Closeness Centrality of Vertices in Graphs Under Some Operations

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Abstract. In this paper, we revisit the concept of (normalized) closeness centrality of a vertex in a graph and investigate it in some graphs under some operations. Specifically, we derive formulas to compute the closeness centrality of vertices in the shadow graph, complementary prism, edge corona, and disjunction of graphs.

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1. Introduction

According to a study in [7], centrality is one of the most studied subjects in the analysis of social networks. As mentioned in [4], the concept was developed by social scientists some decades ago with the aim of quantifying an intuitive perception that some nodes or linkages are regarded central according to some criteria in many particular networks or graphs. For example, in a social network which is often represented as a graph, where each individual is represented as a vertex, the relationship between pairs of individuals are connected by edges, and the weights on the edges indicate the strength of the relationships, centrality gives a way of determining how central an individual is located in this network (see [6]). Within graph theory and network analysis, some of the common measurements of centrality pointed out in [8] are degree centrality, closeness centrality, eigen vector centrality, and betweenness centrality. One may also refer to [1], [2], and [5] for some studies in measure of centrality.

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Closeness centrality measures how close a vertex is to all other vertices in a graph. The closeness centrality of a vertex in a graph is the inverse of the average geodesic distance from the vertex to any other vertex in the graph. The greater value of closeness centrality of a vertex would mean a better position of a vertex in spreading information to other vertices (see [6]).

A study on closeness centrality can be found in [4] where the authors derived the closeness centrality of vertices of some families of graphs such as paths, cycles, fans, wheels, complete bipartite graphs, and complete split graphs. In a more recent study, Eballe et al. in [3] presented the closeness centrality of the vertices in the corona, Cartesian product and lexicographic product of graphs.

In this paper, we derive formulas of the closeness centrality of a vertex in the shadow graph, edge corona of graphs, complementary prism, and disjunction of graphs.

2. Terminology

Definition 1. Let $G$ be a connected graph and let $u, v \in V(G)$. The distance between $u$ and $v$, denoted by $d_G(u, v)$, is the length of a shortest path (called $u$-$v$ geodesic) connecting $u$ and $v$. Vertices $u$ and $v$ are adjacent or neighbors (i.e. $uv \in E(G)$) if and only if $d_G(u, v) = 1$. The open neighborhood of $v$ is the set $N_G(v) = \{w \in V(G) : d_G(v, w) = 1\}$ and its closed neighborhood is the set $N_G[v] = N_G(v) \cup \{v\}$.

Definition 2. Let $G$ be a connected graph. If $v \in V(G)$ and $e \in E(G)$, then the distance $d_G(v, e)$ between $v$ and $e$, is given by $d_G(v, e) = \min_{x \in V(G)} \{d_G(v, x), d_G(v, x)\}$.

Definition 3. Let $G = (V(G), E(G))$ be a nontrivial connected graph of order $m$. If $u \in V(G)$, then the closeness centrality of vertex $u$ is given by

$$C_G(u) = \frac{m - 1}{T_G(u)},$$

where $T_G(u) = \sum_{x \in V(G)} d_G(u, x)$.

Definition 4. The shadow graph $S(G)$ of $G$ is the graph obtained by taking two copies of $G$, say $G_1$ and $G_2$, and then joining each vertex $v \in V(G_1)$ to the neighbors of $v' \in V(G_2)$, where $v'$ is the vertex in $V(G_2)$ corresponding to $v$, i.e., $v$ and $v'$ represent the same vertex in $G$. 
Definition 5. Let $G$ and $H$ be two graphs. The edge corona $G \circ H$ of $G$ and $H$ is the graph obtained by taking one copy of $G$ and $|E(G)|$ copies of $H$, and then joining two end-vertices of the $i$-th edge of $G$ to every vertex in the $i$-th copy of $H$. For every edge $e = uv$ of $G$, denote by $H^e = H^{uv}$ the copy of $H$ where vertices are joined to the vertices $u$ and $v$.

Definition 6. The complementary prism of graph $G$, denoted by $G\overline{G}$, is the graph obtained from the disjoint union of $G$ and $\overline{G}$ by adding the edges $vv'$, where $v \in V(G)$ and $v'$ is the vertex of $\overline{G}$ corresponding to vertex $v$. 

**Definition 7.** The *disjunction* of graphs $G$ and $H$, denoted by $G ∨ H$, is the graph with $V(G ∨ H) = V(G) \times V(H)$ and $(x, p)(y, q) ∈ E(G ∨ H)$ if and only if $xy ∈ E(G)$ or $pq ∈ E(H)$.

### 3. Main Results

In what follows, $G_1$ and $G_2$ are the copies of $G$ in the shadow graph $S(G)$.

**Remark 1.** Let $G$ be a graph and let $v, w ∈ V(G_1)$. Then

$$d_{S(G)}(v, w) = d_{G_1}(v, w) = d_{G_2}(v', w') = d_{S(G)}(v', w').$$

**Lemma 1.** Let $G$ be a nontrivial connected graph. For each $p ∈ V(G_1)$ and $v' ∈ V(G_2)$,

$$d_{S(G)}(p, v') = d_{S(G)}(p', v) = \begin{cases} 2 & \text{if } p = v \\ d_{G_1}(p, v) & \text{if } p ≠ v. \end{cases}$$

**Proof.** If $p = v$, then $p' = v'$ and $pp' ∉ E(S(G))$. Let $w ∈ N_{G_1}(p)$. Then $wp' ∈ E(S(G))$. Hence, $[p, w, v']$ is a $p - v'$ geodesic in $S(G)$. This implies that $d_{S(G)}(p, v') = 2$.

Next, suppose that $p ≠ v$, that is, $p' ≠ v'$. Let $[p_1, p_2, ..., p_k]$ be a $p - v$ geodesic in $G_1$. Then $[p_1', p_2', ..., p_k']$ is a $p' - v'$ geodesic in $G_2$. Also, by definition of $S(G)$, $[p_1, p_2, ..., p_k]$ is a $p - v$ geodesic in $S(G)$. Hence, $d_{S(G)}(p, v) = d_{G_1}(p, v)$. Since $d_{G_1}(p, v) = d_{G_2}(p', v')$, it follows that $d_{S(G)}(p, v') = d_{S(G)}(p', v)$. \hfill $\square$

Consider the shadow graph $S(P_3)$ in Figure 4.

![Figure 4: The shadow graph $S(P_3)$](image)

Clearly, $d_{S(G)}(p, p') = 2$, and $d_{S(G)}(p, v) = d_{S(G)}(p', v') = 3$.

**Theorem 1.** Let $G$ be a non-trivial connected graph. For each $p ∈ V(G_1)$,

$$τ_{S(G)}(p) = 2(τ_{G_1}(p)) + 2 \text{ and } τ_{S(G)}(p') = 2(τ_{G_2}(p')) + 2.$$
Proof. Let \( p \in V(G_1) \). Then

\[
\tau_{S(G)}(p) = \sum_{q \in V(S(G))} d_{S(G)}(p, q)
\]

\[
= \sum_{q \in V(G_1)} d_{S(G)}(p, q) + \sum_{v' \in V(G_2)} d_{S(G)}(p, v')
\]

\[
= \sum_{q \in V(G_1)} d_{G_1}(p, q) + \sum_{v' \in V(G_2) - \{p\}} d_{S(G)}(p, v') + d_{S(G)}(p, p')
\]

\[
= \tau_{G_1}(p) + \sum_{v \in V(G_1) - \{p\}} d_{G_1}(p, v) + 2
\]

\[
= 2\tau_{G_1}(p) + 2.
\]

Similarly, \( \tau_{S(G)}(p') = 2\tau_{G_2}(p') + 2 \). \( \square \)

The next result follows from Theorem 1 and Definition 3.

**Corollary 1.** Let \( G \) be a non-trivial connected graph of order \( m \) and let \( p \in V(G_1) \). Then

\[
C_{S(G)}(p) = \frac{2m - 1}{2(\tau_{G_1}(p)) + 2}
\]

and

\[
C_{S(G)}(p') = \frac{2m - 1}{2(\tau_{G_2}(p')) + 2}.
\]

**Example 1.** Consider the shadow graph \( S(P_5) \) in Figure 5, where \( G = P_5 \).

![Figure 5: The shadow graph \( S(P_5) \) and \( p' \in V(G_2) \)](image)

Then \( m = 5 \) and \( \tau_{G_2}(p') = 7 \). Using Corollary 1, the closeness centrality of \( p' \) is

\[
C_{S(G)}(p') = \frac{2m - 1}{2(\tau_{G_2}(p')) + 2}
\]

\[
= \frac{2(5) - 1}{2(7) + 2}
\]

\[
= \frac{10 - 1}{14 + 2}
\]
\[
\sum_{v \in V(G)} (d_G(v, x)) + \sum_{e \in E_v} \sum_{a \in V(H^e)} d_{G,H}(v, a) + \sum_{e \in E(G) \setminus E_v} \sum_{a \in V(H^e)} d_{G,H}(v, a)
\]

For a non-trivial connected graph \(G\) and \(v \in V(G)\), the set \(E_v\) is given by

\[
E_v = \{ xv \in E(G) : x \in V(G) \setminus \{v\} \}.
\]

**Theorem 2.** Let \(G\) and \(H\) be non-trivial connected graphs. If \(v \in V(G)\), then

\[
\tau_{G\circ H}(v) = \tau_G(v) + |V(H)||E(G)| + |V(H)| \sum_{e \in E(G) \setminus E_v} d_G(v, e).
\]

**Proof.** Let \(v \in V(G)\). Then \(|E_v| = |N_G(v)|\) and \(d_{G,H}(v, a) = d_G(v, e) + 1\) for every \(a \in V(H^e)\) where \(e \in E(G) \setminus E_v\). Hence,

\[
\tau_{G\circ H}(v) = \sum_{x \in V(G)} (d_G(v, x)) + \sum_{e \in E_v} \sum_{a \in V(H^e)} d_{G,H}(v, a) + \sum_{e \in E(G) \setminus E_v} \sum_{a \in V(H^e)} d_{G,H}(v, a)
\]

\[
= \tau_G(v) + |E_v||V(H)| + \sum_{e \in E(G) \setminus E_v} [(d_G(v, e) + 1)|V(H)|]
\]

\[
= \tau_G(v) + |E_v||V(H)| + |V(H)| \sum_{e \in E(G) \setminus E_v} d_G(v, e) + |V(H)||E(G)| - |E_v||V(H)|
\]

\[
= \tau_G(v) + |V(H)||E(G)| + |V(H)| \sum_{e \in E(G) \setminus E_v} d_G(v, e). \quad \Box
\]

The next result is immediate from Theorem 2.

**Corollary 2.** Let \(G\) and \(H\) be connected non-trivial graphs with \(m = |V(G)|, r = |E(G)|,\) and \(n = |V(H)|\). If \(v \in V(G)\), then

\[
C_{G\circ H}(v) = \frac{m + rn - 1}{\tau_G(v) + rn + n \sum_{e \in E(G) \setminus E_v} d_G(v, e)}.
\]

**Example 2.** Consider the edge corona of \(G = C_4, H = P_5,\) and \(v \in V(G)\) in Figure 6. Then \(m = |V(G)| = |E(G)| = r = 4, n = |V(H)| = 5, \tau_G(v) = 4, |E_v| = |E(G) \setminus E_v| = 2\) and \(\sum_{e \in E(G) \setminus E_v} d_G(v, e) = 2\). Using Corollary 2, the closeness centrality of \(v \in V(G)\) is

\[
C_{G\circ H} = \frac{m + qn - 1}{\tau_G(v) + qn + n \sum_{e \in E(G) \setminus E_v} d_G(v, e)}
\]

\[
= \frac{24 - 1}{4 + (5)(4) + (5)(2)}
\]

\[
= \frac{9}{16}.
\]
Theorem 3. Let $e \in E(G)$, the sets $V_{u \leq v}, V_{v < u}, E_{u \leq v}, E_{v < u}$ are given by:

- $V_{u \leq v} = \{ z \in V(G) \setminus \{ u \} : d_G(z, u) \leq d_G(z, v) \}$
- $V_{v < u} = \{ y \in V(G) \setminus \{ v \} : d_G(v, y) < d_G(u, y) \}$
- $E_{u \leq v} = \{ e' \in E(G) \setminus \{ e \} : d_G(u, e') \leq d_G(v, e') \}$
- $E_{v < u} = \{ e' \in E(G) : d_G(v, e') < d_G(u, e') \}$

Note that $|E_{u \leq v}| + |E_{v < u}| = |E(G)| - 1$ and $|V_{u \leq v}| + |V_{v < u}| = |V(G)| - 2$.

Theorem 3. Let $G$ and $H$ be connected non-trivial graphs and let $e = uv \in E(G)$. If $p \in V(H^e)$, then

$$
\tau_{G \bowtie H}(p) = |V(G)| - d_{eg_H}(p) + 2|V(H)||E(G)| - 2 + \sum_{x \in V_{u \leq v}} d_G(u, x) + \sum_{x \in V_{v < u}} d_G(v, x) + |V(H)| \sum_{e' \in E_{u \leq v}} d_G(u, e') + |V(H)| \sum_{e' \in E_{v < u}} d_G(v, e').
$$

Proof. Let $e = uv \in E(G)$ and let $p \in V(H^e)$. Then

$$
\tau_{G \bowtie H}(p) = \sum_{q \in V(G \bowtie H)} d_{G \bowtie H}(p, q)
= \sum_{e' \in E(G)} \sum_{q \in V(H^{e'})} d_{G \bowtie H}(p, q) + \sum_{q \in V(G)} d_{G \bowtie H}(p, q)
= \sum_{q \in V(H^e)} d_{G \bowtie H}(p, q) + \sum_{e' \in E(G)} \sum_{q \in V(H^{e'})} d_{G \bowtie H}(p, q) + \sum_{x \in V_{u \leq v}} (1 + d_G(u, x)) + \sum_{x \in V_{v < u}} (1 + d_G(v, x)) + d_{G \bowtie H}(p, u) + d_{G \bowtie H}(p, v)
$$
Hence,
\[
\tau_{G_0H}(p) = |N_H(p)| + 2|V(H)| - 2|N_H(p)| - 2 + 2 + 2|\mathcal{V}(H)| \geq 1 + 2|G_0H(p, q)| + \sum_{e' \in E_{u \leq v}} d_G(u, e') + |\mathcal{V}(H)| \sum_{e' \in E_{v < u}} d_G(v, e')
\]
\[
+ |\mathcal{V}(H)| \sum_{e' \in E_{u \leq v}} d_G(u, e') + |\mathcal{V}(H)| \sum_{e' \in E_{v < u}} d_G(v, e')
\]
\[
+ (|\mathcal{V}(H)| + |\mathcal{V}(H)|) \sum_{x \in V_u \leq v} d_G(u, x) + \sum_{x \in V_v < u} d_G(v, x)
\]
\[
+ |\mathcal{V}(H)| \sum_{e' \in E_{u \leq v}} d_G(u, e') + |\mathcal{V}(H)| \sum_{e' \in E_{v < u}} d_G(v, e')
\]
\[
= 2|V(H)| - |N_H(p)| + 2|V(H)|(|E_u \leq v| + |E_v < u|) + (|\mathcal{V}(H)| + |\mathcal{V}(H)|) \sum_{x \in V_u \leq v} d_G(u, x) + \sum_{x \in V_v < u} d_G(v, x)
\]
\[
+ |\mathcal{V}(H)| \sum_{e' \in E_{u \leq v}} d_G(u, e') + |\mathcal{V}(H)| \sum_{e' \in E_{v < u}} d_G(v, e')
\]
\[
= 2|V(H)| - |N_H(p)| + 2|V(H)|(|E(G)| - 1) + |V(G)| - 2
\]
\begin{align*}
&+ \sum_{x \in V_u \leq v} d_G(u, x) + \sum_{x \in V_v < u} d_G(v, x) + |V(H)| \sum_{e' \in E_u \leq v} d_G(u, e') \\
&+ |V(H)| \sum_{e' \in E_v < u} d_G(v, e') \\
&= |V(G)| - \deg_H(p) + 2|V(H)||E(G)| - 2 + \sum_{x \in V_u \leq v} d_G(u, x) + \sum_{x \in V_v < u} d_G(v, x) \\
&+ |V(H)| \sum_{e' \in E_u \leq v} d_G(u, e') + |V(H)| \sum_{e' \in E_v < u} d_G(v, e'). \quad \square
\end{align*}

The next result is a direct consequence of Theorem 3.

**Corollary 3.** Let $G$ and $H$ be connected non-trivial graphs of orders $m$ and $n$, respectively, and let $r = |E(G)|$ and $e = uv \in E(G)$. If $p \in V(H^e)$, then

$$C_{G \circ H}(p) = \frac{m + r n - 1}{\tau_{G \circ H}(p)},$$

where $\tau_{G \circ H}(p)$ is given in Theorem 3.

**Example 3.** Consider $p \in V(H^e)$, where $e = uv$, in Figure 7, where $G = P_4$ and $H = P_5$. Then $m = 4$, $r = 4$, $n = 5$, $\deg_H(p) = 1$, $\sum_{x \in V_u \leq v} d_G(u, x) = 1$, $\sum_{x \in V_v < u} d_G(v, x) = 1$, $\sum_{e' \in E_u \leq v} d_G(u, e') = 1$, and $\sum_{e' \in E_v < u} d_G(v, e') = 0$. By Corollary 3, the closeness centrality of $p \in H^e$ is $C_{G \circ H}(p) = \frac{23}{38}$.

![Figure 7: The edge corona graph of $C_4 \circ P_5$ with point $p$ in $H^e$](image)

**Lemma 2.** Let $G$ be a non-trivial connected graph and let $v, x \in V(G)$. Then

$$d_{G^G}(v, x) = \begin{cases} 
0 & \text{if } x = v \\
1 & \text{if } vx \in E(G) \\
2 & \text{if } d_G(v, x) = 2 \\
3 & \text{if } d_G(v, x) \geq 3 
\end{cases}$$
Proof. Clearly, \(d_{\overline{G}}(v, x) = d_G(v, x)\) if \(1 \leq d_G(v, x) \leq 2\). Suppose \(d_G(v, x) \geq 3\). Then \(d_{\overline{G}}(v, x) \geq 3\). Since \(\bar{v} \bar{p} \in E(\overline{G})\), it follows that \([v, \bar{v}, \bar{p}, x]\) is a \(v\)-\(x\) geodesic in \(\overline{G}\). Hence, \(d_{\overline{G}}(v, x) = 3\). \(\square\)

**Lemma 3.** Let \(G\) be a non-trivial connected graph and let \(v, x \in V(G)\). Then

\[
d_{\overline{G}}(v, x) = \begin{cases} 
1 & \text{if } \bar{v} = \bar{p} \\
2 & \text{otherwise}
\end{cases}
\]

Proof. If \(\bar{v} = \bar{p}\), then \(d_{\overline{G}}(v, x) = 1\) by definition of \(\overline{G}\). Suppose \(\bar{v} \neq \bar{p}\), i.e., \(x \neq v\). If \(xv \in E(G)\), then \(\bar{v}\bar{p} \notin E(\overline{G})\) and \([v, x, \bar{p}]\) is a \(v\)-\(x\) geodesic in \(\overline{G}\). If \(xv \notin E(G)\), then \(\bar{v}\bar{p} \in E(\overline{G})\). Hence, \([v, \bar{v}, \bar{p}, x]\) is a \(v\)-\(x\) geodesic in \(\overline{G}\). Thus \(d_{\overline{G}}(v, x) = 2\). \(\square\)

**Lemma 4.** Let \(G\) be a non-trivial connected graph and let \(v, x \in V(G)\). Then

\[
d_{\overline{G}}(\bar{v}, \bar{x}) = \begin{cases} 
0 & x = v \\
1 & \text{if } xv \notin E(G), x \neq v \\
2 & \text{if } xv \in E(G) \text{ and } (V(G) \setminus N_G[x]) \cap (V(G) \setminus N_G[v]) \neq \emptyset, x \neq v \\
3 & \text{if } xv \in E(G) \text{ and } (V(G) \setminus N_G[x]) \cap (V(G) \setminus N_G[v]) = \emptyset, x \neq v
\end{cases}
\]

Proof. Clearly, \(d_{\overline{G}}(\bar{v}, \bar{v}) = 0\) and \(d_{\overline{G}}(\bar{v}, \bar{x}) = 1\) if \(xv \notin E(G)\). Next, suppose that \(xv \in E(G)\). Then \(d_{\overline{G}}(\bar{v}, \bar{v}) \geq 2\). Suppose there exists \(z \in (V(G) \setminus N_G[x]) \cap (V(G) \setminus N_G[v])\). Then \(\bar{v}\bar{z}, \bar{z}\bar{v} \in E(\overline{G})\). Hence, \([\bar{v}, \bar{z}, \bar{p}, x]\) is a \(\bar{v}\)-\(x\) geodesic on \(\overline{G}\). Thus, \(d_{\overline{G}}(\bar{v}, \bar{p}) = 2\). Next, suppose that \((V(G) \setminus N_G[x]) \cap (V(G) \setminus N_G[v]) = \emptyset\). Suppose \([\bar{v}, p, \bar{x}]\) is a \(\bar{v}\)-\(x\) geodesic in \(\overline{G}\). Suppose first that \(p \in V(G)\). Then \(p = v\) by definition of \(\overline{G}\). Similarly, \(p = x\). This is a contradiction because \(x \neq v\). Hence, \(p \in V(\overline{G})\), say \(p = \bar{q}\). However, since \(\bar{v}\bar{q}, \bar{q}\bar{v} \in E(\overline{G})\), \(q \in (V(G) \setminus N_G[v]) \cap (V(G) \setminus N_G[x])\), a contradiction. Thus, \(d_{\overline{G}}(\bar{v}, \bar{x}) \geq 3\). Since \([\bar{v}, v, x, \bar{x}]\) is a \(\bar{v}\)-\(x\) geodesic in \(\overline{G}\), it follows that \(d_{\overline{G}}(\bar{v}, \bar{p}) = 3\). \(\square\)

In what follows, \(N_G^2(v) = \{w \in V(G) : d_G(v, w) = 2\}\).

**Theorem 4.** Let \(G\) and \(\overline{G}\) be non-trivial connected graphs. If \(v \in V(G)\), then

\[
\tau_{\overline{G}}(v) = 5|V(G)| - 2 \cdot \text{deg}_G(v) - |N_G^2(v)| - 4.
\]
This proves the assertion.\hfill\qed

**Corollary 4.** Let $G$ be a non-trivial connected graph of order $m$ such that $\overline{G}$ is connected and let $v \in V(G)$. Then

$$C_{\overline{G}}(v) = \frac{2m - 1}{5|V(G)| - 2 \cdot deg_G(v) - |N_G^2(v)| - 4}.$$

**Example 4.** Consider $G = P_4$, $G\overline{G} = P_4\overline{P}_4$, and $v \in V(G)$ in Figure 8. Then $|V(G)| = 4$, $m = 8$, $N_G(v) = \{x\}$, and $N_G^2(v) = \{y\}$. Using Corollary 4, the closeness centrality of $v$ is

$$C_{\overline{G}}(v) = \frac{2m - 1}{5|V(G)| - 2 \cdot deg_G(v) - |N_G^2(v)| - 4}$$

$$= \frac{8 - 1}{5(4) - 2(1) - 1 - 4}$$

$$= \frac{20 - 2 - 1 - 4}{13}$$

$$= \frac{7}{13}.$$

For a non-trivial connected graph $G$ and $v \in V(G)$, the sets $N_G^0(V)$ and $N_G^2(v)$ are given below:
Clearly, \( N_G(v) = N_G^0(v) \cup N_G^1(v) \).

**Theorem 5.** Let \( G \) and \( \overline{G} \) be non-trivial connected graphs and let \( v \in V(G) \). Then

\[
\tau_{G\overline{G}}(v) = 3|V(G)| + deg_G(v) + |N_G^1(v)| - 2.
\]

**Proof.**

\[
\begin{align*}
\tau_{G\overline{G}}(v) & = \sum_{x \in V(G)} d_{G\overline{G}}(v, x) + \sum_{x \in V(G) - \{v\}} d_{G\overline{G}}(v, x) \\
& = \sum_{x \in V(G) - \{v\}} d_{G\overline{G}}(v, x) + d_{G\overline{G}}(v, v) + \sum_{x \in V(G) - N_G(v)} d_{G\overline{G}}(v, x) \\
& + \sum_{x \in N_G^0(v)} d_{G\overline{G}}(v, x) + \sum_{x \in N_G^1(v)} d_{G\overline{G}}(v, x) \\
& = 2(|V(G)| - 1) + 1 + (|V(G)| - N_G(v)) + 2|N_G^0(v)| + 3|N_G^1(v)| \\
& = 2|V(G)| - 2 + 1 + |V(G)| - |N_G(v)| - 1 + 2(|N_G^0(v)| + |N_G^1(v)|) \\
& + |N_G^1(v)| \\
& = 3|V(G)| + |N_G(v)| + |N_G^1(v)| - 2 \\
& = 3|V(G)| + deg_G(v) + |N_G^1(v)| - 2. \quad \square
\end{align*}
\]

**Corollary 5.** Let \( G \) be a non-trivial connected graph of order \( m \) such that \( \overline{G} \) is connected and let \( v \in V(G) \). Then

\[
C_{G\overline{G}}(v) = \frac{2m - 1}{3|V(G)| + deg_G(v) + |N_G^1(v)| - 2}.
\]
Example 5. Let $G = P_4$ and consider $GG = P_4 P_4$ and $v \in V(G)$ in Figure 9. Then $N_G(v) = \{x\}$ and $N^1_G(v) = \emptyset$. Hence $|V(G)| = 4$, $m = 8$, $|N_G(v)| = deg_G(v) = 1$ and $|N^1_G(v)| = 0$. Using Corollary 5, we have

$$C_{GG}(\overline{v}) = \frac{2m - 1}{3|V(G)| + deg_G(v) + |N^1_G(v)| - 2}$$

$$= \frac{8 - 1}{3(4) + 1 + 0 - 2}$$

$$= \frac{7}{11}.$$

Lemma 5. Let $G$ and $H$ be a non-trivial connected graphs and let $(x, p), (y, q) \in V(G \vee H)$. Then

$$d_{G \vee H}((x, p), (y, q)) = \begin{cases} 0 & \text{if } x = y \text{ and } p = q \\ 1 & \text{if } xy \in E(G) \text{ or } pq \in E(H) \\ 2 & \text{otherwise} \end{cases}$$

Proof. Clearly, $d_{G \vee H}((x, p), (y, q)) = 0$ if $x = y$ and $p = q$. Also, by definition, $d_{G \vee H}((x, p), (y, q)) = 1$ if $xy \in E(G)$ or $pq \in E(H)$. Suppose $x \neq y$, $p \neq q$, $xy \notin E(G)$, and $pq \notin E(H)$. Let $z \in N_G(y)$ and $r \in N_H(p)$. Then $(x, p)(z, r), (z, r)(y, q) \in E(G \vee H)$. This implies that $[(x, p), (z, r), (y, q)]$ is an $(x, p)$-$(y, q)$ geodesic in $G \vee H$. Thus, $d_{G \vee H}((x, p), (y, q)) = 2$. \hfill \Box$

Theorem 6. Let $G$ and $H$ be connected non-trivial graphs of orders $m$ and $n$, respectively, and let $(x, p) \in E(G \vee H)$. Then

$$\tau_{G \vee H}((x, p)) = 2mn + deg_G(x)deg_H(p) - n \cdot deg_G(x) - m \cdot deg_H(p) - 2.$$

Proof. By Definition 3 and Lemma 5,

$$\tau_{G \vee H}((x, p)) = \sum_{q \in V(H)} \sum_{y \in N_G(x)} d_{G \vee H}(x, p)(y, q) + \sum_{q \in N_H(p)} \sum_{y \in V(G) \setminus N_G(x)} d_{G \vee H}(x, p)(y, q).$$
\[ + \sum_{q \in V(H) \setminus N_H[p]} d_{G \lor H}(x, p)(x, q) + \sum_{q \in V(H) \setminus N_H(p) \setminus V(G) \setminus N_G[x]} \sum_{y \in V(H) \setminus N_H(p)} d_{G \lor H}(x, p)(y, q) = |V(H)||N_G(x)| + (|V(G)| - |N_G(x)||N_H(p)| + 2(|V(H)| - |N_H(p)| - 1) + 2(|V(H)| - |N_H(p)||V(G)| - |N_G(x)| - 1) = n|N_G(x)| + (m - |N_G(x)||N_H(p)| + 2(n - |N_H(p)| - 1) + 2(n - |N_H(p)|)(m - |N_G(x)| - 1) = n|N_G(x)| + m|N_H(p)| - |N_G(x)||N_H(p)| - 2|N_H(p)| - 2n - 2 + 2mn - 2n|N_G(x)| - 2m|N_H(p)| + 2|N_G(x)||N_H(p)| - 2n + 2|N_H(p)| = 2mn + |N_G(x)||N_H(p)| - n|N_G(x)| - m|N_H(p)| - 2 = 2mn + \text{deg}_G(x)\text{deg}_H(p) - n \cdot \text{deg}_G(x) - m \cdot \text{deg}_H(p) - 2. \]

**Corollary 6.** Let \( G \) and \( H \) be connected non-trivial graphs of orders \( m \) and \( n \), respectively, and let \((x, p) \in V(G \lor H)\). Then

\[ C_{G \lor H}((x, p)) = \frac{mn - 1}{2mn + \text{deg}_G(x)\text{deg}_H(p) - n \cdot \text{deg}_G(x) - m \cdot \text{deg}_H(p) - 2}. \]

**Example 6.** Let \( G = P_3 = [x, y, z] \), \( H = P_3 = [q, p, s] \), and \((x, p) \in V(G \lor H)\). By Corollary 6 with \( mn = 9 \), \( \text{deg}_G(x) = 1 \), and \( \text{deg}_H(p) = 2 \), we find that \( C_{G \lor H}((x, p)) = \frac{2}{3} \).

### 4. Conclusion

The study revisited the concept of closeness centrality of a vertex in a graph. Formulas that can be used to determine the closeness centrality of vertices in the shadow graph, edge corona of graphs, complementary prism, and disjunction of graphs were generated. The parameter can be studied further for other graphs under some other binary operations.

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### References


