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On B-commutators of B-algebras

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Abstract. In this paper, we investigate some properties of B-commutators of B-algebras. We also characterize solvable B-algebras via B-commutators.

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1. Introduction and Preliminaries

In 1966, Y. Imai and K. Iséki introduced the concept of BCK-algebras [14]. It is known that BCK-algebras are inspired by some implicational logic. From then on, several generalizations of BCK-algebras exist. In [15], K. Iséki introduced BCI-algebras and that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1983, Q.P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras [13]. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. These algebras are of type (2, 0), that is, a nonempty set together with a binary operation and a constant, satisfying some axioms. Up to this day, inspired by BCK/BCI/BCH-algebras, there are more than twenty type (2, 0) algebras introduced and investigated. One of these algebras is the concept of B-algebras.

In [21], J. Neggers and H.S. Kim introduced and established the notion of B-algebras. A *B-algebra* is an algebra $(X; *, 0)$ of type (2, 0) satisfying:
(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$.
X is said to be *commutative* if $x * (0 * y) = y * (0 * x)$ for any $x, y \in X$. Let X be a B-algebra. Recall that for any $x, y, z \in X$, we have the following properties:
(P1) $0 * (0 * x) = x$ [21],
(P2) $x * y = 0 * (y * x)$ [26],
(P3) $x * (y * z) = (x * (0 * z)) * y$ [21],
(P4) $(x * z) * (y * z) = x * y$ [26].

We now present two examples of B-algebras, one is commutative and the other is noncommutative.

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Example 1. Let $X = \{0, 1, 2, 3\}$ be a set with the following table of operations:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X; *, 0)$ is a commutative B-algebra [10].

Example 2. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table of operations:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X;\ *,\ 0)$ is a noncommutative B-algebra [20].

Throughout this paper, let X be a B-algebra $(X; *, 0)$. In [20], a nonempty subset N of X is called a *subalgebra* of X if $x * y \in N$ for any $x, y \in N$. A subalgebra N of X is called *normal* in X if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$. Let N be normal in X. Define a relation $\sim_N$ on X by $x \sim_N y$ if and only if $x * y \in N$, where $x, y \in X$. Then $\sim_N$ is an equivalence relation on X. Denote the equivalence class containing x by xN, that is, $xN = \{y \in X : x \sim_N y\}$. Let $X/N = \{xN : x \in X\}$. The binary operation in X/N is defined by $xN *' yN = (x * y)N$. The B-algebra X/N is called the *quotient B-algebra* of X by N. In [1], $xH = \{x * (0 * h) : h \in H\}$ and $Hx = \{h * (0 * x) : h \in H\}$, called the *left* and *right B-cosets* of H in X, respectively. The subset HK [11] of X is given by $HK = \{x \in X : x = h * (0 * k) \text{ for some } h \in H, k \in K\}$.

Other properties and characterizations of B-algebras can be found in some other papers ([2–7, 9, 10, 12, 16–19], [22, 23], [25, 26].) In particular, R. Soleimani [24] introduced the notion of B-commutators of B-algebras. He also established some basic properties of B-commutators. In [8], J.C. Endam and G.S. Dael introduced the notion of solvable B-algebras. In this paper, we established some basic properties of B-commutators of B-algebras. These properties are used in characterizing solvable B-algebras via B-commutators. As a result, we showed that a B-algebra X is solvable if and only if there is positive integer m such that the mth B-commutator subalgebra $X^{(m)}$ is equal to $\{0\}$.

2. B-commutators

This section presents some identities satisfied by the B-commutators in B-algebras. We recall first from [24] the definition of B-commutators. Let $x, y \in X$. The *B-commutator* of x and y is given by

$$[x, y] = ((0 * x) * y) * ((0 * y) * x).$$

The subalgebra of X generated by $\{[x, y] : x, y \in X\}$ is called the *derived B-algebra*, denoted by $D(X)$.

Example 3. Let $(X; *, 0)$ be the B-algebra in Example 2. We now compute for $[x, y]$ for all $x, y \in X$. These computations are used in the succeeding examples.

$[0,0]=0$	$[1,1]=0$	$[2,2]=0$	$[3,3]=0$	$[4,4]=0$	$[5,5]=0$
$[0,1]=0$	$[1,0]=0$	$[2,0]=0$	$[3,0]=0$	$[4,0]=0$	$[5,0]=0$
$[0,2]=0$	$[1,2]=0$	$[2,1]=0$	$[3,1]=2$	$[4,1]=2$	$[5,1]=2$
$[0,3]=0$	$[1,3]=1$	$[2,3]=2$	$[3,2]=1$	$[4,2]=1$	$[5,2]=1$
$[0,4]=0$	$[1,4]=1$	$[2,4]=2$	$[3,4]=1$	$[4,3]=2$	$[5,3]=1$
$[0,5]=0$	$[1,5]=1$	$[2,5]=2$	$[3,5]=2$	$[4,5]=1$	$[5,4]=2$

A map $\varphi : X \to Y$ is called a *B-homomorphism* [20] if $\varphi(x * y) = \varphi(x) * \varphi(y)$ for any $x, y \in X$.

Lemma 1. [24] *Let $\varphi : X \to Y$ be a B-homomorphism and let $x, y \in X$. Then*

i. $[x, y] = 0$ *if and only if* $x * (0 * y) = y * (0 * x)$,

ii. $\varphi([x, y]) = [\varphi(x), \varphi(y)]$,

iii. if φ is onto, then $\varphi(D(X)) = D(\varphi(X))$.

Lemma 2. *Let $x, y \in X$. Then*

i. $[x, x] = [x, 0] = [0, x] = 0$,

ii. $0 * [x, y] = [y, x]$.

Proof. Clearly, (i) follows from Lemma 1(i) and (P1); (ii) follows from (P2). □

Let $x, w \in X$. We define x^w to be the element $(0 * w) * ((0 * w) * x)$. For instance, let X be the B-algebra in Example 1. Below are some sample computations to illustrate x^w:

$$2^3 = (0 * 3) * ((0 * 3) * 2) = 3 * (3 * 2) = 3 * 1 = 2$$
$$3^2 = (0 * 2) * ((0 * 2) * 3) = 2 * (2 * 3) = 2 * 1 = 3$$
$$1^3 = (0 * 3) * ((0 * 3) * 1) = 3 * (3 * 1) = 3 * 2 = 1$$
$$3^1 = (0 * 1) * ((0 * 1) * 3) = 1 * (1 * 3) = 1 * 2 = 3$$
$$1^2 = (0 * 2) * ((0 * 2) * 1) = 2 * (2 * 1) = 2 * 3 = 1$$
$$2^1 = (0 * 1) * ((0 * 1) * 2) = 1 * (1 * 2) = 1 * 3 = 2$$

The following lemma presents the basic properties of x^w.

Lemma 3. *Let $x, y, w \in X$. Then the following properties hold:*

i. $0 * x^w = (0 * x)^w$,

ii. $(x * y)^w = 0 * (y * x)^w$,

iii. $(0 * x)^x = 0 * x$,

iv. $x^x = x$,

v. $x^{0*x} = x$,

vi. $x * y^x = (0 * y) * (0 * x)$,

vii. $x^y = x * [y, x]$,

viii. $x^{0*y} = y * (y * x)$,

ix. $[x^y, 0 * y] = [y, x]$.

Proof. Let $x, y, w \in X$.
i. By (P2) and (III), we have

$$\begin{aligned} 0 * x^w &= 0 * ((0 * w) * ((0 * w) * x)) \\ &= ((0 * w) * x) * (0 * w) \\ &= (0 * w) * ((0 * w) * (0 * x)) \\ &= (0 * x)^w. \end{aligned}$$

ii. By (i), we have $(x * y)^w = (0 * (y * x))^w = 0 * (y * x)^w$.
iii. By (I) and (II), we have

$$\begin{aligned} (0 * x)^x &= (0 * x) * ((0 * x) * (0 * x)) \\ &= (0 * x) * 0 \\ &= 0 * x. \end{aligned}$$

iv. By P3, (I), and P1, we have

$$\begin{aligned} x^x &= (0 * x) * ((0 * x) * x) \\ &= ((0 * x) * (0 * x)) * (0 * x) \\ &= 0 * (0 * x) \\ &= x. \end{aligned}$$

v. By P1, (I), and (II), we have

$$\begin{aligned} x^{0*x} &= (0 * (0 * x)) * ((0 * (0 * x)) * x) \\ &= x * (x * x) \\ &= x * 0 \\ &= x. \end{aligned}$$

vi. By P3, P2, (III), and (I), we have

$$\begin{aligned} x * y^x &= x * ((0 * x) * ((0 * x) * y)) \\ &= (x * (0 * ((0 * x) * y))) * (0 * x) \\ &= (x * (y * (0 * x))) * (0 * x) \\ &= ((x * x) * b) * (0 * x) \\ &= (0 * y) * (0 * x). \end{aligned}$$

vii. By P3, P2, (III), and (I), we have

$$\begin{aligned} x * [y, x] &= x * (((0 * y) * x) * ((0 * x) * y)) \\ &= (x * (0 * ((0 * x) * y))) * ((0 * y) * x) \\ &= (x * (y * (0 * x))) * ((0 * y) * x) \\ &= ((x * x) * y) * ((0 * y) * x) \\ &= (0 * y) * ((0 * y) * x) \\ &= x^y. \end{aligned}$$

viii. This follows from P1.
ix. By P1, (vi), P4, (vii), P2, and (II), we get

$$\begin{aligned} [x^y, 0 * y] &= ((0 * x^y) * (0 * y)) * ((0 * (0 * y)) * x^y) \\ &= ((0 * x^y) * (0 * y)) * (y * x^y) \\ &= ((0 * x^y) * (0 * y)) * ((0 * x) * (0 * y)) \\ &= (0 * x^y) * (0 * x) \\ &= (0 * (x * [y, x])) * (0 * x) \\ &= ([y, x] * x) * (0 * x) \\ &= [y, x] * 0 \\ &= [y, x]. \end{aligned}$$

□

The following lemma is used to prove the succeeding theorems.

Lemma 4. *Let $a, b, c \in X$. Then the following properties hold:*

i. $((a * b) * a) * (a * (a * c)) = a * (a * ((0 * b) * c))$,

ii. $(a * (a * b)) * (a * (a * c)) = a * (a * (b * c))$,

iii. $[((a * b) * ((0 * b) * (0 * a))) * c] * (c * (c * b)) = (a * b) * (c * (0 * a))$.

Proof. i. By (III), P2, P4, and P3, we have

$$((a * b) * a) * (a * (a * c)) = (a * b) * [(a * (a * c)) * (0 * a)]$$

$$
\begin{aligned}
&= a*[((a*(a*c))*(0*a))*(0*b)] \\
&= a*[(a*((0*a)*(0*(a*c))))*(0*b)] \\
&= a*[(a*((0*a)*(c*a)))*(0*b)] \\
&= a*[(a*(0*c))*(0*b)] \\
&= a*(a*((0*b)*c)).
\end{aligned}
$$

ii. By (III), P2, and P4, we have

$$
\begin{aligned}
(a*(a*b))*(a*(a*c)) &= a*[(a*(a*c))*(0*(a*b))] \\
&= a*((a*(a*c))*(b*a)) \\
&= a*[a*((b*a)*(0*(a*c)))] \\
&= a*(a*((b*a)*(c*a))) \\
&= a*(a*(b*c)).
\end{aligned}
$$

iii. By (III), P2, P4, P3, (I), and (II), we have
$[((a*b)*((0*b)*(0*a)))*c]*(c*(c*b))$

$$
\begin{aligned}
&= [(a*b)*(c*(0*((0*b)*(0*a))))]*(c*(c*b)) \\
&= [(a*b)*(c*((0*a)*(0*b)))]*(c*(c*b)) \\
&= (a*b)*[(c*(c*b))*(0*(c**((0*a)*(0*b))))] \\
&= (a*b)*[(c*(c*b))*(((0*a)*(0*b))*c)] \\
&= (a*b)*[c*((((0*a)*(0*b))*c)*(0*(c*b)))] \\
&= (a*b)*[c*((((0*a)*(0*b))*c)*(b*c))] \\
&= (a*b)*[c*(((0*a)*(0*b))*b)] \\
&= (a*b)*[c*((0*a)*(b*b))] \\
&= (a*b)*(c*((0*a)*0)) \\
&= (a*b)*(c*(0*a)).
\end{aligned}
$$

□

Theorem 1. *Let $w, x, y \in X$. Then $[x,y]^w = [x^w, y^w]$.*

Proof. By P2, we have

$$
\begin{aligned}
[x^w, y^w] &= ((0*x^w)*y^w)*((0*y^w)*x^w) \\
&= [(0*((0*w)*((0*w)*x)))*y^w]*[(0*((0*w)*((0*w)*y)))*x^w] \\
&= \underbrace{[(((0*w)*x)*(0*w))*y^w]}_{(1)}*\underbrace{[(((0*w)*y)*(0*w))*x^w]}_{(2)}
\end{aligned}
$$

We first consider (1), by Lemma 4(i) [with $a = 0*w, b = x, c = y$], we have

$$
\begin{aligned}
(((0*w)*x)*(0*w))*y^w &= (((0*w)*x)*(0*w))*((0*w)*((0*w)*y)) \\
&= (0*w)*((0*w)*((0*x)*y)).
\end{aligned}
$$

Similarly for (2), by Lemma 4(i) [with $a = 0*w, b = y, c = x$], we have

$$
(((0*w)*y)*(0*w))*x^w = (((0*w)*y)*(0*w))*((0*w)*((0*w)*x))
$$

$$= (0*w)*((0*w)*((0*y)*x)).$$

Thus,

$$\begin{aligned}[x^w, y^w] &= \underbrace{[(((0*w)*x)*(0*w))*y^w]}_{(1)} * \underbrace{[(((0*w)*y)*(0*w))*x^w]}_{(2)} \\ &= [(0*w)*((0*w)*((0*x)*y))]*[(0*w)*((0*w)*((0*y)*x))].\end{aligned}$$

Applying Lemma 4(ii) [with $a = 0*w, b = (0*x)*y, c = (0*y)*x$], we have

$$\begin{aligned}[x^w, y^w] &= [(0*w)*((0*w)*((0*x)*y))]*[(0*w)*((0*w)*((0*y)*x))] \\ &= (0*w)*[(0*w)*(((0*x)*y)*((0*y)*x))] \\ &= (0*w)*((0*w)*[x,y]) \\ &= [x,y]^w.\end{aligned}$$

□

Theorem 2. *Let $w, x, y, z \in X$. Then $[x*(0*y), z] = [x,z]^y * [z,y]$.*

Proof. By (III) and P2, we have

$$\begin{aligned}[x,z]^y * [z,y] &= ((0*y)*((0*y)*[x,z]))*[z,y] \\ &= (0*y)*([z,y]*(0*((0*y)*[x,z]))) \\ &= (0*y)*([z,y]*([x,z]*(0*y))) \\ &= (0*y)*[(((0*z)*y)*((0*y)*z))*((((0*x)*z)*((0*z)*x))*(0*y))]\end{aligned}$$

For simplicity, we write $x' = 0*x, y' = 0*y, z' = 0*z$. Thus, by (III), P1, and P2, we get

$$\begin{aligned}[x,z]^y * [z,y] &= y'*[((z'*y)*(y'*z))*(((x'*z)*(z'*x))*y')] \\ &= y'*[(z'*((y'*z)*y'))*(((x'*z)*(z'*x))*y')] \\ &= y'*[z'*((((x'*z)*(z'*x))*y')*(y'*(y'*z)))]\end{aligned}$$

Applying Lemma 4(iii) [with $a = x', b = z, c = y'$], P2, and (III), we get

$$\begin{aligned}[x,z]^y * [z,y] &= y'*[z'*((x'*z)*(y'*x))] \\ &= y'*[z'*((x'*z)*(0*(x*y')))] \\ &= y'*[(z'*(x*y'))*(x'*z)] \\ &= y'*[(z'*(x*y'))*(0*(z*x'))] \\ &= (y'*(z*x'))*(z'*(x*y')) \\ &= ((0*y)*(z*(0*x)))*((0*z)*(x*(0*y))) \\ &= (((0*y)*x)*z)*((0*z)*(x*(0*y))) \\ &= ((0*(x*(0*y)))*z)*((0*z)*(x*(0*y))) \\ &= [x*(0*y), z].\end{aligned}$$

□

Corollary 1. *Let* $x, y, z \in X$*. Then* $[x, y * (0 * z)] = [x, z] * [y, x]^z$.

Proof. By Lemma 2(ii), Theorem 2, and P2, we get

$$\begin{aligned}[x, y * (0 * z)] &= 0 * [y * (0 * z), x] \\ &= 0 * ([y, x]^z * [x, z]) \\ &= [x, z] * [y, x]^z.\end{aligned}$$

□

Theorem 3. *Let* $x, y \in X$*. Then* $[x, 0 * y] = [y, x]^{0*y}$.

Proof. By Theorem 1, Lemma 3(v, vi), P1, P4, Lemma 3($viii$), P2, and P3, we get

$$\begin{aligned}[y, x]^{0*y} &= [y^{0*y}, x^{0*y}] \\ &= [y, x^{0*y}] \\ &= ((0 * y) * x^{0*y}) * ((0 * x^{0*y}) * y) \\ &= ((0 * x) * (0 * (0 * y))) * ((0 * x^{0*y}) * y) \\ &= ((0 * x) * y) * ((0 * x^{0*y}) * y) \\ &= (0 * x) * (0 * x^{0*y}) \\ &= (0 * x) * (0 * (y * (y * x))) \\ &= (0 * x) * ((y * x) * y) \\ &= ((0 * x) * (0 * y)) * (y * x) \\ &= ((0 * x) * (0 * y)) * ((0 * (0 * y)) * x) \\ &= [x, 0 * y].\end{aligned}$$

□

Corollary 2. *Let* $x, y \in X$*. Then* $[0 * x, y] = [y, x]^{0*x}$.

Proof. By Lemma 2(ii), Theorem 3, and Lemma 3(i), we get

$$\begin{aligned}[0 * x, y] &= 0 * [y, 0 * x] \\ &= 0 * [x, y]^{0*x} \\ &= (0 * [x, y])^{0*x} \\ &= [y, x]^{0*x}.\end{aligned}$$

□

3. kth B-commutators

We recall first the concept of solvable B-algebras [8]. Let

$$X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n = \{0\}$$

be a series of subalgebras of X. The series is called a *subnormal B-series* if each H_i is normal in H_{i-1}. The series is called a *normal B-series* if each H_i is normal in X. Since $\{0\}$ is normal in X, every B-algebra has a normal B-series. If X has a subnormal B-series $X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}$ such that H_i/H_{i+1} is commutative, $i = 0, 1, \ldots, n-1$, then we say that X is *solvable.* Such a subnormal B-series is called a *solvable B-series* for X.

For simplicity, we write the derived B-algebra $D(X)$ as X'.

Definition 1. *Set $X^{(1)} = X'$ and define inductively $X^{(k+1)} = X^{(k)\prime}$, the B-commutator subalgebra of $X^{(k)}$, $k > 0$. For any positive integer k, $X^{(k)}$ is called the kth B-commutator subalgebra of X.*

By Lemma 1, a B-algebra X is commutative if and only if $X' = \{0\}$.

Example 4. Let $(X; *, 0)$ be the noncommutative B-algebra in Example 2. Then from the computations in Example 3, we see that $X' = \{0, 1, 2\}$ and $X^{(2)} = \{0, 1, 2\}' = \{0\}$. Thus, $X^{(k)} = \{0\}$ for all $k \geq 2$.

The following theorem characterizes solvable B-algebra.

Theorem 4. *X is solvable if and only if there is positive integer m such that $X^{(m)} = \{0\}$.*

Proof. Suppose that X is solvable. Then X has a solvable series, say,

$$X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}.$$

Since H_{i+1} is normal in H_i and H_i/H_{i+1} is commutative, $H_i' \subseteq H_{i+1}$ by [24, Theorem 4.14]. Hence,

$$H_1 \supseteq H_0' = X^{(1)},\ H_2 \supseteq H_1' \supseteq X^{(2)},\ \ldots,\{0\} = H_n \supseteq H_{n-1}' \supseteq X^{(n)}.$$

Thus, $X^{(n)} = \{0\}$.
Conversely, suppose that $X^{(m)} = \{0\}$. The series $X \supseteq X^{(1)} \supseteq \cdots \supseteq X^{(m-1)} = \{0\}$ is a solvable B-series. Thus, X is solvable. □

Proposition 1. *Let $H \neq \{0\}$ be a subalgebra of a solvable B-algebra X. Then $H' \neq H$.*

Proof. Suppose $H' = H$. Then $H^{(2)} = (H')' = H' = H \neq \{0\}$. By induction, $H^{(n)} = H \neq \{0\}$ for any positive integer n. By [8, Theorem 12], H is solvable. Thus, by Theorem 4, there exists a positive integer n such that $H^{(n)} = \{0\}$, a contradiction. Hence, $H' \neq H$. □

Theorem 5. *A finite B-algebra X is solvable if and only if $H' \neq H$ for any subalgebra $H \neq \{0\}$ of X.*

Proof. Let X be a finite B-algebra. Suppose that X is solvable. By Proposition 1, $H' \neq H$ for any subalgebra $H \neq \{0\}$ of X. Conversely, suppose that $H' \neq H$ for any subalgebra $H \neq \{0\}$ of X. Then $X \neq X'$. Thus, $X' \subset X$. If $X^{(n)} \neq \{0\}$, then $X^{(n)} \neq X^{(n+1)}$, that is $X^{(n+1)} \subset X^{(n)}$. Hence, we have the following strictly descending series of subalgebras:

$$X \supset X' \supset \cdots \supset X^{(n)} \supset X^{(n+1)} \supset \cdots .$$

Since X is finite and $H' \neq H$ for any subalgebra $H \neq \{0\}$ of X, there exists a positive integer n such that $X^{(n)} = \{0\}$. Hence, X is solvable. □

Example 5. Let $(X; *, 0)$ be the noncommutative B-algebra in Example 2. The nontrivial subalgebras of X are the following: $H_1 = \{0, 3\}$, $H_2 = \{0, 4\}$, $H_3 = \{0, 5\}$, $H_4 = \{0, 1, 2\}$. Clearly, from the computations in Example 3, we get $H_1' = \{0\} \neq H_1$, $H_2' = \{0\} \neq H_2$, $H_3' = \{0\} \neq H_3$, and $H_4' = \{0\} \neq H_4$. In Example 4, $X' = \{0, 1, 2\} \neq X$. Hence, $H' \neq H$ for any subalgebra $H \neq \{0\}$ of X. Therefore, by Theorem 5, X is solvable, which confirms the result in [8, Example 11].

4. Conclusion

We established some basic properties of B-commutators of B-algebras. These properties are used in characterizing solvable B-algebras via B-commutators. As a result, we showed that a B-algebra X is solvable if and only if there is positive integer m such that the mth B-commutator subalgebra $X^{(m)}$ is equal to $\{0\}$.

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