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The Generator Graph of a Group

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Abstract. This paper presents a way to represent a group using a graph, which involves the concept of a generator element of a group. The graph representing a group is called the generator graph. In the generator graph, the vertices correspond to the elements of the group, and two vertices, x and y, are connected by an edge if either x or y serves as a generator for the group. The paper investigates some properties of these generator graphs and obtains the generator graphs for specific groups. Additionally, it explores the relationship between the generator graph of a group and the generating graph introduced by Lucchini et al. in their work [7].

2020 Mathematics Subject Classifications: 05C25

Key Words and Phrases: Graph, Group, Generator graph, Generating graph

1. Introduction

In graph theory, a graph is a collection of points called vertices, and edges that connect the vertices. It is widely known for its applications in networks such as Facebook. Graphs can be used to model relationships or connections between objects, describe events and even in analysis and problem solving. In most cases, objects are represented by vertices while the relationship between two objects is represented by an edge in the graph.

A group can be briefly defined as a set with an associative binary operation, an identity element, and inverses with respect to that identity element. For example, the set of real numbers is a group under the usual addition operation. Group theory also finds application in chemistry in the study of the group of symmetries of crystals and molecules. Basic concepts in graph theory and group theory may be found in [4], and in [3] and [5] respectively.

The interplay between group theory and graph theory has been explored in literature for many years. Some researchers represent groups as graphs, while others do it the other way around. The concept of identity graph of a group is introduced in [6]. The identity graph is a graph with elements of group as vertices and adjacency of vertices is defined in terms of the identity element of a group. Two vertices x and y can be joined by an

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edge in the identity graph if $xy = e$ ($e-$ being the identity of a group). Lucchini et al. introduced the concept of generating graph of a group in [7]. The generating graph of a group is also a graph with the elements of group as vertices, and adjacency of vertices is restricted to subgroup generated by the given pair of vertices. The generating graph of symmetric groups is considered in [2]. Other graphs associated with finite groups are discussed in [8].

The above ideas of group representations have motivated the present paper to introduce a graph with the elements of the group as vertices, and adjacency of vertices is defined in terms of the generator element of the group. The graph will be called the generator graph of a group.

2. Basic Concepts

2.1. Graphs

Althroughout this paper, we only consider undirected graphs with no loops. The basic definitions and concepts used in this study are adopted from [1]. Given a graph $G = (V(G), E(G))$, the sets $V(G)$ and $E(G)$ are, respectively, the *vertex set* and *edge set* of G. The cardinality of $V(G)$, denoted $|V(G)|$, is called the *order* of G and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. Each element of $V(G)$ is considered a *vertex* of G. If $u, v \in V(G)$ and $e = [u, v]$ is in $E(G)$, then e is an *edge* of G, and e is said to *join* u and v. In this case, it is customary to write $e = uv$ and say that u and v are *adjacent*, while u and e are *incident*, as v and e are. Adjacent vertices are also called neighbors. The degree $deg_G(v)$ of a vertex v is the number of edges incident to v.

A *subgraph* of a graph G is a graph having all its vertices and edges in G. It is a *spanning subgraph* if it contains all the vertices of G. The *induced subgraph* $\langle S \rangle$ is the maximal subgraph with vertex set S. The *path* of order n, denoted by $P_n = [v_1, v_2, ..., v_n]$, is the graph with vertices $v_1, v_2, ..., v_n$, and edges $v_1v_2, v_2v_3, ..., v_{n-1}v_n$. We also call P_n as a u-v path of length $n - 1$, where $u = v_1$ and $v = v_n$. A graph G is *connected* if there is a path that joins each pair of vertices.

A graph G is called *complete* if every pair of its vertices is adjacent. K_n denotes the complete graph of order n. The graph K_1 is the *trivial graph*. It is trivial to see that K_n contains $n(n - 1)/2$ edges. A graph with an empty edge set is called a *null graph*. A null graph of order n is denoted $\overline{K_n}$. The join $G_1 + G_2$ of two graphs G_1 and G_2 is their disjoint union together with all the edges that connect all the vertices of G_1 with all the vertices of G_2.

2.2. Groups

A binary operation or law of composition on a set G is a function $G \times G \to G$ that assigns to each pair $(a, b) \in G \times G$ a unique element $a \circ b$, or $ab \in G$, called the composition of a and b. A group $(G, \circ)$ is a set G together with a law of composition $(a, b) \mapsto a \circ b$ that satisfies the following axioms: (i) the law of composition is associative, i.e., $(a \circ b) \circ c = a \circ (b \circ c)$ for $a, b, c \in G$; (ii) there exists an element $e \in G$ called the identity element, such that for any element $a \in G$, $e \circ a = a \circ e = a$; and for each element $a \in G$, there exists an inverse element in G, denoted by a^{-1}, such that $a \circ a^{-1} = a^{-1} \circ a = e$.

A group G with the property that $a \circ b = b \circ a$ for all $a, b \in G$ is called abelian or commutative. Groups not satisfying this property are said to be non−abelian or noncommutative.

A subgroup H of a group G is a subset H of G such that when the group operation of G is restricted to H, H is a group in its own right. The subgroup $H = e$ of a group G is called the trivial subgroup. A subgroup that is a proper subset of G is called a proper subgroup.

Let G be a group, and let a be any element in G. Then the set $(a) = \{a^k : k \text{ is an integer}\}$ is a subgroup of G. Furthermore, (a) is the smallest subgroup of G that contains a.

For $a \in G$, we call (a) the cyclic subgroup generated by a. If G contains some element a such that $G = (a)$, then G is a cyclic group. In this case, a is a generator of G. The order of a is the smallest positive integer n such that $a^n = e$, and we write $|a| = n$. If there is no such integer n, we say that the order of a is infinite and write $|a| = \infty$ to denote the order of a.b Note that a cyclic group can have more than a single generator.

For example, from the group of integers modulo 6, both 1 and 5 generate Z_6; hence, Z_6 is a cyclic group. Not every element in a cyclic group is necessarily a generator of the group. The order of $2 \in Z_6$ is 3. The cyclic subgroup generated by 2 is $(2) = \{0, 2, 4\}$. The Klien 4-group V is an abelian noncyclic group consisting of 3 elements a, b, c and an identity element e, with the property that $a^2 = b^2 = c^2 = e$.

3. Generator Graph of Group

Definition 1. *The generator graph $gg(G)$ of a group G is a graph whose vertices are the elements of G, and two vertices x and y in $gg(G)$ are joined by an edge if either x or y is a generator of G.*

Figure 1 illustrates the generator graphs $gg(Z_6)$ and $gg(Z_5)$ of groups Z_6 under addition modulo 6 and Z_5 under addition modulo 5, respectively. The set $S_{Z_6} = \{1, 5\}$ is the set of all generators of Z_6 while the set $S_{Z_5} = \{1, 2, 3, 4\}$ is the set of all generators of Z_5. Observe that in Z_6, $deg_{Z_6}(1) = deg_{Z_6}(5) = 5 = |V(gg(G))| - 1$ while $deg_{Z_6}(0) = deg_{Z_6}(2) = deg_{Z_6}(3) = deg_{Z_6}(4) = 2 = |S_{Z_6}|$.

Theorem 1. *Let G be a group of order $n > 1$ and, S_G be the set of all generators of G. Let $x \in V(gg(G))$. Then*

(i) $deg_G(x) = n - 1$ if $x \in S_G$ and $deg_G(x) = |S_G|$ if $x \notin S_G$; and

(ii) the size of $gg(G)$ is given by

$$|E(gg(G))| = \frac{|S_G|}{2}(2n - |S_G| - 1).$$

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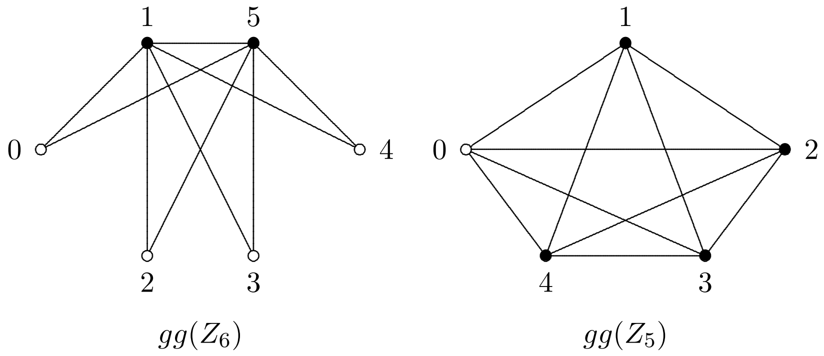


Figure 1: The generator graphs $gg(Z_6)$ and $gg(Z_5)$.

Proof.

(i) If $x \in S_G$, then x is adjacent to the other $n-1$ vertices of $gg(G)$. In this case, $deg_G(x) = n-1$. If $x \notin S_G$, then x is adjacent only to the $|S_G|$ vertices in $gg(G)$. In this case, $deg_G(x) = |S_G|$.

(ii) The size of $gg(G)$ is

$$|E(gg(G))| = \frac{1}{2}\sum_{x \in V(gg(G))}(deg_G(x)) = \frac{1}{2}[|S_G|(n-1) + (n-|S_G|)|S_G|]$$

$$= \frac{|S_G|}{2}[(n-1) + (n-|S_G|)]$$

$$= \frac{|S_G|}{2}(2n-|S_G|-1).$$

□

Theorem 2. *Let G be a group of order $n > 1$. The generator graph $gg(G)$ of G is complete if and only if n is prime.*

Proof. Assume that $gg(G)$ is complete, and suppose the order $n > 1$ of G is not prime. This implies that for some integer $k \neq n$ there exists an integers $m \neq n$ such that $km = n$, and k is the order of some vertex $a \in V(gg(G))$, i.e., $(a) \neq G$. This means that a is not a generator of G. Consequently, vertex a is not adjacent to the identity element of G in $gg(G)$. Thus, $deg_G(a) \leq n-2$ in $gg(G)$. This is a contradiction to the assumption that $gg(G)$ is complete. Therefore, the order n of G must be prime.

Suppose that the order of G is prime, and let e be the identity element of G. Then for all $x \in G\backslash\{e\}$, $(x) = G$. This means that x is a generator of G, and x is adjacent to all other $n-1$ vertices of $gg(G)$. Thus, $deg_{gg(G)}(x) = n-1$ for all $x \in G\backslash\{e\}$. Since all elements of $x \in G\backslash\{e\}$ are generators, all $|x \in G\backslash\{e\}| = n-1$ vertices of $gg(G)$ are adjacent to the identity element e of G in $gg(G)$. So, $deg(e) = n-1$. This proves that for all $x \in V(gg(G))$, $deg_{gg(G)}(x) = n-1$. Therefore, $gg(G)$ is a complete graph. □

Example 1. *Let Z_p be the group of integers under addition modulo p, where p is prime. Then $gg(Z_p) = K_p$.The graph $gg(Z_5)$ of Z_5 in Figure 1 is a complete graph K_5.*

Remark 1. *Let G be a nontrivial group. The generator graph of G is connected if and only if G is cyclic.*

Example 2. *In Figure 1, the generator graphs $gg(Z_6)$ of Z_6 under addition modulo 6 and $gg(Z_5)$ of Z_5 under addition modulo 5 are connected. It is easy to verify that the generator graph $gg(V)$ of Klien 4-group V is a null graph of order 4 which is disconnected.*

Theorem 3. *Let G be a cyclic group of order $n > 1$, and S_G be the set of all generators of G. Then each of the following is true:*

(i) the induced subgraph $< S_G >$ generated by S_G of $gg(G)$ is the complete graph $K_{|S_G|}$; and

(ii) the induced subgraph $< G\backslash S_G >$ generated by $G\backslash S_G$ of $gg(G)$ is the empty graph $\overline{K}_{n-|S_G|}$.

Proof.

(i) Since G is cyclic and $n > 1$, there exist $a \in S_G$ such that $(a) = G$, i.e., $S_G \neq \varnothing$. Then for every pair x and y in S_G, x and y generate each other, i.e., $x \in (y)$ and $y \in (x)$. Hence, x and y are adjacent in $gg(G)$ for all x and y in S_G. This means also that x and y are adjacent in $< S_G >$ for all x and y in S_G. Therefore, $< S_G >$ is a complete graph, i.e., $< S_G >= K_{|S_G|}$.

(ii) For every $x, y \in G\backslash S_G$, x and y are not adjacent in $gg(G)$. Thus, x and y are also not adjacent in $< G\backslash S_G >$. Therefore, $E(< G\backslash S_G >) = \varnothing$ and $< G\backslash S_G >$ is an empty graph.

□

Example 3. *In Z_6, $\langle S_{Z_6}\rangle = \langle\{1,5\}\rangle$ is a complete subgraph while $\langle Z_6\backslash S_{Z_6}\rangle = \langle\{0,2,3,4\}\rangle$ is a null subgraph of Z_6. These are illustrated in Figure 2 and Figure 3, respectively.*

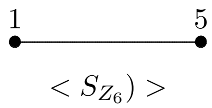
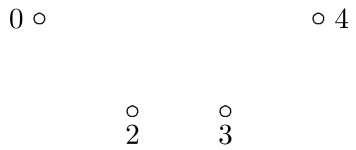


Figure 2: The induced subgraph of $S_{Z_6} = \{1,5\}$ of Z_6 under addition modulo 6.



$< Z_6 \backslash (S_{Z_6}) >$

Figure 3: The induced subgraph of $Z_6 \backslash S_{Z_6}$under addition modulo 6.

Theorem 4. *Let G be a cyclic group of order $n > 1$, and S_G be the set of all generators of G. Then*

$$G = K_{|S_G|} + \overline{K_{n-|S_G|}}.$$

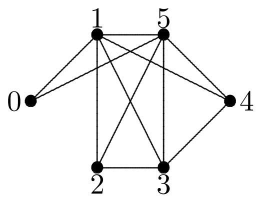
Proof. Let G be a cyclic group of order $n > 1$,and let S_G be the set of all generators of G. For $x, y \in V(ggG)$, if $x, y \in G \backslash S_G$, then $xy \notin E(ggG)$. This case is the same as connecting the generator element of the group to all other elements of the group whether they are another generator or non-generators. In other words, all gerators of G are adjacent in $gg(G)$, forming a complete subgroup. The other edges are formed by connecting each genrator by an edge to each of the non-generators. Therefore, the generator graph ggG can be described as the join of complete graph of order $|S_G|$ and null graph of order $n - |S_G|$. □

4. Relationship between generating graph and generator graph of group

The concept of the generating graph of a group was introduced by Luchini et al in [7]. In their study, two non-generator elements of a group G are adjacent in the generating graph of G if they generate G, while they are not adjacent in the generator graph of G.

Definition 2. *[7] The generating graph $\Gamma(G)$ of a group G is the graph defined on the elements of G, with edge between two vertices if and only if they generate G.*

Figure 4 illustrates the generating graph of the group of integers modulo 6, Z_6. Both sets $\{2, 3\})$ and $\{3, 4\}$ generate the group Z_6, that is, $(\{2, 3\}) = Z_6$ and $(\{3, 4\}) = Z_6$. Thus, $(2, 3), (3, 4) \in E(Z_6)$. Note that 2, 3 and 4 are not generators of G, hence none of these vertices are adjacent in the generator graph of G. This can be verified in Figure 1. This observation is generalized in the next result.



$\Gamma(Z_6)$

Figure 4: The generating graph of $\Gamma(Z_6)$

Theorem 5. *Let G be a group. Then the generator graph of G is a spanning subgraph of the generating graph of G.*

Proof. Let G be a group. By definition of the generator graph and the generating graph, $V(ggG) = V(\Gamma(G))$. Let $x, y \in V(ggG)$. If both or one of x and y are generators, then the set {x, y} clearly generates G. In this case, $xy \in E(ggG)$ and $xy \in E(\Gamma(G))$. If both x and y are non-generators, then the set {x, y} either generates G or does not generate G. In this case, $xy \notin E(ggG)$ but either $xy \in E(\Gamma(G))$ or $xy \notin E(\Gamma(G))$. It follows that $E(ggG) \subseteq E(\Gamma G)$. Therefore, $gg(G)$ is a spanning subgraph of $\Gamma(G)$. □

Example 4. *The dihedral group D_n is the symmetry group, which includes rotations and reflections, of an n-sided regular polygon for $n > 1$. Dihedral groups D_n are non-Abelian permutation groups for $n > 2$. The group D_n has elements $r_0, ..., r_{n-1}$ that represents rotations and $s_0, ..., s_{n-1}$ that represents reflections. Their compositions are given by: $r_i r_j = r_{(i+j)(mod\, n)}$, $r_i s_j = s_{(i+j)(mod\, n)}$, $s_i r_j = r_{(i-j)(mod\, n)}$, and $s_i s_j = r_{(i-j)(mod\, n)}$ where $i, j \in \{0, 1, 2\}$. For n_3, $D_3 = \{r_0, r_1, r_2, s_0, s_1, s_2\}$, representing the symmetric group of triangle with the identity r_0. Note that the generator graph of D_3 is the empty graph of order 6, i.e., $gg(D_3) = \overline{K_6}$, since D_3 has no generator. It is easy to verify that $\Gamma(D_3) = (\overline{K_2} + K_3) \cup K_1$. Clearly, $V(gg(D_3)) \subset V(K_2 + K_3) \cup K_1) = \Gamma(D_3)$. Therefore, $gg(D_3)$ is a spanning subgraph of $\Gamma(D_3)$.*

5. Conclusion

This paper introduces the concept of the generator graph of a group. The properties of the generator graph are presented in terms its structure, the degree of a vertex, and size of the induced subgraph of the set of generators or non-generators. Additionally, the generator graphs of some special graphs are also presented. It has been shown that the generator graph of a group forms a spanning subgraph of the generating graph.

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