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Multigrid Methods for The Solution of Nonlinear Variational Inequalities

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Abstract. In this research, we investigate the numerical solution of second member problems that depend on the solution obtained through a multigrid method. Specifically, we focus on the application of multigrid techniques for solving nonlinear variational inequalities. The main objective is to establish the uniform convergence of the multigrid algorithm. To achieve this, we employ elementary subdifferential calculus and draw insights from the convergence theory of nonlinear multigrid methods.

2020 Mathematics Subject Classifications: 65M55, 65N30, 49N05, 65K15, 35J86

Key Words and Phrases: Multigrid method, nonlinear variational inequality, finite element, iterative method, HJB equation

1. Introduction

The recent literature offers a wide range of diverse computational methods [6, 9, 20, 21, 24, 29, 31, 32, 34, 35] that are employed to solve complex real-world problems across

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various fields of science and engineering [1, 2, 5, 8, 23, 33]. These methods have been developed and applied to tackle challenging problems and provide effective solutions in their respective domains. Researchers have explored and utilized these computational techniques to address a multitude of problems, advancing our understanding and facilitating progress in numerous scientific and engineering disciplines [3, 4, 7, 27, 28, 30].

The numerical methods commonly employed to solve boundary problems typically lead, after discretization, to the resolution of a system of large algebraic equations. These numerical algorithms, including iteration methods such as Jacobi, Gauss-Seidel iteration, and relaxation methods, are often used due to their customary nature. However, they may exhibit slow convergence for small mesh sizes and complexity when adapted to general elliptical problems. In contrast, multigrid methods offer a distinct advantage. They are algorithms with a linear cost based on the number of discretization points, irrespective of the problem's dimension. These methods are particularly efficient in solving both linear and nonlinear partial differential equations (PDEs) and linear variational inequalities (VIs) [16, 19]. Their linear complexity makes them a powerful tool for large-scale problems, significantly reducing computational efforts while providing accurate solutions.

Multigrid techniques are widely acknowledged as rapid methods for solving various types of variational equations and inequalities [22], especially for elliptic problems with discretization leading to an M-matrix [15].

In the second section, we present an overview of nonlinear variational inequality (VI) problems and their discretization using a conforming finite element method P_1 [13]. Additionally, we introduce an algorithm that formulates the VI as stationary Hamilton-Jacobi-Bellman equations, drawing inspiration from the Hoppe multigrid method [18, 25]. We refer to this algorithm as the multigrid Hierarchy Jacobi (MGHJB) and provide the iteration matrices associated with the algorithm.

Firstly, we present the original results concerning the approximation and smoothing properties in the L^∞ norm. Subsequently, we demonstrate the uniform convergence of the MGHJB algorithm. Finally, we apply the numerical methods to a specific application where the operator is linear, and the second member is nonlinear, dependent on the solution. In this context, we implement the Gauss-Seidel method and the multigrid methods V and W-cycle. The numerical experiments aim to assess the effectiveness and performance of these methods in solving the given problem.

2. Multigrid method

2.1. Symbols and assumptions

Let Ω be an open set in $\mathbb{R}^N$ with sufficiently regular bounds $\partial\Omega$. For $u, v \in H^1(\Omega)$, we define second-order operators

$$\mathcal{A} = \sum_{1\le j,k\le N} a_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{k=1}^{N} b_k(x)\frac{\partial}{\partial x_k} + a_0(x).$$

The coefficients $a_{jk}(x)$, $b_k(x)$, and $b_0(x)$ are required to possess sufficient regularity, ensuring:

$$a_{kj}(x) = a_{jk}(x); \quad b_0(x) \geqq \beta > 0, \quad x \in \Omega,$$
$$\sum_{1 \leq j,k \leq N} a_{kj}(x)\xi_k\xi_j \geqq \alpha|\xi|^2; \quad \left(x \in \bar{\Omega}, \xi \in \mathbb{R}^N, \alpha > 0\right).$$

We also define the associated coercive continuous bilinear form:

$$a(u,v) = \int_\Omega \left(\sum_{1 \leq j,k \leq N} a_{jk} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^{N} b_k \frac{\partial u}{\partial x_k} v + a_0(x) uv \right) dx.$$

In addition, we consider the second member f is l-Lipschitzian such that:

$$f(u) \in L^\infty(\Omega); \quad f \geq 0 \quad \text{and} \quad \frac{l}{\beta} < 1.$$

An obstacle $\psi \in W^{2,\infty}$, such that $\psi \geq 0$.

2.2. Continuous problem

The objective is to find the solution u for the given variational inequalities:
Find u as the solution to:

$$\begin{cases} a(u, v-u) \geq (f(u), v-u), \quad \forall v \in H^1(\Omega) \\ u \leq \psi; v \leq \psi \end{cases} \tag{2.3}$$

It is established that this problem possesses a unique solution, as proven by the fixed point theorem under the preceding assumption (see [10]).

2.3. Discretize

To build a multigrid loop, we consider a sequence of discretization steps, denoted as $0 < h_{k+1} < h_k < 1$, such that the grids are nested $h_{k+1} = \frac{h_k}{2}$.
Next, we define $\Omega_k = \Omega_{h_k}$, $V_k = V_{h_k}$, and $\mathcal{A}k = \mathcal{A}h_k$, and establish a series of uniform regular triangulations denoted as $\{T_k, k \in \mathbb{N}_0\}$. For all T_k, we have the following estimates:

$$\Omega_k \subset \Omega_{k+1} \subset \Omega,$$
$$\operatorname{dist}(\partial\Omega_k, \partial\Omega) \leq c_0 h_k^2,$$
$$h_k h_{k+1} \leq c_1.$$

We introduce $\mathbb{V}_{h_k} = \left\{ v_{h_k} \in C(\Omega) \cap H^1 \text{ So } v_{h_k}/_T \in P_1 \right\}$, we simply write:

$$\mathbb{V}_k = \left\{ v_k \in C(\Omega) \cap H^1 \quad \text{this} \quad v_k/r \in P_1 \right\}.$$

The function of the usual basis $\varphi_k^i, i \in (1, \ldots, m(h_k))$ is defined as: $\varphi_k^i\left(x_k^j\right) = \delta_{ij}, x_k^j$ is a node of the triangulation T_k.
We define the normal restriction operator as follows:

$$r_k v(x) = \sum_{i=1}^{m(h_k)} v\left(M_k^i\right) \varphi_k^i(x). \tag{2.4}$$

If we write $U_k = \mathbb{R}^{m_k}, r_k$ is a bijection from U_k to $\mathbb{V}_k$.
In U_k, we define the scalar product:

$$< u, v >= h_k^2 \sum_{i=1}^{m(h_k)} u_i v_i, \quad \|u\|_k =< u, v >_k^{1/2} .$$

The maximum norm $\|\cdot\|_\infty$ (in U_k) and $\|\cdot\|_{L^\infty}$ (in $\mathbb{V}_k$) are equivalent standards, we denote them $\|\cdot\|_\infty$. We have the following lemma (see [11]).
Lemma 1: There exists C_1, C_2 independent of k so:

$$\begin{aligned} &\|r_k(u)\|_{L^\infty} = \|u\|_{L^\infty}, \forall u \in U_k. \\ &C_1\|v\|_{L^\infty} \leq \|r_k^*(v)\|_{L^\infty} \leq C_2\| v\|_{L^\infty}, \forall v \in \mathbb{V}_k. \end{aligned} \tag{2.5}$$

2.4. Discrete problem

In a natural progression, we introduce the discretization matrices $\mathcal{A}k$ and the generic coefficient matrices $a\left(\varphi k^1, \varphi_k^s\right)$, where $\varphi_s, s = 1, 2, \ldots m(h_k)$ represent the customary basic functions. With these definitions in place, we can now define the discrete problem as follows:
Find $u_k \in \mathbb{V}_k$, which serves as the solution of:

$$\begin{cases} < \mathcal{A}_k u_k, v_k - u_k > \geq < f_k(u_k), v_k - u_k >, \quad \forall v_k \in \mathbb{V}_k. \\ u_k \leq r_k\psi; v_k \leq r_k\psi. \end{cases} \tag{2.6}$$

We assume that the matrices $\mathcal{A}_k$ are M-matrices (see [12]).

2.5. Formulation in HJB:

The equivalence between the finite-dimensional variational inequality (2.3) and a formulation in Hamilton-Jacobi-Bellman (HJB) can be readily observed (see [19]). We describe the numerical algorithm chosen to solve the stationary HJB equations.
In the classic framework, we recall some convergence results that will be instrumental in establishing the convergence of the MGHJB algorithm described in the subsequent section.
Iterative diagram:
Step 1: Choose an initial vector $u_k^0 \in \mathbb{R}^{n_k}$.
Step 2: Let $u_k^{(\nu)} \in \mathbb{R}^{n_k}, \nu \geq 0$, and calculate $u_k^{(\nu+1)} \in \mathbb{R}^{n_k}$ as the solution of the equation:

$$\mathcal{A}_k^\nu u_k^{\nu+1} - Z_k^\nu = 0, \tag{2.7}$$

such that:

$$Z_k^\nu = F_k^\nu(u_k^\nu),$$

where:

$$\mathcal{A}_{k,i}^\nu = \begin{cases} \mathcal{A}_{k,i}(u_k) & \text{if} \quad \mathcal{A}_{k,i}u_{k,i}^\nu - Z_{k,i} > u_{k,i}^\nu - \psi_{k,i}. \\ u_{k,i} & \text{if} \quad 1 \le i \le N \end{cases} . \tag{2.8a}$$

$$Z_{k,i}^\nu = \begin{cases} Z_{k,i} & \text{if} \quad \mathcal{A}_{k,i}u_{k,i}^\nu - Z_{k,i} > u_{k,i}^\nu - \psi_{k,i} \\ u_{k,i} & \text{if} \quad 1 \le i \le N \end{cases} \tag{2.8b}$$

Let u_k^* be the unique solution of the discrete HJB equation

$$\max_{1\le i\le N}\left(\mathcal{A}_{k,i}u_k^* - Z_{k,i}, u_{k,i}^* - \psi_{k,i}\right) = 0. \tag{2.9}$$

We shall state the following theorem and present our problem formulated based on the Hamilton-Jacobi-Bellman (HJB) equation, which is an adaptation of Hoppe's work [19].
Theorem 2: Let u_k^ν be the iteration defined, and it satisfies the HJB equation. Additionally, we assume that $\mathcal{A}k$ is continuously differentiable. Under these conditions, the sequence $(u_k^\nu)_{\nu\ge 0}$ converges and approaches u_k^*.
Before proceeding to present the results, it is pertinent to recall the following theorem.
Theorem 3: ([10] , [14]) Under the previous assumptions and notation, we have:

$$\|u - u_k^*\|_\infty \le Ch_k^2 \left|\mathrm{Logh}_k\right|^2 \|f(u)\|_\infty. \tag{2.10}$$

2.6. Description of the multigrid method for VIs

Choosing an iteration $u_k^\nu, \nu > 0$ for the multigrid method, we obtain $\bar{u}_k^\nu$, by applying an iterative method to solve the system (2.7) by α, expressing for

$$\bar{u}_k^\nu = S_k^\alpha\left(u_k^\nu\right) \tag{2.11}$$

where S_k represents the iteration or smoothing operator, and α denotes the number of iterations performed. We denote the solution to (2.7) by u_k^*. The error setting $e_k^\nu = \bar{u}_k^\nu - u_k^*$, and the residual $d_k^{(\nu)} = Z_k^\nu - \mathcal{A}_k^\nu \bar{u}_k^\nu$, we can write the equation (2.7) as

$$\mathcal{A}_k^\nu\left(\bar{u}_k^\nu + e_k^\nu\right) = Z_k^\nu.$$

This leads to the residual equation

$$\mathcal{A}_k^\nu e_k^\nu = Z_k^\nu - \mathcal{A}_k^\nu \bar{u}_k^\nu = d_k^{(\nu)}.$$

On the fine grid, after relaxation on $\mathcal{A}_k^\nu \bar{u}_k^\nu = Z_k^\nu$, the error will exhibit smooth behavior. However, on the coarse grid, the error appears to be more oscillatory, leading to efficient relaxation. To completely determine e_k^ν, we need to compute e_{k-1}^ν at the $(k-1)$ level, as it serves as the solution for the coarse grid system.

$$\mathcal{A}_{k-1}^\nu e_{k-1}^\nu = d_{k-1}^{(\nu)}. \tag{2.12}$$

We can interpret e_{k-1}^{ν} (and its corresponding operators $\mathcal{A}_{k-1}^{\nu}$, $d_{k-1}^{(\nu)}$) as the approximation operator at level $k-1$, while e_{k}^{ν} (and its corresponding operators $\mathcal{A}_{k}^{\nu}$, $d_{k}^{(\nu)}$) represent the approximation operator at level k. Additionally, we have the restriction operator $\mathcal{R}_k$ and its inverse $\mathcal{P}_k$.
Therefore, we identify an improved iteration at the k level

$$u_k^{\nu+1} = \bar{u}_k^{\nu} + \mathcal{P}_k \left(e_{k-1}^{\nu} \right). \tag{2.13}$$

Due to the nestedness, we use the well-defined identity operator

$$\begin{aligned} \Pi : \mathbb{V}_{k-1} &\longrightarrow \mathbb{V}_k, \\ \Pi v &= v, \end{aligned}$$

to define prolongation and restriction operators, that is:

$$\mathcal{P}_k = r_k^{-1} r_{k-1}, \quad \mathcal{R}_k = \mathcal{P}_k^t. \tag{2.14}$$

Note 4: The preceding algorithm describes a loop of two mesh iterations to solve (2.7) for two mesh levels Ω_{k-1}. It is clear that the coarse grid system (2.12) has the same form as the system (2.7). Therefore, we can approximate the solution of the system (2.12) by recursively doing two-grid iterations at all grid levels $\{\Omega_k, k = 0, \ldots, m_k\}$.

2.7. Matrix associated with the MGHJB algorithm:

The matrix of iterations for the two-grid method with α_1 presmoothing and α_2 post-smoothing at level k is expressed as follows:

$$TG_k\left(\alpha_1, \alpha_2\right) = S_k^{\alpha_2} \left(\left(\mathcal{A}_k^{\nu}\right)^{-1} - \mathcal{P}_k \left(\mathcal{A}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k \right) \left(\mathcal{A}_k^{\nu}\right) S_k^{\alpha_1}. \tag{2.15}$$

Theorem 5: ([26]) The multigrid approach constitutes a linear iterative method, with the iteration matrix denoted as MG_k.

$$\begin{aligned} MG_0 &= 0, \\ MG_k &= S_k^{\alpha_2} \left(I_k - \mathcal{P}_k \left(I_k - MG_{k-1}\right) \left(\mathcal{A}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k \right) \left(\mathcal{A}_k^{\nu}\right) S_k^{\alpha_1}, \\ &= TG_k + S_k^{\alpha_2} \mathcal{P}_k MG_{k-1} \left(\mathcal{A}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k \left(\mathcal{A}_k^{\nu}\right) S_k^{\alpha_1}, \quad k = 1, 2, .. \end{aligned} \tag{2.16}$$

3. Convergence analysis of multigrid method in L^{∞} norm

In this section, we provide a comprehensive convergence analysis of the multigrid algorithm. The algorithm was previously described using the maximum norm in the preceding section.

3.1. Approximation property

Theorem 6: ([17]) The matrix $\Upsilon_k = \left[(\mathcal{A}_k^\nu)^{-1} - \mathcal{P}_k \left(\mathcal{A}_{k-1}^\nu\right)^{-1} \mathcal{R}_k\right]$ has the approximation property

$$\|\Upsilon_k\|_\infty \leq C h_k^2 \left|\ln h_k\right|^2. \tag{3.1}$$

Proof. The proof of the approximation property was introduced by Arnold in [25], relying on Theorem (2).

3.2. Smoothing property

To demonstrate the smoothness property, we decompose $\mathcal{A}_k^\nu = E_k - N_k$ and use the following assumptions:

$$E_k \text{ is regular and } \left\|E_k^{-1} N_k\right\|_\infty \leq 1, \text{ for all } \quad k. \tag{3.2}$$

$$\|E_k\|_\infty \leq C h_k^{-2}, \text{for all} \quad k, \text{ with C independent of k.} \tag{3.3}$$

For the smoothing process, we employ a relaxation method with an iterative matrix.

$$S_k = I_k - \omega E_k^{-1} N_k, \quad \omega \in (0,1).$$

In the following theorem, Arnold Reusken applied a concept that is relevant to our work. The theorem's application is appropriate as it is closely linked to the operator used in our study.

Theorem 7: ([25]) Under the given assumptions and notations, it can be inferred that there exists a constant C, which is independent of both k and α. With this constant, the following smoothness properties can be established:

$$\|(\mathcal{A}_k^\nu)\, S_k^\alpha\|_\infty \leq C \frac{1}{\sqrt{\alpha}} h_k^{-2}. \tag{3.4}$$

By passing to the norm in (2.13), and taking into account (3.1) and (3.4), we have to prove the the following stability limit:

$$\exists C_s : \|S_k^\alpha\|_\infty \leq C_s, \text{ for all } k \text{ and } \alpha. \tag{3.5}$$

The convergence analysis is performed based on the split of the following two lattices: the iterate matrix with $\alpha_2 = 0$.

$$\begin{aligned} \|TG_k(\alpha_1, 0)\|_\infty &= \left\| \left((\mathcal{A}_k^\nu)^{-1} - \mathcal{P}_k \left(\mathcal{A}_{k-1}^\nu\right)^{-1} \mathcal{R}_k \right) (\mathcal{A}_k^\nu)\, S_k^{\alpha_1} \right\|_\infty \\ &\leq \left\| \left((\mathcal{A}_k^\nu)^{-1} - \mathcal{P}_k \left(\mathcal{A}_{k-1}^\nu\right)^{-1} \mathcal{R}_k \right) \right\|_\infty \left\|(\mathcal{A}_k^\nu)\, S_k^{\alpha_1}\right\|_\infty. \end{aligned}$$

Typically, a hierarchy of more than two grids is selected. In this scenario, iterative matrices (2.16) can be defined using recursion of iterative matrices (2.15) for all levels k. If we assume that (3.5) holds, then L^{∞}-convergence results can be readily deduced from the previous findings.

Theorem 8: ([26]) Consider a multigrid method for a given iterative matrix (2.16). Then under the previous assumption, for the parameter value $\alpha_2 = 0, \alpha_1 = \alpha > 0, \tau \geq 2$. For each $\zeta \in (0,1)$ there is α, α^*such that for all $\alpha \geq \alpha^*$

$$\|MG_k\|_{\infty} \leq \zeta, k = 0, 1, ... \tag{3.7}$$

hold.

Proof. If the approximation and smoothness properties are combined with (3.5), then we can apply the same parameters as in [26].

The following theorem represents the main result of our work.

Theorem 9: Under the previous assumptions and notations the iterated $u_k^{\nu}, \nu \geq 0$ for two meshes k and $k-1$ satisfy:

$$\left\|u_k^{\nu+1} - u_k^*\right\|_{\infty} \leq \left(\frac{C}{\sqrt{\alpha}} \left|\mathrm{Logh}_k\right|^2\right) \left\|u_k^{\nu} - u_k^*\right\|_{\infty}. \tag{3.6}$$

Proof. We have

$$\begin{aligned}
\left\|u_k^{\nu+1} - u_k^*\right\|_{\infty} = & \left\|\left(\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{A}_{k-1}^{\nu}\right)^{-1}\mathcal{R}_k\right)\left(\mathcal{A}_k^{\nu}\right)S_k^{\alpha_1}\right)\left(u_k^{\nu} - u_k^*\right)\right\|_{\infty} \\
\leq & \left\|\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{A}_{k-1}^{\nu}\right)^{-1}\mathcal{R}_k\right)\right\|_{\infty}\left\|\mathcal{A}_k^{\nu}S_k^{\alpha_1}\right\|_{\infty}\left\|u_k^{\nu} - u_k^*\right\|_{\infty} \\
\leq & \left(\frac{C_2}{\sqrt{\alpha}}h_k^{-2}\right)\left(C_1 h_k^2\left|\log h_k\right|^2\right)\left\|\left(u_k^{\nu} - u_k^*\right)\right\|_{\infty} \\
\leq & \left(\frac{C_1 C_2}{\sqrt{\alpha}}\right)\left|\log h_k\right|^2\left\|u_k^{\nu} - u_k^*\right\|_{\infty}
\end{aligned}$$

4. Numerical simulation

In this section, we present a numerical example of a nonlinear variational inequality. To apply the proposed method to the example, we assume that the data of the problem is sufficiently smooth. We then employ Bellman's principle of dynamic programming to tackle the problem and proceed with solving (2.3), as discussed previously, using the following specified data:

$$\begin{cases}
Au \leq f(u), \quad \text{in } \Omega = \left\{(x,y) \mid x^2 + y^2 \leq 1\right\}, \\
\langle Au - f(u), u - \psi\rangle = 0, \\
u \leq \psi, \\
u = 0, \quad \text{in } \partial\Omega,
\end{cases} \tag{4.1}$$

where

$$Au = -\Delta u,$$
$$f(u) = \cos u,$$
$$\psi = 0.$$

We are confine ourselves to the FEM discretization with a uniform triangulation and $P1$ nested finite element function spaces. For the discretization of the domain, we have used the PDE toolbox in MATLAB (R2017b) to generate the mesh and then the multigrid FEM can be used to efficiently solve 4.1 as mentioned above.

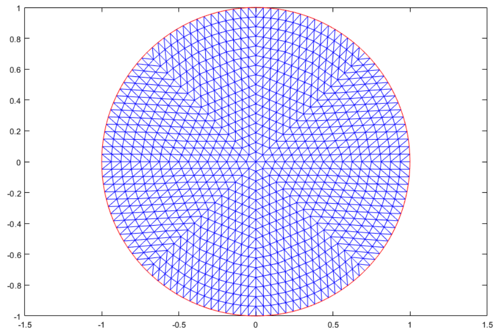

Figure 1: Domain of our problem with 2048 triangle and 1089 nodes.

This numerical example is conducted to demonstrate the high efficiency of the multigrid method. In this study, we utilize the Gauss-Seidel method for pre/post smoothing within the multigrid code. With respect to recursion in the multigrid method, the recursive algorithm is terminated when the degrees of freedom, represented by the number of interior grid points, become less than 5. Figure 2 illustrates the convergence behavior of the

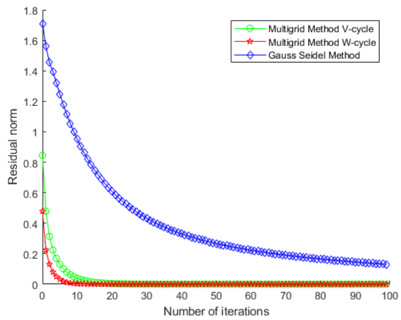

Figure 2: Comparison between the convergence behavior of Multigrid method and Gauss-Seidel methods.

multigrid solver (green and red curves for the maximum norm of the residual of multigrid (V and W cycle)) with respect to the number of iterations performed. For comparison, the convergence behavior of Gauss-Seidel (blue curves) are included. The multigrid V -cycle is carried out on the finest grid with 1089 nodes and 4 nodes on the coarsest one, and then we have applied thethe Matlab backslash-operator and Gauss-Seidel on this finest grid to get the solutions in figures 3-6.

$$\begin{cases} \text{Norm of residual obtained by Gauss Seidel method after 100 iteratons } := 0.1303. \\ \text{Norm of residual obtained by multigrid V-cycle after 100 iterations} := 3.7937\text{e} - 11. \\ \text{Norm of residual obtained by multigrid W-cycle after 100 iterations} := 1.6209\text{e} - 14. \end{cases}$$

Notting that, if we perform more than 20 iterations, the multigrid solution is better than the Matlab backslash-operator (MBO) solution.

$$\begin{cases} \text{Norm of residual obtained by Gauss Seidel method after 100 iteratons } := 0.1303. \\ \text{Norm of residual obtained by multigrid V-cycle after 20 iterations} := 0.0049. \\ \text{Norm of residual obtained by multigrid W-cycle after 20 iterations} := 4.5777\text{e} - 5. \end{cases}$$

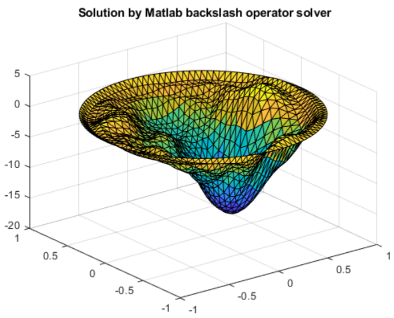

Figure 3: Solution of the problem 4.1 on fine grid with 1089 DOFs using Matlab backslash operator solver after 100 iterations.

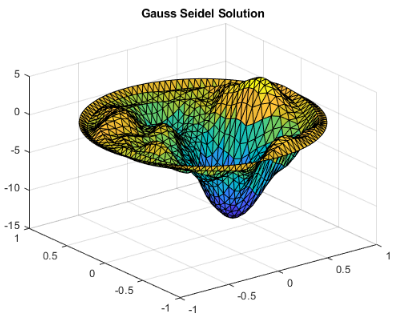

Figure 4: Solution of the problem 4.1 on fine grid with 1089 DOFs using Gauss Seidel Method after 100 iterations.

5. Conclusion

In this study, we utilize the algebraic multigrid method, specifically the efficient iterative solutions for discretizing elliptic variational inequalities, to handle the discretization

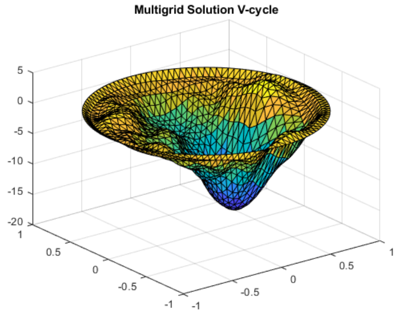

Figure 5: Solution of the problem 4.1 on fine grid with 1089 DOFs using Multigrid Method V-cycle after 100 iterations.

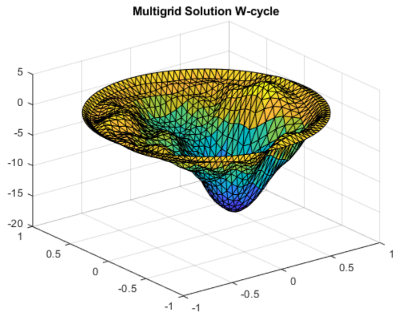

Figure 6: Solution of the problem 4.1 on fine grid with 1089 DOFs using Multigrid Method W-cycle after 100 iterations.

of a loop domain using adaptive finite element approximation. After discretization, we implement the multigrid method to efficiently solve the discrete problems. We investigate the uniform convergence of our approach and demonstrate that the multigrid method exhibits a significant reduction in the number of iterations compared to the maximum norm. In our numerical experiments, we present an example of a variational inequality. The results indicate that the Gauss-Seidel method does not perform well even after a large number of iterations. In contrast, the multigrid method, with its debug function that reduces high-frequency errors through relaxation and maps low-frequency errors to the coarse grid for reduction, achieves convergence in a small number of iterations. We acknowledge the potential for many extensions of these techniques. An interesting future direction could involve applying a parallel full multigrid method to solve unconstrained elliptical inequalities. This approach may further enhance the efficiency and scalability of our numerical solution for a broader range of problems.

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References

[1] M Adel, NH Sweilam, MM Khader, SM Ahmed, H Ahmad, and T Botmart. Numerical simulation using the non-standard weighted average FDM for 2Dim variable-order Cable equation. *Results in Physics*, 39:105682, 2022.

[2] H Ahmad AE Abouelregal and S-W Yao. Functionally graded piezoelectric medium exposed to a movable heat flow based on a heat equation with a memory-dependent derivative. *Materials*, 13(18):3953, 2020.

[3] H Ahmad, , and TA Khan. Variational iteration algorithm I with an auxiliary parameter for the solution of differential equations of motion for simple and damped mass–spring systems. *Noise & Vibration Worldwide*, 51(1-2):12–20, 2020.

[4] H Ahmad, TA Khan, and S-W Yao. An efficient approach for the numerical solution of fifth-order KdV equations. *Open Mathematics*, 18(1):738–748, 2020.

[5] I Ahmad, AR Seadawy, H Ahmad, P Thounthong, and F Wang. Numerical study of multi-dimensional hyperbolic telegraph equations arising in nuclear material science via an efficient local meshless method. *International Journal of Nonlinear Sciences and Numerical Simulation*, 23(1):115–122, 2022.

[6] M Ahsan, AA Khan, S Dinibutun, I Ahmad, H Ahmad, N Jarasthitikulchai, and W Sudsutad. The Haar wavelets based numerical solution of Reccati equation with integral boundary condition. *Thermal Science*, 27(Spec. issue 1):93–100, 2023.

[7] A Akgül and H Ahmad. Reproducing kernel method for Fangzhu's oscillator for water collection from air. *Mathematical Methods in the Applied Sciences*, 2020.

[8] R Alharbi, R Jan, S Alyobi, A Yousif, and Z Khan. Mathematical modeling and stability analysis of the dynamics of monkeypox via fractional-calculus. *Fractals*, 30(10):2240266, 2022.

[9] B Almutairi, I Ahmad, B Almohsen, H Ahmad, and DU Ozsahin. Numerical simulations of time-fractional PDEs arising in mathematics and physics using the local Meshless differential quadrature method. *Thermal Science*, 27(Spec. issue 1):263–272, 2023.

[10] M Boulbrachene and M Haiour. The finite element approximation of Hamilton-Jacobi-Bellman equations. *Computers & Mathematics with Applications*, 41(7-8):993–1007, 2001.

[11] F Brezzi and LA Caffarelli. Convergence of the discrete free boundaries for finite element approximations. *RAIRO. Analyse numérique*, 17(4):385–395, 1983.

[12] PG Ciarlet and P-A Raviart. Maximum principle and uniform convergence for the finite element method. *Computer methods in applied mechanics and engineering*, 2(1):17–31, 1973.

[13] P. Cortey-Dumont. On the finite element approximation in the l^{∞} norm of variational inequalities with nonlinear operators. *Numerische Mathematik*, 47:45–57, 1985.

[14] P Cortey-Dumont and S I'Analyse. Numérique des Equations de Hamilton- Jacobi-Bellman. *Mathematical Methods in the Applied Sciences*, 9:198–209, 1987.

[15] W Hackbusch. *Multi-grid methods and applications.*, volume 4. Springer Science & Business Media, 2013.

[16] W Hackbusch and HD Mittelmann. On multi-grid methods for variational inequalities. *Numerische Mathematik*, 42:65–76, 1983.

[17] M HAIOUR. *Etude de la convergence uniforme de la méthode multigrilles appliquées aux problèmes frontières libres.* PhD thesis, Université de Annaba-Badji Mokhtar, 2004.

[18] R HW Hoppe. Multi-grid methods for Hamilton-Jacobi-Bellman equations. *Numerische Mathematik*, 49:239–254, 1986.

[19] R HW Hoppe. Multigrid algorithms for variational inequalities. *SIAM journal on numerical analysis*, 24(5):1046–1065, 1987.

[20] H Irshad, M Shakeel, I Ahmad, H Ahmad, C Tearnbucha, and W Sudsutad. Simulation of generalized time fractional Gardner equation utilizing in plasma physics for non-linear propagation of ion-acoustic waves. *Thermal Science*, 27(Spec. issue 1):121–128, 2023.

[21] R Jan, A Khan, S Boulaaras, and SA Zubair. Dynamical behaviour and chaotic phenomena of HIV infection through fractional calculus. *Discrete Dynamics in Nature and Society*, 2022, 2022.

[22] D Kinderlehrer and G Stampacchia. *An introduction to variational inequalities and their applications.* SIAM, 2000.

[23] J-F Li, I Ahmad, H Ahmad, DShah, Y-M Chu, P Thounthong, and M Ayaz. Numerical solution of two-term time-fractional PDE models arising in mathematical physics using local meshless method. *Open Physics*, 18(1):1063–1072, 2020.

[24] H Ahmad M Adel, ME Ramadan and T Botmart. Sobolev-type nonlinear Hilfer fractional stochastic differential equations with noninstantaneous impulsive. *AIMS Mathematics*, 7(11):20105–20125, 2022.

[25] A. Reusken. On maximum norm convergcnce of multigrids methods for elliptic boundary value problems. *SIAM Journal on Numerical Analysis*, 29(6):1569–1578, 1992.

[26] A Reusken. *Introduction to multigrid methods for elliptic boundary value problems.* Inst. für Geometrie und Praktische Mathematik, 2008.

[27] NA Shah, I Ahmad, O Bazighifan, AE Abouelregal, and H Ahmad. Multistage optimal homotopy asymptotic method for the nonlinear Riccati ordinary differential equation in nonlinear physics. *Applied Mathematics*, 14(6):1009–1016, 2020.

[28] M Shakeel, I Hussain, H Ahmad, I Ahmad, P Thounthong, Y-F Zhang, and Ying-Fang. Meshless technique for the solution of time-fractional partial differential equations having real-world applications. *Journal of Function Spaces*, 2020:1–17, 2020.

[29] M Shakeel, MN Khan, I Ahmad, H Ahmad, N Jarasthitikulchai, and W Sudsutad. Local meshless collocation scheme for numerical simulation of space fractional PDE. *Thermal Science*, 27(Spec. issue 1):101–109, 2023.

[30] TA Sulaiman, A Yusuf, S Abdel-Khalek, M Bayram, and H Ahmad. Nonautonomous complex wave solutions to the (2+ 1)-dimensional variable-coefficients nonlinear Chiral Schrödinger equation. *Results in Physics*, 19:103604, 2020.

[31] R Jan T-Q TangZ Shah and E Alzahrani. Modeling the dynamics of tumor–immune cells interactions via fractional calculus. *The European Physical Journal Plus*, 137(3):367, 2022.

[32] T-Q Tang, Z Shah, R Jan, W Deebani, and M Shutaywi Meshal. A robust study to conceptualize the interactions of CD4+ T-cells and human immunodeficiency virus via fractional-calculus. *Physica Scripta*, 96(12):125231, 2021.

[33] F Wang, I Ahmad, H Ahmad, MD Alsulami, KS Alimgeer, C Cesarano, and TA Nofal. Meshless method based on RBFs for solving three-dimensional multi-term time fractional PDEs arising in engineering phenomenons. *Journal of King Saud University-Science*, 33(8):101604, 2021.

[34] F Wang, NA Ali, I Ahmad, H Ahmad, KM Alam, and P Thounthong. Solution of Burgers' equation appears in fluid mechanics by multistage optimal homotopy asymptotic method. *Thermal Science*, 26(1 Part B):815–821, 2022.

[35] F Wang, E Hou, I Ahmad, H Ahmad, and Y Gu. An efficient meshless method for hyperbolic telegraph equations in (1+ 1) dimensions. *CMES-Computer Modeling in Engineering and Sciences*, 128(2):687–98, 2021.