{S,M}((x_n)_n) = P_S((P_M(x_n))) = \sup \{ \sum_{n=1}^\infty |\alpha_n P_M(x_n)| : (\alpha_n)_n \in S \}.$$

For the topology so defined, Ronald C. Rosier in [10] proved that the dual space $(\lambda\{E\})^*$ of $\lambda\{E\}$ is $\lambda^\times(E')$ and that a subset of $(\lambda\{E\})^*$ is equicontinuous if and only if it is contained in some $S(M)$ for $S \in S$ and $M \in M$.

Starting from this setting, Theorem 2 gives

**Theorem 4.** $\lambda\{E\}$ is nuclear if and only if $\lambda$ and $E$ are nuclear.

Also, Theorem 3 gives

**Theorem 5.** If $E$ (resp. $\lambda$) is nuclear, then $\lambda\{E\}$ is a Schwartz space if and only if $\lambda$ (resp. $E$) is a Schwartz space.

5. Conclusion

In this paper we have characterized the bornological structure, the completeness and the nuclearity of $\lambda(E)$ in terms of that of $\lambda$ and $E$. An application to the nuclearity of the locally convex space $\lambda\{E\}$ is given.

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References


