**Abstract.** We came up with the concept $b^*$-open set which has stricter condition with respect to the notion $b$-open sets, introduced by Andrijevic [2] as a generalization of Levine’s [7] generalized closed sets. The condition imposes equality instead of inclusion. In this study, we gave some important properties of $b^*$-open sets with respect to an ideal, and $b^*$-compact spaces.

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1. Introduction


It was in the year 1970, when Levine [7] presented the concept of generalized closed sets, and anchoring on this notion, Andrijevic [2] presented yet another generalization of open sets called $b$-open sets. This study uses the notion of $b$-open sets to come up with a new concept called $b^*$-open sets.

The concept ideal topological spaces (or simply, ideal space) was first seen in [5]. Vaidyanathaswamy [19] investigated this concept in point set topology. Tripathy and Shravan [13, 14], Tripathy and Acharjee [17], Tripathy and Ray [18], Catalan et al. [4] among others, also made investigations in ideal topological spaces.

Several concepts in topology were generalized using this structure. One of which is the concept $b^*$-open sets. Consequently, using the notion of $b^*$-open sets, we introduced...
the concepts $b^*$-compact sets, compatible $b_j^*$-compact sets, countably $b_j^*$-compact sets, $b_j^*$-connected sets, in ideal generalized topological spaces.

Let $W$ be a non-empty set. An ideal $J$ on a set $W$ is a non-empty collection of subsets of $W$ which satisfies:

1. $B \in J$ and $D \subseteq B$ implies $D \in J$.
2. $B \in J$ and $D \in J$ implies $B \cup D \in J$.

Let $W$ be a topological space and $B$ be a subset of $W$. We say that $B$ is $b^*$-open set if $B = \text{cl}(\text{int}(B)) \cup \text{int}(\text{cl}(B))$. For example, consider $W = \{a, b, c\}$ and the topology $\mathcal{J} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, W\}$ on $W$. Then the $b^*$-open subsets are $\emptyset$, $\{a, b\}$, $\{c\}$ and $W$.

Let $W$ be a topological space and $B$ be a subset of $W$. The set $B$ is called $b^*$-open relative to an ideal $J$ (or $b_j^*$-open), if there is an open set $P$ with $P \subseteq \text{Int}(B)$, and a closed set $S$ with $\text{Cl}(B) \subseteq S$ such that

1. $(\text{Int}(S) \cup \text{Cl}(\text{Int}(B))) \setminus B \in J$, and
2. $B \setminus (\text{Int}(\text{Cl}(B)) \cup \text{Cl}(P)) \in J$.

In addition, we say that a set $B$ is a $b_j^*$-close set if $B^C$ is $b_j^*$-open.

Consider the ideal space $\langle \{q, r, s\}, \{\emptyset, \{q\}, \{r\}, \{q, r\}, \{q, r, s\}, \{\emptyset, \{r\}\} \rangle$. Then $B = \{r, s\}$ is a $b^*$-open with respect to the ideal $J = \{\emptyset, \{r\}\}$. To see this, we let $P$ be the open set $\{r\}$ and $S$ be the closed set $\{r, s\}$. Then $\text{Int}(S) \cup \text{cl}(\text{int}(\{r, s\})) \setminus \{r, s\} = \text{Int}(\{r, s\}) \cup \text{cl}(\{r\}) \setminus \{r, s\} = \{r\} \cup \{r, s\} \setminus \{r, s\} = \emptyset \in J$. Also, $\text{Int}(\text{Cl}(\{r, s\}) \cup \text{Cl}(P)) \setminus \{r, s\} = \text{Int}(\{r, s\}) \cup \text{Cl}(\{r\}) \setminus \{r, s\} = \{r\} \cup \{r, s\} \setminus \{r, s\} = \emptyset \in J$. This shows that $B = \{r, s\}$ is a $b_j^*$-open.

Let $W$ be a topological space and $B$ be a subset of $W$. The set $B$ is called nearly $b^*$-open relative to an ideal $J$ (or nearly $b_j^*$-open) if there is an open set $P$ with $P \subseteq \text{Int}(B)$, and a closed set $S$ with $\text{Cl}(B) \subseteq S$ such that

1. $(\text{Int}(S) \cup \text{Cl}(\text{Int}(B))) \setminus \text{Cl}(B) \in J$, and
2. $B \setminus (\text{Int}(\text{Cl}(B)) \cup \text{Cl}(P)) \in J$.

Consider the ideal topological space $\langle \{1, 2, 3\}, \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{\emptyset, \{2\}\} \rangle$. Then $B = \{2, 3\}$ is a nearly $b^*$-open with respect to the ideal $J$ (or nearly $b_j^*$-open). To see this, we let $P$ be the open set $\{2\}$ and $S$ be the closed set $\{2, 3\}$. Then $\text{Int}(S) \cup \text{cl}(\text{int}(\{2, 3\})) \setminus \text{cl}(\{2, 3\}) = \text{Int}(\{2, 3\}) \cup \text{cl}(\{2\}) \setminus \text{cl}(\{2, 3\}) = \{2, 3\} \setminus \{2, 3\} = \emptyset \in J$. Also, $\text{Int}(\text{Cl}(\{2, 3\}) \cup \text{Cl}(P)) \setminus \{2, 3\} = \text{Int}(\{2, 3\}) \cup \text{Cl}(\{2\}) \setminus \{2, 3\} = \{2\} \cup \{2, 3\} \setminus \{2, 3\} = \emptyset \in J$. This shows that $B = \{2, 3\}$ is a nearly $b_j^*$-open.

The set $B$ is said to be $b^*$-compact if every cover of $B$ by $b^*$-open sets, containing $W$, has a smaller finite sub-cover. The space $W$ is said to be a $b^*$-compact space if $W$ is $b^*$-compact set. Consider the topological space $(W = \{a, b, c\}, \{\emptyset, \{a\}, \{b, c\}, W\}, J = \{\emptyset, \{a\}\})$. Then $B = \{a\}$ is a $b^*$-compact set, while $D = \{a, b\}$ is not. To see this, we note that the
$b^*$-open sets of $W$ are $\emptyset$, $\{a\}$, $\{b, c\}$ and $W$. Observe that the covering of $B$ containing $W$ is $\{\{a\}, W\}$. Thus, $\{\{a\}\}$ is a smaller cover. Hence, $B = \{a\}$ is a $b^*$-compact set.

On the other hand, observe that the covering of $D$ containing $W$ are $\{\{a\}, \{b, c\}, W\}$ and $\{\{b, c\}, W\}$. Since $\{\{b, c\}, W\}$ has no smaller subcover, $D = \{a, b\}$ is not a $b^*$-compact set.

The set $B$ is called $b^*_J$-compact if every cover of $B$ by $b^*_J$-open sets which contains $W$, has a smaller finite sub-cover. The space $W$ is called $b^*_J$-compact space if it is $b^*_J$-compact set. Consider the ideal topological space $(W = \{x, y, z\}, \emptyset, \{\emptyset, \{x\}, \{y, x, y\}, W\}, \emptyset, \{\emptyset, \{y\}\})$. Then $B = \{y, z\}$ is a $b^*_J$-compact set where $J = \{\emptyset, \{y\}\}$. To see this, we note that the $b^*_J$-open sets of $W$ are $\emptyset$, $\{y, z\}$ and $W$. Hence, every cover $\{P_\psi : \psi \in \Psi\}$ of $B$ by $b^*_J$-open set must contain $\{y, z\}$ or $W$. Thus, each of the following is a covering of $B$: $\{\{y, z\}\}$; $\{\{y, z\}, W\}$; and $\{W\}$. Note that $\{\{y, z\}, W\}$ is a covering of $B$ which has a smaller subcover $\{\{y, z\}\}$. This shows that $B = \{y, z\}$ is a $b^*_J$-compact set.

Now, consider the ideal topological space $(W = \{l, m, n\}, \emptyset, \{\emptyset, \{l\}, \{m\}, \{l, m\}, W\}, \emptyset, \{\emptyset, \{m\}\})$. Then $B = \{l, m\}$ is a not $b^*_J$-compact set where $J = \{\emptyset, \{m\}\}$. To see this, we note again that the $b^*_J$-open sets of $W$ are $\emptyset$, $\{m, n\}$ and $W$. Hence, every cover $\{P_\psi : \psi \in \Psi\}$ of $B$ by $b^*_J$-open set must contain $W$. Thus, each of the following is a covering of $B$: $\{\{m, n\}, W\}$; and $\{W\}$. Note that $\{\{m, n\}, W\}$ has no smaller. This shows that $B = \{l, m\}$ is not a $b^*_J$-compact set.

The set $B$ is said to be compatible $b^*_J$-compact (or simply $cb^*_J$-compact) if any cover $\{P_\psi : \psi \in \Psi\}$ of $B$ by $b^*_J$-open sets containing $W$, $\Psi$ has a smaller finite subset $\Psi_0$ such that $B \setminus \bigcup \{U_\psi : \psi \in \Psi_0\} \in J$. The topological space $W$ is said to be a $cb^*_J$-compact space if it is $cb^*_J$-compact as a set. Consider the ideal topological space $(Z, \zeta, J) = ((h, i, j), \emptyset, \{h, \{i, j\}, Z\}, \emptyset, \{i\})$. Then $(h, i)$ is a compatible $b^*_J$-compact where $J = \{\emptyset, \{i\}\}$. To see this, we observe that the $b^*_J$-open sets of $Z$ are $\emptyset$, $\{h, i\}$ and $Z$. Hence, every cover $\{P_\psi : \psi \in \Psi\}$ of $Z$ by $b^*_J$-open set must contain $\{h, i\}$ or $Z$. Thus, $\{P_\psi : \psi \in \Psi\}$ is $\{\{h, \{i, j\}\} \text{ or } \\{\{h\}, Z\} \text{ or } \{Z, \{i, j\}, \{h\} \text{ or } \{Z, \{i, j\}\} \}$. In the first 3 cases, there is a smaller subset $\{\{h\}\}$ such that $\{h, i\}\{\{h\}\} = \{i\} \in J$, and for the last case, there exist a smaller subset $\{\{h, i\}\}$ such that $\{h, i\}\{\{h, i\}\} = \emptyset \in J$. This shows that $\{h, i\}$ is a compatible $b^*_J$-compact set. Next, consider the ideal topological space $(V = \{q, r, s\}, \emptyset, \{\emptyset, \{q\}, \{r, s\}, V\}, \emptyset, \{\emptyset, \{s\}\})$. Then $\{q, r\}$ is not compatible $b^*_J$-compact. To see this, we note that the $b^*_J$-open sets of $V$ are $\emptyset$, $\{q, \{r, s\}\}$ and $V$. Hence, every cover $\{P_\psi : \psi \in \Psi\}$ of $\{q, r\}$ by $b^*_J$-open set must contain $\{q\}$, $\{r, s\}$ or $V$. Thus, $\{P_\psi : \psi \in \Psi\}$ is $\{\{q\}, \{r, s\}\}$ or $\{\{q\}, V\}$ or $\{V, \{r, s\}, \{q\}\}$ or $\{V, \{r, s\}\}$. Consider the open cover $\{\{q\}, \{r, s\}\}$. Note that its smaller covers are $\{\{q\}\}$ and $\{\{r, s\}\}$. Observe that $\{q, r\}\{q\} = \{r\} \notin J$ and $\{q, r\}\{r, s\} = \{q\} \notin J$. This shows that $\{q, r\}$ is not a compatible $b^*_J$-compact set.

2. Results

We present some of the important properties of $b^*$-open sets and $b^*_J$-open sets. Lemma 1 is a characterization of $b^*$-open sets.
Lemma 1. Let \((Y, \varsigma, J)\) be an ideal space and \(B\) be a subset of \(Y\). Then \(B\) is an \(b^*\)-open set precisely when there is an open set \(P\) with \(P \subseteq \text{Int}(B)\) and there is a close set \(S\) with \(\text{Cl}(B) \subseteq S\) such that \(\text{Int}(S) \cup \text{Cl}(\text{Int}(B)) \subseteq B \subseteq \text{Int}(\text{Cl}(B)) \cup \text{Cl}(P)\).

Proof. Necessity. Let \(B\) be a \(b^*\)-open set. Then \(B = \text{Int}(\text{Cl}(B)) \cup \text{Cl}(\text{Int}(B))\). Take the open set \(P = \text{Int}(B)\) and the close set \(S = \text{Cl}(B)\). Note that \(\text{Int}(S) \cup \text{Cl}(\text{Int}(B)) \subseteq \text{Int}(\text{Cl}(B)) \cup \text{Cl}(\text{Int}(B)) = B\), and \(\text{Int}(\text{Cl}(B)) \cup \text{Cl}(P) \supseteq \text{Int}(\text{Cl}(B)) \cup \text{Cl}(\text{Int}(B)) = B\). Hence, \(\text{Int}(S) \cup \text{Cl}(\text{Int}(B)) \subseteq B \subseteq \text{Int}(\text{Cl}(B)) \cup \text{Cl}(P)\).

Sufficiency. Next, let \(P\) be an open set with \(P \subseteq \text{Int}(B)\) and let \(S\) be a closed set with \(\text{Cl}(B) \subseteq S\) such that \(\text{Int}(S) \cup \text{Cl}(\text{Int}(B)) \subseteq B \subseteq \text{Int}(\text{Cl}(B)) \cup \text{Cl}(P)\). Then \(B \supseteq \text{Int}(S) \cup \text{Cl}(\text{Int}(B)) \supseteq \text{Int}(\text{Cl}(B)) \cup \text{Cl}(\text{Int}(B))\), and \(B \supseteq \text{Int}(\text{Cl}(B)) \cup \text{Cl}(P) \subseteq \text{Int}(\text{Cl}(B)) \cup \text{Cl}(\text{Int}(B))\).

Therefore, \(B = \text{Int}(\text{Cl}(B)) \cup \text{Cl}(\text{Int}(B))\), that is \(B\) is a \(b^*\)-open set. \(\square\)

An open set is nearly \(b^*_J\)-open. The next lemma, Lemma 2, shows this idea.

Lemma 2. Let \((Y, \varsigma, J)\) be an ideal space. Then every open set is a \(b^*_J\)-open set.

Proof. Let \(B\) be an open set, and consider \(S = \emptyset = P\). Then \(S\) and \(P\) are both open and closed. Observed that \(\text{int}(\text{cl}(B)) \cup \text{cl}(P) \supseteq \text{Int}(B) \cup \text{Cl}(\emptyset) = \text{Int}(B) \cup \emptyset = \text{Int}(B) = B\), and \(\text{int}(S) \cup \text{cl}(\text{int}(B)) = \text{int}(\emptyset) \cup \text{cl}(\text{int}(B)) \subseteq \emptyset \cup \text{cl}(B) = \text{cl}(B)\).

Hence, we have \(B \setminus \text{int}(\text{cl}(B)) \cup \text{cl}(P) = \emptyset \in J\), and \(\text{int}(S) \cup \text{cl}(\text{int}(B)) \setminus \text{cl}(B) = \emptyset \in J\), that is, \(B\) is nearly \(b^*_J\)-open. \(\square\)

An element of ideal \(J\) is nearly \(b^*_J\)-open set. The next lemma, Lemma 3, shows this idea. Please see [9] and [4] to have more insights.

Lemma 3. Let \((Y, \varsigma, J)\) be an ideal space. Then each element of \(J\) is \(b^*_J\)-open.

Proof. Let \(B \in J\). Since \(B = \text{Int}(\text{cl}(B)) \cup \text{cl}(B) \subseteq B\), we have \(\text{int}(\text{cl}(B)) \cup \text{cl}(B) \in J\). Next, consider \(S = \emptyset\). Then \(\text{Int}(S) \cup \text{cl}(\text{int}(B)) \setminus \text{cl}(B) = \emptyset \cup \text{cl}(\text{int}(B)) \setminus \text{cl}(B) = \text{cl}(\text{int}(B)) \setminus \text{cl}(B) = \emptyset \in J\). Therefore, \(B\) is nearly \(b^*_J\)-open. \(\square\)

Lemma 4 says that each \(b^*\)-open set is \(b^*_J\)-open.

Lemma 4. Let \((Y, \varsigma, J)\) be an ideal space. Then a \(b^*\)-open set is \(b^*_J\)-open.

Proof. Let \(B\) be a \(b^*\)-open set. Then \(\text{int}(\text{cl}(B)) \cup \text{cl}(\text{int}(B)) = B\). Consider \(P = \text{int}(B)\) and \(S = \text{cl}(B)\). Then \(P\) is open with \(P \subseteq \text{int}(B)\), and \(S\) is closed with \(S \subseteq \text{cl}(B)\). Observed that \(\text{int}(\text{cl}(B)) \cup \text{cl}(P) = \text{int}(B) \cup \text{cl}(\text{int}(B)) = B\), and \(\text{Int}(S) \cup \text{Cl}(\text{Int}(B)) = \text{Int}(\text{Cl}(B)) \cup \text{Cl}(\text{Int}(B)) = B\).

Hence, we have \(B \setminus \text{int}(\text{cl}(B)) \cup \text{cl}(P) = \emptyset \in J\), and \(\text{Int}(S) \cup \text{Cl}(\text{Int}(B)) \setminus B = \emptyset \in J\), that is, \(B\) is \(b^*_J\)-open. \(\square\)
Lemma 5. Let \((Y, \varsigma, J)\) be an ideal space with \(J = \{\emptyset\}\). Then \(B\) is \(b^\ast\)-open precisely if \(B\) is \(b^\ast_J\)-open.

Proof. Necessity. Let \(B\) be \(b^\ast_J\)-open. Then there is an open set \(P\) such that \(P \subseteq \text{int}(B)\), and there is a close set \(S\) such that \(S \subseteq \text{cl}(B)\). Hence, \(B \subseteq \text{int}(\text{cl}(B)) \cup \text{cl}(P)\), and \(\text{int}(S) \cup \text{cl}(\text{int}(B)) \subseteq B\). Thus, \(\text{int}(\text{cl}(B)) \cup \text{cl}(\text{int}(B)) = \text{int}(S) \cup \text{cl}(\text{int}(B)) \subseteq B\), and \(\text{int}(\text{cl}(B)) \cup \text{cl}(\text{int}(B)) = \text{int}(\text{cl}(B)) \cup \text{cl}(P) \supseteq B\). Therefore, \(\text{int}(\text{cl}(B)) \cup \text{cl}(\text{int}(B)) = B\), that is, \(B\) is \(b^\ast\)-open.

Sufficiency. The converse follows from Lemma 4. \(\square\)

If \(J\) is the minimal ideal, then the notions \(b^\ast\)-compact, \(b^\ast_J\)-compact and \(cb^\ast_J\)-compact are the same. Theorem 1 shows this idea.

Theorem 1. Let \((Y, \varsigma, J)\) be an ideal space with \(J = \{\emptyset\}\). Then the following are equivalent.

(i). \((Y, \varsigma, J)\) is a \(b^\ast\)-compact ideal space.

(ii). \((Y, \varsigma, J)\) is a \(b^\ast_J\)-compact ideal space.

(iii). \((Y, \varsigma, J)\) is a \(cb^\ast_J\)-compact ideal space.

Proof. (i) implies (ii): Let \(\{U_\psi : \psi \in \Psi\}\) be a \(b^\ast_J\)-open covering \(Y\). By Lemma 5, \(\{U_\psi : \psi \in \Psi\}\) is also a \(b^\ast\)-open covering \(Y\). Since \(Y\) is a \(b^\ast\)-compact ideal space, \(\Psi\) has a smaller finite subset, say \(\Psi_0\), with \(\{U_\psi : \psi \in \Psi_0\}\) still covering \(Y\). Thus, by Lemma 5, \(\{U_\psi : \psi \in \Psi_0\}\) is a smaller finite \(b^\ast_J\)-covering of \(Y\). This shows that \(Y\) is a \(b^\ast_J\) compact set.

(ii) implies (iii): Let \(\{U_\psi : \psi \in \Psi\}\) be a \(b^\ast_J\)-open covering \(Y\). Since \(Y\) is a \(b^\ast_J\)-compact ideal space, \(\Psi\) has a smaller finite subset, say \(\Psi_0\), with \(\{U_\psi : \psi \in \Psi_0\}\) still covering \(Y\). Thus, \(Y - \bigcup_{\psi \in \Psi_0} U_\psi = \emptyset \in J\). Therefore, \(Y\) is \(cb^\ast_J\) compact set.

(iii) implies (i): Let \(\{U_\psi : \psi \in \Psi\}\) be a \(b^\ast\)-open covering \(Y\). By Lemma 5, \(\{U_\psi : \psi \in \Psi\}\) is also a \(b^\ast\)-open covering \(Y\). Since \(Y\) is a \(b^\ast\)-compact ideal space, \(\Psi\) has a smaller finite subset, say \(\Psi_0\), with \(Y - \bigcup_{\psi \in \Psi_0} U_\psi = \emptyset \in J\), that is, \(\{U_\psi : \psi \in \Psi_0\}\) is a smaller finite \(b^\ast\)-covering of \(Y\). Therefore, \(Y\) is \(b^\ast\) compact set. \(\square\)

Another characterization of \(b^\ast_J\)-compact topological spaces is presented in Theorem 2.

Theorem 2. Let \((Y, \varsigma, J)\) be an ideal space. Then statement (i) is a necessary and sufficient condition for statement (ii).

i. \((Y, \varsigma, J)\) is a \(b^\ast_J\)-compact space.

ii. If \(\{S_\psi : \psi \in \Psi\}\) is a class of \(b^\ast_J\)-closed sets with \(\bigcap\{S_\psi : \psi \in \Psi\} = \emptyset\), then \(\Psi\) has a smaller finite subset, say \(\Psi_0\), with \(\bigcap\{S_\psi : \psi \in \Psi_0\} = \emptyset\).
Note that 
\[\{T \\}\ \text{sufficient condition for statement (i)}\]

**Definition 1.** Let \( \Psi \) be a mapping. Then:

**Remark 1.** \( \{11\} \)

\(M. \text{ Baldado Jr.} / \text{Eur. J. Pure Appl. Math,} \text{ 16 (3) (2023),} \text{ 1809-1816} \)

\(\text{Proof.} \ (i) \implies (ii): \) Let \( \{S_\psi : \psi \in \Psi\} \) be a class of \( b^*_Y \)-closed sets with \( \bigcap \{S_\psi : \psi \in \Psi\} = 0. \) Then \( Y = 0^C = (\bigcap \{S_\psi : \psi \in \Psi\})^C = \bigcup \{S_\psi^C : \psi \in \Psi\}. \) Hence, \( \{S_\psi^C : \psi \in \Psi\} \) is a class of \( b^*_Y \)-open sets which covers of \( Y. \) By assumption, \( \Psi \) has a smaller finite subset, say \( \Psi_0, \) with the property \( \bigcup \{S_\psi^C : \psi \in \Psi_0\} = X. \) Hence, \( (\bigcap \{S_\psi : \psi \in \Psi_0\}) = \bigcup \{S_\psi^C : \psi \in \Psi_0\})^C = Y^C = 0. \)

\((ii) \implies (i): \) Let \( \{P_\psi : \psi \in \Psi\} \) be a \( b^*_Y \)-open covering of \( Y, \) i.e. \( \bigcup \{P_\psi : \psi \in \Psi\} = Y. \) Then \( \bigcap \{P_\psi^C : \psi \in \Psi\} = (\bigcup \{P_\psi : \psi \in \Psi\})^C = 0. \) Note that \( P^C \) is \( b^*_Y \)-close since \( P \) is \( b^*_Y \)-open. By assumption, \( \Psi \) has a smaller finite subset, say \( \Psi_0, \) with the property that \( \bigcap \{P_\psi^C : \psi \in \Psi_0\} = 0. \) Note that \( \bigcup \{P_\psi : \psi \in \Psi_0\} = (\bigcap \{P_\psi^C : \psi \in \Psi_0\})^C = Y. \) Hence, \( \{P_\psi : \psi \in \Psi_0\} \) is a class of \( b^*_Y \)-open sets that covers \( Y. \)

Another characterization of \( cb^*_Y \)-compact topological spaces is presented in Theorem 3.

**Theorem 3.** Let \( (Y, \zeta, J) \) be an ideal topological space. Then \( (i) \) is a necessary and sufficient condition for statement \( (ii) \).

\(i. \) \( (Y, \zeta, J) \) is \( cb^*_Y \)-compact.

\(ii. \) If \( \{S_\psi : \psi \in \Psi\} \) is a class of \( b^*_Y \)-closed sets with \( \bigcap \{S_\psi : \psi \in \Psi\} = 0, \) then \( \Psi \) has a smaller finite subset, say \( \Lambda_0, \) with the property that \( \bigcap \{P_\lambda : \lambda \in \Lambda_0\} \in J. \)

\(\text{Proof.} \ (i) \implies (ii): \) Let \( \{S_\psi : \psi \in \Psi\} \) be a class of \( b^*_Y \)-closed sets such that \( \bigcap \{S_\psi : \psi \in \Psi\} = 0. \) Note that \( \bigcup \{S_\psi^C : \psi \in \Psi\} = (\bigcap \{S_\psi : \psi \in \Psi\})^C = Y. \) Hence, \( \{S_\psi^C : \psi \in \Psi\} \) is a class of \( b^*_Y \)-open sets covering \( Y. \) By assumption, \( \Psi \) has a finite subset, say \( \Psi_0, \) with \( Y - \bigcup \{S_\psi^C : \psi \in \Psi_0\} \in J, \) i.e. \( \bigcap \{S_\psi : \psi \in \Psi_0\} \in J. \)

\((ii) \implies (i): \) Let \( \{P_\psi : \psi \in \Psi\} \) be a \( b^*_Y \)-open covering of \( Y, \) i.e. \( \bigcup \{P_\psi : \psi \in \Psi\} = Y. \) Note that \( \bigcap \{P_\psi^C : \psi \in \Psi\} = (\bigcup \{P_\psi : \psi \in \Psi\})^C = 0. \) By assumption, \( \Psi \) has a smaller finite subset, say \( \Psi_0, \) with \( \bigcap \{P_\psi^C : \psi \in \Psi_0\} \in J, \) i.e. \( Y - \bigcup \{P_\psi : \psi \in \Psi_0\} \in J. \)

**Remark 1.** \( [11] \) Let \( (Y, \zeta, J) \) and \( (W, \xi, K) \) be ideal topological spaces, and \( \zeta : Y \to W \) be a mapping. Then:

\(i. \) \( \zeta(J) = \{\zeta(B) : B \in J\} \) is an ideal in \( W; \) And,

\(i. \) if \( \zeta \) is a one to one correspondence, then \( \zeta^{-1}(K) = \{\zeta^{-1}(D) : D \in K\} \) is an ideal in \( Y. \)

**Definition 1.** Let \( (Y, \zeta, J) \) and \( (W, \xi, K) \) be ideal spaces. A mapping \( \zeta : Y \to W \) is

\(i. \) \( b^*_Y \)-open if \( \zeta(B) \) is \( b^*_K \)-open for every \( b^*_Y \)-open set \( B \) in \( Y, \) and

\(ii. \) \( b^*_Y \)- irresolute if \( \zeta^{-1}(D) \) is \( b^*_Y \)-open for each \( b^*_K \)-open set \( D \) in \( W. \)

If the domain of a \( b^* \)- irresolute map is \( cb^*_Y \)-compact with respect to an ideal, then so is the image. We show this idea in Theorem 4.
Theorem 4. Let $(Y, \varsigma, J)$ and $(W, \xi, K)$ be ideal spaces, and $\zeta : Y \to W$ be a $b^*_J$-irresolute function with $\zeta(J) = K$. If $Y$ is a $cb^*_J$-compact, then $\zeta(Y)$ is $cb^*_K$-compact.

Proof. Let $\{P_\psi : \psi \in \Psi\}$ be a $b^*_K$-open covering of $\zeta(Y)$. Since $\zeta$ is $b^*_J$-irresolute, $\{\zeta^{-1}(P_\psi) : \psi \in \Psi\}$ is a $b^*_J$-open covering $Y$. By assumption, $\Psi$ has a smaller finite subset, say $\Psi_0$, with $Y - \bigcup \{\zeta^{-1}(P_\psi) : \psi \in \Psi_0\} \in J$. And so by Remark 1 $\zeta(Y) \setminus \bigcup \{P_\psi : \psi \in \Psi_0\} = \zeta(Y - \bigcup \{\zeta^{-1}(P_\psi) : \psi \in \Psi_0\}) \in K$. □

If the co-domain of a $b^*$-open and onto map is $cb^*_J$-compact with respect to an ideal, then so is the domain. We show this idea in Theorem 5.

Theorem 5. Let $(Y, \varsigma, J)$ and $(W, \xi, K)$ be ideal spaces, and $\zeta : Y \to W$ be a $b^*_J$-open and onto map with $\zeta(J) = K$. If $W$ is $cb^*_K$-compact, then $Y$ is $cb^*_J$-compact.

Proof. Let $\{P_\psi : \psi \in \Psi\}$ be a $b^*_J$-open covering of $Y$. Since $\zeta$ is a $b^*_J$-open and onto, $\{\zeta(P_\psi) : \psi \in \Psi\}$ is a $b^*_K$-open covering of $W$. By assumption, $\Psi$ has a smaller finite subset, say $\Psi_0$, with $W - \bigcup \{\zeta(P_\psi) : \psi \in \Psi_0\} \in K$. Thus, $Y - \bigcup \{P_\psi : \psi \in \Psi_0\} = \zeta^{-1}(W - \bigcup \{\zeta(P_\psi) : \psi \in \Psi_0\}) \in J$. □

References


