On B-commutators of B-algebras

Joel G. Adanza

Mathematics Department, Negros Oriental State University, Dumaguete City, Philippines

Abstract. In this paper, we investigate some properties of B-commutators of B-algebras. We also characterize solvable B-algebras via B-commutators.

2020 Mathematics Subject Classifications: 08A05, 03G25
Key Words and Phrases: Solvable B-algebras, B-commutators, kth B-commutators

1. Introduction and Preliminaries

In 1966, Y. Imai and K. Iséki introduced the concept of BCK-algebras [14]. It is known that BCK-algebras are inspired by some implicational logic. From then on, several generalizations of BCK-algebras exist. In [15], K. Iséki introduced BCI-algebras and that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1983, Q.P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras [13]. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. These algebras are of type (2, 0), that is, a nonempty set together with a binary operation and a constant, satisfying some axioms. Up to this day, inspired by BCK/BCI/BCH-algebras, there are more than twenty type (2, 0) algebras introduced and investigated. One of these algebras is the concept of B-algebras.

In [21], J. Neggers and H.S. Kim introduced and established the notion of B-algebras. A B-algebra is an algebra \((X; *, 0)\) of type \((2, 0)\) satisfying:

(I) \(x * x = 0\),

(II) \(x * 0 = x\),

(III) \((x * y) * z = x * (z * (0 * y))\), for any \(x, y, z \in X\).

\(X\) is said to be commutative if \(x * (0 * y) = y * (0 * x)\) for any \(x, y \in X\). Let \(X\) be a B-algebra. Recall that for any \(x, y, z \in X\), we have the following properties:

(P1) \(0 * (0 * x) = x\) [21],

(P2) \(x * y = 0 * (y * x)\) [26],

(P3) \(x * (y * z) = (x * (0 * z)) * y\) [21],

(P4) \((x * z) * (y * z) = x * y\) [26].

We now present two examples of B-algebras, one is commutative and the other is noncommutative.

DOI: https://doi.org/10.29020/nybg.ejpam.v16i3.4841

Email address: joel.adanza@norsu.edu.ph (J. Adanza)
Example 1. Let $X = \{0, 1, 2, 3\}$ be a set with the following table of operations:

<table>
<thead>
<tr>
<th>$*$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X; *, 0)$ is a commutative $B$-algebra [10].

Example 2. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table of operations:

<table>
<thead>
<tr>
<th>$*$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X; *, 0)$ is a noncommutative $B$-algebra [20].

Throughout this paper, let $X$ be a $B$-algebra $(X; *, 0)$. In [20], a nonempty subset $N$ of $X$ is called a subalgebra of $X$ if $x \ast y \in N$ for any $x, y \in N$. A subalgebra $N$ of $X$ is called normal in $X$ if $(x \ast a) \ast (y \ast b) \in N$ for any $x \ast y, a \ast b \in N$. Let $N$ be normal in $X$. Define a relation $\sim_N$ on $X$ by $x \sim_N y$ if and only if $x \ast y \in N$, where $x, y \in X$. Then $\sim_N$ is an equivalence relation on $X$. Denote the equivalence class containing $x$ by $x_N$, that is, $x_N = \{y \in X : x \sim_N y\}$. Let $X/N = \{x_N : x \in X\}$. The binary operation in $X/N$ is defined by $x_N \ast y_N = (x \ast y)N$. The $B$-algebra $X/N$ is called the quotient $B$-algebra of $X$ by $N$. In [1], $xH \ast h = \{x \ast (0 \ast h) : h \in H\}$ and $Hx = \{h \ast (0 \ast x) : h \in H\}$, called the left and right $B$-cosets of $H$ in $X$, respectively. The subset $HK$ [11] of $X$ is given by $HK = \{x \in X : x = h \ast (0 \ast k) \text{ for some } h \in H, k \in K\}$.

Other properties and characterizations of $B$-algebras can be found in some other papers ([2–7, 9, 10, 12, 16–19], [22, 23], [25, 26].) In particular, R. Soleimani [24] introduced the notion of $B$-commutators of $B$-algebras. He also established some basic properties of $B$-commutators. In [8], J.C. Endam and G.S. Dael introduced the notion of solvable $B$-algebras. In this paper, we established some basic properties of $B$-commutators of $B$-algebras. These properties are used in characterizing solvable $B$-algebras via $B$-commutators. As a result, we showed that a $B$-algebra $X$ is solvable if and only if there is positive integer $m$ such that the $m$-th $B$-commutator subalgebra $X^{(m)}$ is equal to $\{0\}$.

2. $B$-commutators

This section presents some identities satisfied by the $B$-commutators in $B$-algebras. We recall first from [24] the definition of $B$-commutators. Let $x, y \in X$. The $B$-commutator of $x$ and $y$ is given by
\[ [x, y] = ((0 \ast x) \ast y) \ast ((0 \ast y) \ast x). \]

The subalgebra of \( X \) generated by \( \{ [x, y] : x, y \in X \} \) is called the \textit{derived B-algebra}, denoted by \( D(X) \).

**Example 3.** Let \((X; \ast, 0)\) be the B-algebra in Example 2. We now compute for \([x, y]\) for all \( x, y \in X \). These computations are used in the succeeding examples.

\[
\begin{array}{cccccccc}
0,0 & 1,1 & 2,2 & 3,3 & 4,4 & 5,5 \\
0,1 & 1,0 & 2,0 & 3,0 & 4,0 & 5,0 \\
0,2 & 1,2 & 2,1 & 3,1 & 4,1 & 5,1 \\
0,3 & 1,3 & 2,3 & 3,2 & 4,2 & 5,2 \\
0,4 & 1,4 & 2,4 & 3,4 & 4,3 & 5,3 \\
0,5 & 1,5 & 2,5 & 3,5 & 4,5 & 5,4 \\
\end{array}
\]

A map \( \varphi : X \to Y \) is called a \textit{B-homomorphism} \([20]\) if \( \varphi(x \ast y) = \varphi(x) \ast \varphi(y) \) for any \( x, y \in X \).

**Lemma 1.** \([24]\) Let \( \varphi : X \to Y \) be a B-homomorphism and let \( x, y \in X \). Then

i. \( [x, y] = 0 \) if and only if \( x \ast (0 \ast y) = y \ast (0 \ast x) \),

ii. \( \varphi([x, y]) = [\varphi(x), \varphi(y)] \),

iii. if \( \varphi \) is onto, then \( \varphi(D(X)) = D(\varphi(X)) \).

**Lemma 2.** Let \( x, y \in X \). Then

i. \( [x, x] = [x, 0] = [0, x] = 0 \),

ii. \( 0 \ast [x, y] = [y, x] \).

**Proof.** Clearly, (i) follows from Lemma 1(i) and (P1); (ii) follows from (P2). \( \square \)

Let \( x, w \in X \). We define \( x^w \) to be the element \((0 \ast w) \ast ((0 \ast w) \ast x)\). For instance, let \( X \) be the B-algebra in Example 1. Below are some sample computations to illustrate \( x^w \):

\[
\begin{align*}
2^3 &= (0 \ast 3) \ast ((0 \ast 3) \ast 2) = 3 \ast (3 \ast 2) = 3 \ast 1 = 2 \\
3^2 &= (0 \ast 2) \ast ((0 \ast 2) \ast 3) = 2 \ast (2 \ast 3) = 2 \ast 1 = 3 \\
1^3 &= (0 \ast 3) \ast ((0 \ast 3) \ast 1) = 3 \ast (3 \ast 1) = 3 \ast 2 = 1 \\
3^1 &= (0 \ast 1) \ast ((0 \ast 1) \ast 3) = 1 \ast (1 \ast 3) = 1 \ast 2 = 3 \\
1^2 &= (0 \ast 2) \ast ((0 \ast 2) \ast 1) = 2 \ast (2 \ast 1) = 2 \ast 3 = 1 \\
2^1 &= (0 \ast 1) \ast ((0 \ast 1) \ast 2) = 1 \ast (1 \ast 2) = 1 \ast 3 = 2
\end{align*}
\]

The following lemma presents the basic properties of \( x^w \).

**Lemma 3.** Let \( x, y, w \in X \). Then the following properties hold:

i. \( 0 \ast x^w = (0 \ast x)^w \),

ii. \( (x \ast y)^w = 0 \ast (y \ast x)^w \),

iii. \( (0 \ast x)^x = 0 \ast x \),

iv. \( x^x = x \),

v. \( x^{0 \ast x} = x \),

vi. \( x \ast y^x = (0 \ast y) \ast (0 \ast x) \),

vii. \( x^y = x \ast [y, x] \),

viii. \( x^{0 \ast y} = y \ast (y \ast x) \),

ix. \([x^y, 0 \ast y] = [y, x] \).

**Proof.** Let \( x, y, w \in X \).

i. By (P2) and (III), we have
\[
0 \ast x^w = 0 \ast ((0 \ast w) \ast ((0 \ast w) \ast x))
= ((0 \ast w) \ast x) \ast (0 \ast w)
= (0 \ast w) \ast ((0 \ast w) \ast (0 \ast x))
= (0 \ast x)^w.
\]

ii. By (i), we have \((x \ast y)^w = (0 \ast (y \ast x))^w = 0 \ast (y \ast x)^w\).

iii. By (I) and (II), we have
\[
(0 \ast x)^x = (0 \ast x) \ast ((0 \ast x) \ast (0 \ast x))
= (0 \ast x) \ast 0
= 0 \ast x.
\]

iv. By P3, (I), and P1, we have
\[
x^x = (0 \ast x) \ast ((0 \ast x) \ast x)
= ((0 \ast x) \ast (0 \ast x)) \ast (0 \ast x)
= 0 \ast (0 \ast x)
= x.
\]

v. By P1, (I), and (II), we have
\[
x^{0 \ast x} = (0 \ast (0 \ast x)) \ast ((0 \ast (0 \ast x)) \ast x)
= x \ast (x \ast x)
= x \ast 0
= x.
\]
vi. By P3, P2, (III), and (I), we have
\[ x * y = x * ((0 * x) * ((0 * x) * y)) = (x * (0 * ((0 * x) * y))) * (0 * x) = (x * (y * (0 * x))) * (0 * x) = ((x * x) * b) * (0 * x) = (0 * y) * (0 * x). \]

vii. By P3, P2, (III), and (I), we have
\[ x * [y, x] = x * (((0 * y) * x) * ((0 * x) * y)) = (x * (0 * ((0 * x) * y))) * ((0 * y) * x) = (x * (y * (0 * x))) * ((0 * y) * x) = ((x * x) * y) * ((0 * y) * x) = (0 * y) * ((0 * y) * x) = x^y. \]

viii. This follows from P1.
ix. By P1, (vi), P4, (vii), P2, and (II), we get
\[ [x^y, 0 * y] = ((0 * x^y) * (0 * y)) * ((0 * (0 * y)) * x^y) = ((0 * x^y) * (0 * y)) * (y * x^y) = ((0 * x^y) * (0 * y)) * ((0 * x) * (0 * y)) = (0 * x^y) * (0 * x) = (0 * (x * [y, x])) * (0 * x) = ([y, x] * x) * (0 * x) = [y, x] * 0 = [y, x]. \]

The following lemma is used to prove the succeeding theorems.

**Lemma 4.** Let \( a, b, c \in X \). Then the following properties hold:

i. \((a \ast b) \ast a) \ast (a \ast (a \ast c)) = a \ast (a \ast ((0 \ast b) \ast c))\),

ii. \((a \ast (a \ast b)) \ast (a \ast (a \ast c)) = a \ast (a \ast (b \ast c))\),

iii. \[((a \ast b) \ast ((0 \ast b) \ast (0 \ast a))) \ast c \ast (c \ast (c \ast b)) = (a \ast b) \ast (c \ast (0 \ast a))\).

**Proof.**

i. By (III), P2, P4, and P3, we have
\[ ((a \ast b) \ast a) \ast (a \ast (a \ast c)) = (a \ast b) \ast [(a \ast (a \ast c)) \ast (0 \ast a)] \]
Similarly for (2), by Lemma 4(i) we have
\[ ((0 \ast (0 \ast y)) \ast (0 \ast y)) \ast (0 \ast y) = (0 \ast (0 \ast y) \ast (0 \ast y)) \ast ((0 \ast y) \ast (0 \ast y)) \]
\[ = (0 \ast (0 \ast y) \ast (0 \ast y)). \]

Similarly for (2), by Lemma 4(ii) we have
\[ (((0 \ast w) \ast y) \ast (0 \ast w)) \ast x^w = (((0 \ast w) \ast y) \ast (0 \ast w)) \ast ((0 \ast w) \ast (0 \ast w) \ast x^w) \]
Applying Lemma 4(ii), thus,

\[ [x^w, y^w] = \left[ \left( \left( (0 \ast w) \ast x \right) \ast (0 \ast w) \right) \ast y^w \right] \ast \left[ \left( \left( (0 \ast w) \ast y \right) \ast (0 \ast w) \right) \ast x^w \right] \]

Therefore,

\[ [0 \ast w] \ast \left( \left( (0 \ast w) \ast (0 \ast y) \right) \ast x \right) \]

For simplicity, we write

\[ (0 \ast w) \ast \left( (0 \ast y) \ast (0 \ast w) \right) \ast \left( \left( (0 \ast w) \ast (0 \ast y) \right) \ast x \right) \]

Applying Lemma 4(ii) [with \( a = 0 \ast w, b = (0 \ast x) \ast y, c = (0 \ast y) \ast x \)], we have

\[ [x^w, y^w] = \left[ (0 \ast w) \ast \left( ((0 \ast w) \ast (0 \ast x) \ast y) \right) \ast (0 \ast w) \ast (0 \ast y) \ast x \right] \]

\[ = (0 \ast w) \ast \left( (0 \ast w) \ast ((0 \ast x) \ast y) \ast (0 \ast y) \ast x \right) \]

\[ = (0 \ast w) \ast \left( (0 \ast w) \ast (0 \ast x) \ast y \right) \]

\[ = [x, y]^w. \]

\[ \square \]

**Theorem 2.** Let \( w, x, y, z \in X \). Then \( [x \ast (0 \ast y), z] = [x, z]^y \ast [z, y] \).

**Proof.** By (III) and P2, we have

\[ [x, z]^y \ast [z, y] = \left( (0 \ast y) \ast ((0 \ast y) \ast (0 \ast w) \ast (0 \ast x) \ast z) \right) \ast [z, y] \]

\[ = (0 \ast y) \ast \left( (0 \ast y) \ast \left( (0 \ast w) \ast \left( (0 \ast x) \ast z \right) \right) \right) \ast [z, y] \]

\[ = (0 \ast y) \ast \left( (0 \ast y) \ast \left( (0 \ast w) \ast \left( (0 \ast x) \ast z \right) \right) \right) \]

\[ = (0 \ast y) \ast \left( \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \ast \left( (0 \ast w) \ast (0 \ast y) \ast x \right) \right) \]

For simplicity, we write \( x' = 0 \ast x, y' = 0 \ast y, z' = 0 \ast z \). Thus, by (III), P1, and P2, we get

\[ [x, z]^y \ast [z, y] = y' \ast \left( \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \ast \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \right) \ast [z, y] \]

\[ = y' \ast \left( \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \ast \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \right) \]

\[ = y' \ast \left( \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \ast \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \right) \]

Applying Lemma 4(iii) [with \( a = x', b = z, c = y' \)], P2, and (III), we get

\[ [x, z]^y \ast [z, y] = y' \ast \left( \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \ast \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \right) \]

\[ = y' \ast \left( \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \ast \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \right) \]

\[ = y' \ast \left( \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \ast \left( (0 \ast y) \ast (0 \ast x) \ast z \right) \right) \]

\[ = [x, y]^w. \]

\[ \square \]
Corollary 1. Let \( x, y, z \in X \). Then \([x, y * (0 * z)] = [x, z] * [y, x]^z\).

Proof. By Lemma 2(ii), Theorem 2, and P2, we get
\[
[x, y * (0 * z)] = 0 * [y * (0 * z), x] = 0 * ([y, x]^z * [x, z]) = [x, z] * [y, x]^z.
\]

\[ \square \]

Theorem 3. Let \( x, y \in X \). Then \([x, 0 * y] = [y, x]^{0 * y}\).

Proof. By Theorem 1, Lemma 3(v, vi), P1, P4, Lemma 3(viii), P2, and P3, we get
\[
[y, x]^{0 * y} = [y^{0 * y}, x^{0 * y}] = [y, x^{0 * y}]
= ((0 * y) * x^{0 * y}) * ((0 * x^{0 * y}) * y)
= (0 * x) * (0 * (0 * y)) * ((0 * x^{0 * y}) * y)
= ((0 * x) * y) * ((0 * x^{0 * y}) * y)
= (0 * x) * (0 * x^{0 * y})
= (0 * x) * (0 * (y * (y * x)))
= (0 * x) * ((y * x) * y)
= ((0 * x) * (0 * y)) * (y * x)
= ((0 * x) * (0 * y)) * ((0 * (0 * y)) * x)
= [x, 0 * y].
\]

\[ \square \]

Corollary 2. Let \( x, y \in X \). Then \([0 * x, y] = [y, x]^{0 * x}\).

Proof. By Lemma 2(ii), Theorem 3, and Lemma 3(i), we get
\[
[0 * x, y] = 0 * [y, 0 * x] = 0 * [y, 0 * x]^{0 * x} = 0 * [y, x]^{0 * x} = (0 * [x, y])^{0 * x} = [y, x]^{0 * x}.
\]

\[ \square \]
3. \(k\)th B-commutators

We recall first the concept of solvable B-algebras [8]. Let

\[ X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n = \{0\} \]

be a series of subalgebras of \(X\). The series is called a subnormal B-series if each \(H_i\) is normal in \(H_{i-1}\). The series is called a normal B-series if each \(H_i\) is normal in \(X\). Since \(\{0\}\) is normal in \(X\), every B-algebra has a normal B-series. If \(X\) has a subnormal B-series \(X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}\) such that \(H_i/H_{i+1}\) is commutative, \(i = 0, 1, \ldots, n - 1\), then we say that \(X\) is solvable. Such a subnormal B-series is called a solvable B-series for \(X\).

For simplicity, we write the derived B-algebra \(D(X)\) as \(X'\).

**Definition 1.** Set \(X^{(1)} = X'\) and define inductively \(X^{(k+1)} = X'(X'^{(k)})\), the B-commutator subalgebra of \(X^{(k)}\), \(k > 0\). For any positive integer \(k\), \(X^{(k)}\) is called the \(k\)th B-commutator subalgebra of \(X\).

By Lemma 1, a B-algebra \(X\) is commutative if and only if \(X' = \{0\}\).

**Example 4.** Let \((X; *, 0)\) be the noncommutative B-algebra in Example 2. Then from the computations in Example 3, we see that \(X' = \{0, 1, 2\}\) and \(X^{(2)} = \{0, 1, 2\}' = \{0\}\). Thus, \(X^{(k)} = \{0\}\) for all \(k \geq 2\).

The following theorem characterizes solvable B-algebra.

**Theorem 4.** \(X\) is solvable if and only if there is positive integer \(m\) such that \(X^{(m)} = \{0\}\).

**Proof.** Suppose that \(X\) is solvable. Then \(X\) has a solvable series, say,

\[ X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}. \]

Since \(H_{i+1}\) is normal in \(H_i\) and \(H_i/H_{i+1}\) is commutative, \(H_i' \subseteq H_{i+1}\) by [24, Theorem 4.14]. Hence,

\[ H_1 \supseteq H_0' = X^{(1)}, H_2 \supseteq H_1' \supseteq X^{(2)}, \ldots, 0 = H_n \supseteq H_{n-1}' \supseteq X^{(n)}. \]

Thus, \(X^{(n)} = \{0\}\).

Conversely, suppose that \(X^{(m)} = \{0\}\). The series \(X \supseteq X^{(1)} \supseteq \cdots \supseteq X^{(m-1)} = \{0\}\) is a solvable B-series. Thus, \(X\) is solvable. \(\square\)

**Proposition 1.** Let \(H \neq \{0\}\) be a subalgebra of a solvable B-algebra \(X\). Then \(H' \neq H\).

**Proof.** Suppose \(H' = H\). Then \(H^{(2)} = (H')' = H' = H \neq \{0\}\). By induction, \(H^{(n)} = H \neq \{0\}\) for any positive integer \(n\). By [8, Theorem 12], \(H\) is solvable. Thus, by Theorem 4, there exists a positive integer \(n\) such that \(H^{(n)} = \{0\}\), a contradiction. Hence, \(H' \neq H\). \(\square\)
Theorem 5. A finite B-algebra $X$ is solvable if and only if $H' \neq H$ for any subalgebra $H \neq \{0\}$ of $X$.

Proof. Let $X$ be a finite B-algebra. Suppose that $X$ is solvable. By Proposition 1, $H' \neq H$ for any subalgebra $H \neq \{0\}$ of $X$. Conversely, suppose that $H' \neq H$ for any subalgebra $H \neq \{0\}$ of $X$. Then $X \neq X'$. Thus, $X' \subset X$. If $X^{(n)} \neq \{0\}$, then $X^{(n)} \neq X^{(n+1)}$, that is $X^{(n+1)} \subset X^{(n)}$. Hence, we have the following strictly descending series of subalgebras:

$$X \supset X' \supset \cdots \supset X^{(n)} \supset X^{(n+1)} \supset \cdots$$

Since $X$ is finite and $H' \neq H$ for any subalgebra $H \neq \{0\}$ of $X$, there exists a positive integer $n$ such that $X^{(n)} = \{0\}$. Hence, $X$ is solvable.

Example 5. Let $(X; *, 0)$ be the noncommutative B-algebra in Example 2. The nontrivial subalgebras of $X$ are the following: $H_1 = \{0, 3\}$, $H_2 = \{0, 4\}$, $H_3 = \{0, 5\}$, $H_4 = \{0, 1, 2\}$. Clearly, from the computations in Example 3, we get $H'_1 = \{0\} \neq H_1$, $H'_2 = \{0\} \neq H_2$, $H'_3 = \{0\} \neq H_3$, and $H'_4 = \{0\} \neq H_4$. In Example 4, $X' = \{0, 1, 2\} \neq X$. Hence, $H' \neq H$ for any subalgebra $H \neq \{0\}$ of $X$. Therefore, by Theorem 5, $X$ is solvable, which confirms the result in [8, Example 11].

4. Conclusion

We established some basic properties of B-commutators of B-algebras. These properties are used in characterizing solvable B-algebras via B-commutators. As a result, we showed that a B-algebra $X$ is solvable if and only if there is positive integer $m$ such that the $m$th B-commutator subalgebra $X^{(m)}$ is equal to $\{0\}$.

Acknowledgements

The author would like to thank the referees for the comments and suggestions which were incorporated into this revised version.

References


REFERENCES


