Grundy Total Hop Dominating Sequences in Graphs

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Abstract. Let \( G = (V(G), E(G)) \) be an undirected graph with \( \gamma(C) \neq 1 \) for each component \( C \) of \( G \). Let \( S = (v_1, v_2, \ldots, v_k) \) be a sequence of distinct vertices of a graph \( G \), and let \( \hat{S} = \{v_1, v_2, \ldots, v_k\} \). Then \( S \) is a legal open hop neighborhood sequence if \( N_2^G(v_i) \setminus \bigcup_{j=1}^{i-1} N_2^G(v_j) \neq \emptyset \) for every \( i \in \{2, \ldots, k\} \). If, in addition, \( \hat{S} \) is a total hop dominating set of \( G \), then \( S \) is a Grundy total hop dominating sequence. The maximum length of a Grundy total hop dominating sequence in a graph \( G \), denoted by \( \gamma_{gh}(G) \), is the Grundy total hop domination number of \( G \). In this paper, we show that the Grundy total hop domination number of a graph \( G \) is between the total hop domination number and twice the Grundy hop domination number of \( G \). Moreover, determine values or bounds of the Grundy total hop domination number of some graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: total hop domination, total hop domination number, open hop neighborhood sequence, Grundy total hop dominating sequence, Grundy total hop domination number

1. Introduction

Domination has attracted many researchers because of its nice applications in various fields and in networks. A number of variations of the domination concept (see for example, [7, 18, 19, 21]) have been introduced and studied. Recently, hop domination was defined and studied by Natarajan and Ayyaswamy in [17]. From then on a lot of investigations of the concept and some of its variants have been done (see [1, 2, 8–15, 20]).

In 2014, Bresar et al. [4] introduced another concept called Grundy dominating sequence in a graph. The newly defined concept has subsequently attracted other researchers in the area to study and generate more interesting results (see [6, 16]) on it.
In 2016, the concept of Grundy total domination in graphs was investigated by Bresar et al. [5]. Bresar [3] studied further the concept on the product of graphs.

In this study, the concept of Grundy total hop domination number of a graph will be introduced and investigated. Its relationship with total hop domination, and Grundy hop domination numbers of a graph will be given. Bounds for the parameter will be determined for the shadow graph as well as the join and the corona of two graphs.

2. Terminology and Notation

Two vertices $u, v$ of a graph $G$ are adjacent, or neighbors, if $uv$ is an edge of $G$. Moreover, an edge $uv$ of $G$ is incident to two vertices $u, v$ of $G$. The set of neighbors of a vertex $u$ in $G$, denoted by $N_G(u)$, is called the open neighborhood of $u$ in $G$. The closed neighborhood of $u$ in $G$ is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the open neighborhood of $X$ in $G$ is the set $N_G(X) = \bigcup_{u \in X} N_G(u)$. The closed neighborhood of $X$ in $G$ is the set $N_G[X] = N_G(X) \cup X$.

Let $G$ be a graph. A set $D \subseteq V(G)$ is a total dominating set of $G$ if for every $v \in V(G)$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G(D) = V(G)$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. Any total dominating set with cardinality equal to $\gamma_t(G)$ is called a $\gamma_t$-set.

Let $S = (v_1, v_2, \ldots, v_k)$ be a sequence of distinct vertices of a graph $G$, and let $\hat{S} = \{v_1, v_2, \ldots, v_k\}$. Then $S$ is a legal open neighborhood sequence if $N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \emptyset$ for every $i \in \{2, \ldots, k\}$. If, in addition, $\hat{S}$ is a total dominating set of $G$, then $S$ is called a Grundy total dominating sequence. The maximum length of a Grundy total dominating sequence in a graph $G$ is called the Grundy total domination number of $G$, and is denoted by $\gamma_{gr}^t(G)$.

A vertex $v$ in $G$ is a hop neighbor of vertex $u$ in $G$ if $d_G(u, v) = 2$. The set $N^2_G(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the open hop neighborhood of $u$. The closed hop neighborhood of $u$ in $G$ is given by $N^2_G[u] = N^2_G(u) \cup \{u\}$. The open hop neighborhood of $X \subseteq V(G)$ is the set $N^2_G(X) = \bigcup_{u \in X} N^2_G(u)$. The closed hop neighborhood of $X$ in $G$ is the set $N^2_G[X] = N^2_G(X) \cup X$.

A set $S \subseteq V(G)$ is a hop dominating set of $G$ if $N^2_G[S] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets of $G$, denoted by $\gamma_h(G)$, is called the hop domination number of $G$. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a $\gamma_h$-set.

Let $S = (v_1, v_2, \ldots, v_k)$ be a sequence of distinct vertices of $G$ and let $\hat{S} = \{v_1, \ldots, v_k\}$. Then $S$ is a legal closed hop neighborhood sequence of $G$ if $N^2_G[v_i] \setminus \bigcup_{j=1}^{i-1} N^2_G[v_j] \neq \emptyset$ for each $i \in \{2, \ldots, k\}$. If, in addition, $\hat{S}$ is a hop dominating set of $G$, then $S$ is called a Grundy hop dominating sequence. The maximum length of a Grundy hop dominating sequence in a graph $G$, denoted by $\gamma_{gr}^h(G)$, is called the Grundy hop domination number of $G$. Any Grundy hop dominating sequence $S$ with $|\hat{S}| = \gamma_{gr}^h(G)$ is called a maximum....
Grundy hop dominating sequence or a $\gamma_{gr}^h$-sequence of $G$. In this case, we call $\hat{S}$ a $\gamma_{gr}^h$-set of $G$.

A subset $S$ of $V(G)$ is a total hop dominating set of $G$ if for every $v \in V(G)$, there exists $u \in S$ such that $d_G(u, v) = 2$. The smallest cardinality of a total hop dominating set of $G$ denoted by $\gamma_t(G)$, is called the total hop domination number of $G$. Any hop dominating set with cardinality equal to $\gamma_t(G)$ is called a $\gamma_t$-set.

Let $G$ be any graph with $\gamma(C) \neq 1$ for each component $C$ of $G$. Let $S = (v_1, v_2, \ldots, v_k)$ be a sequence of distinct vertices of a graph $G$, and let $\hat{S} = \{v_1, v_2, \ldots, v_k\}$. Then $S$ is a legal open hop neighborhood sequence if $N^2_G(v_i) \cup \bigcup_{j=1}^{i-1} N^2_G(v_j) \neq \emptyset$ for every $i \in \{2, \ldots, k\}$. If, in addition, $\hat{S}$ is a total hop dominating set of $G$, then $S$ is called a Grundy total hop dominating sequence. The maximum length of a Grundy total hop dominating sequence in a graph $G$ is called the Grundy total hop domination number of $G$, and is denoted by $\gamma_{gr}^h(G)$. Any Grundy total hop dominating sequence $S$ with $|\hat{S}| = \gamma_{gr}^h(G)$ is called a maximum Grundy total hop dominating sequence or a $\gamma_{gr}^h$-sequence of $G$. In this case, we call $\hat{S}$ a $\gamma_{gr}^h$-set of $G$.

A legal open hop neighborhood sequence $S = (v_1, v_2, \ldots, v_k)$ with maximum length, i.e., $k = \max\{p \in \mathbb{N} : \exists$ a legal open hop neighborhood sequence $(x_1, \ldots, x_p)$ of $G\}$, will be referred to as a maximum legal open hop neighborhood sequence. We say that vertex $v_i$ hop-footprints the vertices from $N^2_G(v_i) \cup \bigcup_{j=1}^{i-1} N^2_G(v_j)$ (resp. $N^2_G(v_i) \cup \bigcup_{j=1}^{i} N^2_G(v_j)$), and that $v_i$ is their hop-footprinter. Two sequences $S$ and $S'$ in $G$ are loh-identical if they are legal open hop neighborhood sequences (or Grundy total hop dominating sequences) and $\hat{S} = \hat{S}'$ (i.e., one is a rearrangement of the terms of the other).

A sequence $S = (v_1, v_2, \ldots, v_k)$ of distinct vertices of a graph $G$ is a co-legal open neighborhood sequence in $G$ if $[V(G) \setminus N_G[v_i]] \setminus \bigcup_{j=1}^{i-1} [V(G) \setminus N_G[v_j]] \neq \emptyset$ for each $i \in \{2, \ldots, k\}$. A co-legal open neighborhood sequence $S = (v_1, v_2, \ldots, v_k)$ is a co-Grundy total dominating sequence if $V(G) = \bigcup_{i=1}^{k} [V(G) \setminus N_G[v_i]]$. The maximum length of a co-Grundy total dominating sequence in a graph $G$ is called the co-Grundy total domination number of $G$, and is denoted by $\gamma_{cogr}(G)$.

Let $S_1 = (v_1, \ldots, v_n)$ and $S_2 = (u_1, \ldots, u_m)$, $n, m \geq 1$ be two sequences of distinct vertices of $G$. The concatenation of $S_1$ and $S_2$, denoted by $S_1 \oplus S_2$, is the sequence given by $S_1 \oplus S_2 = (v_1, \ldots, v_n, u_1, \ldots, u_m)$.

The shadow graph $S(G)$ of a graph $G$ is constructed by taking two copies of $G$, say $G_1$ and $G_2$ and joining each vertex $u \in G_1$ to the neighbors of the corresponding vertex $u' \in G_2$.

Let $G$ and $H$ be any two graphs. The join of $G$ and $H$, denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The corona $G$ and $H$, denoted by $G \circ H$, is the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. We denote by $H^v$ the copy of $H$ in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\{v\} + H^v$. 

3. Results

Theorem 1. Let $G$ be a graph of order $n$ with $\gamma(C) \neq 1$ for each component $C$ of $G$. Then the following statements hold.

(i) If $\gamma_{th}(G) = t$ and $D = \{u_1, u_2, \ldots, u_t\}$ is a minimum total hop dominating set of $G$, then $S = (u_1, u_2, \ldots, u_t)$ is a Grundy total hop dominating sequence. In particular, $\gamma_{th}(G) \leq \gamma_{gr}^{th}(G)$.

(ii) If $S = (u_1, u_2, \ldots, u_s)$ is a minimum Grundy total hop dominating sequence, then $\gamma_{th}(G) = |\hat{S}|$.

Proof. (i) Suppose that there exists $i \in \{2, 3, \ldots, t\}$ such that $N_G^2(u_i) \setminus \bigcup_{j=1}^{i-1} N_G^2(u_j) = \emptyset$. Then $N_G^2(u_i) \subseteq \bigcup_{j=1}^{i-1} N_G^2(u_j)$. This means that $D \setminus \{u_i\}$ is a total hop dominating set of $G$, which is a contradiction to the minimality of $D$. Hence, $N_G^2(u_i) \setminus \bigcup_{j=1}^{i-1} N_G^2(u_j) \neq \emptyset$ for each $i \in \{2, 3, \ldots, t\}$, and so $S$ is Grundy total hop dominating sequence. Consequently, $\gamma_{th}(G) \leq \gamma_{gr}^{th}(G)$.

(ii) From (i), every $\gamma_{th}$-set of $G$ forms a Grundy total hop dominating sequence. Since $S$ is a minimum Grundy total hop dominating sequence, it follows that $|\hat{S}| \leq \gamma_{th}(G)$. On the other hand, since every Grundy total hop dominating sequence forms a total hop dominating set by definition, it follows that $\gamma_{th}(G) \leq |\hat{S}|$. Consequently, $\gamma_{th}(G) = |\hat{S}|$. □

Theorem 2. Let $G$ be a graph of order $n$ with $\gamma(C) \neq 1$ for each component $C$ of $G$. Then $S = (u_1, u_2, \ldots, u_t)$ is a maximum legal open hop neighborhood sequence of $G$ if and only if $S$ is a Grundy total hop dominating sequence of $G$ and $\gamma_{gr}^{th}(G) = l$.

Proof. Let $S = (u_1, \ldots, u_t)$ be a maximum legal open hop neighborhood sequence of $G$. Suppose on the contrary that $\hat{S}$ is not a total hop dominating set of $G$. Then there exists $v \in V(G)$ such that $v \notin N_G^2(\hat{S})$. This means that $u \notin N_G^2(v)$ for every $v \in \hat{S}$. Since $u$ is not hop dominated by any $v \in \hat{S}$, $u \in N_G^2(t)$ for some $t \in V(G) \setminus \hat{S}$. This means that $N_G^2(t) \setminus \bigcup_{i=1}^{k} N_G^2(u) \neq \emptyset$. Thus, $S' = (u_1, \ldots, u_t, t)$ is a legal open hop neighborhood sequence of $G$, a contradiction to the maximality of $S$. Therefore, $\hat{S}$ is a total hop dominating set of $G$. Consequently, $S$ is a Grundy total hop dominating sequence of $G$ and $\gamma_{gr}^{th}(G) = l$.

The converse is clear. □

The next result follows from Theorem 2.

Corollary 1. Let $G$ be a graph of order $n$ with $\gamma(C) \neq 1$ for each component $C$ of $G$ and let $T = (x_1, \cdots, x_j)$ be a legal open hop neighborhood sequence of $G$. Then $|\hat{T}| = j \leq \gamma_{gr}^{th}(G)$.

Theorem 3. Let $G$ be a graph of order $n$ with $\gamma(C) \neq 1$ for each component $C$ of $G$. Then $4 \leq \gamma_{gr}^{th}(G) \leq n$ and these bounds cannot be improved.
Proof. Clearly \( \gamma_{gr}^h(G) = 1 \) is not possible. Suppose \( \gamma_{gr}^h(G) = 2 \), say, \( S = (v_1, v_2) \) is a Grundy total hop dominating sequence. Since \( \hat{S} \) is a total hop dominating set, \( v_1 \in N^2_G(v_2) \). Let \( v \in N_G(v_1) \cap N_G(v_2) \). Then \( v \notin N^2_G(v_1) \cup N^2_G(v_2) \), a contradiction. Next, suppose \( \gamma_{gr}^h(G) = 3 \), say \( S = (v_1, v_2, v_3) \) is a Grundy total hop dominating sequence. Since \( \hat{S} \) is a total hop dominating set, \( v_1 \) is hop dominated by \( v_2 \) or \( v_3 \). Suppose \( d_G(v_1, v_2) = 2 \) and let \( p \in N_G(v_1) \cap N_G(v_2) \). Then \( p \in N^2_G(v_3) \) and \( v_3 \in N^2_G(v_1) \cup N^2_G(v_2) \). \( \hat{S}^* = (p, v_1, v_2, v_3) \). Then \( \hat{S}^* \) is a total hop dominating set. Moreover, observe that \( v_3 \in N^2_G(p) \), \( v_2 \in N^2_G(v_1) \setminus N^2_G(p) \), \( v_1 \in [N^2_G(v_2) \setminus (N^2_G(v_1) \cup N^2_G(p))] \) and \( p \in [N^2_G(v_3) \setminus (N^2_G(v_2) \cup N^2_G(v_1) \cup N^2_G(p))] \). Hence, \( \hat{S}^* \) is a legal open hop neighborhood sequence, and so \( \hat{S}^* \) is a Grundy total hop dominating sequence of \( G \), contrary to our assumption that \( \gamma_{gr}^h(G) = 3 \). Therefore, \( \gamma_{gr}^h(G) \geq 4 \).

For tightness of the bounds, consider \( G = C_4 \) and \( H = P_8 \). Then \( \gamma_{gr}^h(G) = 4 \) and \( \gamma_{gr}^h(H) = 8 \).

\textbf{Theorem 4.} Let \( G \) be a connected graph of order \( n \) such that \( \gamma(G) \neq 1 \). If \( n = 2m \), \( m \geq 2 \) and the vertices of \( G \) can be labeled as \( u_1, \ldots, u_m, v_1, \ldots, v_m \) in such a way that

\begin{enumerate}[(i)]
  \item \( d_G(u_i, v_i) = 2 \) for each \( i \),
  \item \( \{u_1, \ldots, u_m\} \) is a hop independent set of \( G \), and
  \item \( d_G(u_i, v_j) = 2 \) implies that \( i \geq j \),
\end{enumerate}

then \( \gamma_{gr}^h(G) = n \).

Proof. Suppose the vertices of \( G \) can be labeled as described. Clearly, \( \hat{S} = \{u_1, \ldots, u_m, v_1, \ldots, v_1\} \) is a total hop dominating set of \( G \). Observe that \( v_i \in N^2_G(u_i) \setminus \bigcup_{j=1}^{i-1} N^2_G(u_j) \) for each \( i \in \{2, \ldots, m\} \) by (i) and (iii) and \( u_m \in N^2_G(v_m) \setminus \bigcup_{j=1}^{m-1} N^2_G(v_j) \) by (i) and (ii). Now, for any \( k < m \),

\[ u_k \in N^2_G(v_k) \setminus \left[ \bigcup_{j=1}^{m} N^2_G(u_j) \right] \cup \bigcup_{i=m}^{k+1} N^2_G(v_i) \]

by (i), (ii), and (iii).

Therefore, \( S \) is a Grundy total hop dominating sequence, showing that \( \gamma_{gr}^h(G) = n \). \( \square \)

\textbf{Proposition 1.} Let \( n \) and \( m \) be any positive integers such that \( n \geq 4 \). Then

\[ \gamma_{gr}^h(P_n) = \begin{cases} 
  n - 2 & \text{if } n = 4m + 2 \\
  n - 1 & \text{if } n \geq 5 \text{ and odd} \\
  n & \text{if } n = 4m 
\end{cases} \]

Proof. Let \( P_n = [v_1, v_2, \ldots, v_n] \). Clearly, \( \gamma_{gr}^h(P_6) = 4 \). For \( n = 4m + 2 \geq 10 \), let \( S = (v_1, v_2, \ldots, v_{n-2}) \). Clearly, \( S \) is a Grundy total hop dominating sequence of \( P_n \). Thus,
\[ \gamma_{gr}^h(P_n) \geq n - 2. \] On the other hand, let \( S' = (w_1, w_2, \ldots, w_k) \) be a Grundy total hop dominating sequence of \( P_n \). Notice that one of the vertices \( v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1} \) is not in \( S' \). Suppose all the vertices \( v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1} \) are in \( S' \). If \( v_5 \) comes before \( v_1 \), then \( N_{P_n}^2(v_1) \subset N_{P_n}^2(v_5) \), a contradiction. So, \( v_5 \) comes after \( v_1 \). Next, suppose \( v_9 \) comes before \( v_5 \), then \( N_{P_n}^2(v_5) \subset N_{P_n}^2(v_9) \). Thus, \( v_9 \) comes after \( v_5 \). Continuing in this manner, we find that the following order of appearance (not necessarily consecutive) of the given vertices in the Grundy total hop dominating sequence \( S' \): \( v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1} \). However, \( N_{P_n}^2(v_n-1) \subset N_{P_n}(v_n-5) \). Hence, \( v_{n-1} \notin S' \), a contradiction. Similarly, one of the vertices \( v_2, v_6, v_{10}, \ldots, v_{n-4}, v_n \) is not in \( S' \). Therefore, \( \gamma_{gr}^h(P_n) = k \leq n - 2 \). Consequently, \( \gamma_{gr}^h(P_n) = n - 2 \) for all \( n = 4m + 2 \).

Next, let \( n \geq 5 \) and odd. Clearly, \( \gamma_{gr}^h(P_5) = 4 \). Suppose \( n \geq 7 \) and odd. For \( n \in \{7, 11, 15, \ldots\} \), let

\[ \gamma_{gr}^h(P_n) \geq n - 1. \]

Next, for \( n \in \{9, 13, 17, \ldots\} \), let

\[ \gamma_{gr}^h(P_n) \geq n - 1. \]

Suppose \( \gamma_{gr}^h(P_n) = n \), say \( S_0 = (w_1, w_2, \ldots, w_n) \) is a Grundy total hop dominating sequence of \( P_n \). Observe that for \( n \in \{7, 11, 15, \ldots\} \), one of the vertices \( v_2, v_6, v_{10}, \ldots, v_{n-4}, v_n \) is not in \( S_0 \). Suppose all vertices \( v_2, v_6, v_{10}, \ldots, v_{n-4}, v_n \) are in \( S_0 \). If \( v_6 \) comes before \( v_2 \), then \( N_{P_n}^2(v_2) \subset N_{P_n}^2(v_6) \), a contradiction. So, \( v_6 \) comes after \( v_2 \). Next, suppose \( v_{10} \) comes before \( v_6 \), then \( N_{P_n}^2(v_6) \subset N_{P_n}^2(v_2) \cup N_{P_n}^2(v_{10}) \), a contradiction. Thus, \( v_{10} \) comes after \( v_6 \). Continuing in this manner, we find that the following order of appearance (not necessarily consecutive) of the given vertices in the Grundy total hop dominating sequence \( S_0 \): \( v_2, v_6, v_{10}, \ldots, v_{n-4}, v_n \). However, \( N_{P_n}^2(v_n) \subset N_{P_n}(v_{n-4}) \). Hence, \( v_n \notin S_0 \), a contradiction. Similarly, for \( n \in \{9, 13, 17, \ldots\} \), one of the vertices \( v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1} \) is not in the Grundy total hop dominating sequence, say \( S'' \). Thus, \( \gamma_{gr}^h(P_n) \leq n - 1 \). Consequently, \( \gamma_{gr}^h(P_n) = n - 1 \) for all \( n \geq 5 \) and odd.

Lastly, assume that \( n = 4m \). Clearly, \( \gamma_{gr}^h(P_4) = 4 \). For \( n = 4m \geq 8 \), let

\[ C = (v_1, v_2, v_5, v_6, \ldots, v_{n-3}, v_{n-2}, v_{n-1}, v_n, v_{n-5}, v_{n-4}, \ldots, v_3, v_4). \]

Then \( C \) is a Grundy total hop dominating sequence of \( P_n \). Thus, \( \gamma_{gr}^h(P_n) = n \) for all \( n = 4m \).  \( \square \)
Proposition 2. Let $G$ be a graph of order $n$ with $\gamma(C) \neq 1$ for each component $C$ of $G$. If $|N^2_G(v)| \geq m$ for every $v \in V(G)$, then $\gamma_{gr}^th(G) \leq n - (m - 1)$.

Proof. Suppose $\gamma_{gr}^th(G) = k$, say $S = (w_1, w_2, \ldots, w_k)$ is a Grundy total hop dominating sequence of $G$. Assume $w_1 = v_i$ for some $i \in \{1, \ldots, n\}$. Then $|N^2_G(v_1)| = |N^2_G(v_i)| \geq m$ for some $i \in \{1, \ldots, n\}$. It follows that there are at most $n - m$ remaining vertices of $G$ that could be hop footprinted by the next terms of $S$. Therefore,

$$\gamma_{gr}^th(G) = k \leq n - m + |\{v_i\}| = n - m + 1 = n - (m - 1).$$

The next result follows from Proposition 2.

Corollary 2. Let $G$ be a graph of order $n$ with $\gamma(C) \neq 1$ for each component $C$ of $G$. If $|N^2_G(u)| \geq 2$ for every $u \in V(G)$, then $\gamma_{gr}^th(G) \leq n - 1$.

Proposition 3. Let $n$ and $m$ be any positive integers such that $n \geq 4$. Then

$$\gamma_{gr}^th(C_n) = \begin{cases} 
4 & \text{if } n = 4, 5, 6 \\
6 & \text{if } n = 8 \\
n - 4 & \text{if } n = 4m \geq 12 \\
n - 2 & \text{if } n = 4m + 2 \geq 10 \\
n - 1 & \text{if } n \geq 7 \text{ and odd}
\end{cases}$$

Proof. Clearly for $n = 4, 5, 6$ and $n = 8$, $\gamma_{gr}^th(C_n) = 4$ and $\gamma_{gr}^th(C_n) = 6$, respectively. For $n = 4m \geq 12$, let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Observe that $S = (v_1, v_2, \ldots, v_{n-4})$ is a Grundy total hop dominating sequence of $C_n$. Thus, $\gamma_{gr}^th(C_n) \geq n - 4$. On the other hand, let $S' = (w_1, w_2, \ldots, w_k)$ be a Grundy total hop dominating sequence of $C_n$. Notice that one of the vertices $v_1, v_2, v_3, \ldots, v_{n-7}, v_{n-3}$ is not in $S'$. Suppose all the vertices $v_1, v_5, v_9, \ldots, v_{n-7}, v_{n-3}$ are in $S'$. WLOG, assume that $w_1 = v_1$. If $v_9$ comes before $v_9$, then $N^2_{C_n}(v_9) \subseteq N^2_{C_n}(v_1) \cup N^2_{C_n}(v_9)$, a contradiction. So, $v_9$ comes after $v_9$. Next, suppose $v_{13}$ comes before $v_9$, then $N^2_{C_n}(v_{13}) \subseteq N^2_{C_n}(v_1) \cup N^2_{C_n}(v_9) \cup N^2_{C_n}(v_{13})$, a contradiction. Thus, $v_{13}$ comes after $v_9$. Continuing in this manner, we find that the following order of appearance (not necessarily consecutive) of the given vertices in the Grundy total hop dominating sequence $S'$: $v_1, v_5, v_9, \ldots, v_{n-7}, v_{n-3}$. However, $N^2_{C_n}(v_{n-3}) \subseteq N^2_{C_n}(v_1) \cup N_{C_n}(v_{n-7})$. Hence, $v_{n-3} \notin S'$, a contradiction. Similarly, one of the vertices $v_2, v_6, v_{10}, \ldots, v_{n-6}, v_{n-2}, v_3, v_7, v_{11}, \ldots, v_{n-5}, v_{n-1}$, and $v_4, v_8, v_{12}, \ldots, v_{n-4}, v_n$, respectively, is not in $S'$. Therefore, $\gamma_{gr}^th(P_n) = k \leq n - 4$. Consequently, $\gamma_{gr}^th(P_n) = n - 4$ for all $n = 4m$.

Next, for $n = 4m + 2 \geq 10$, let

$$S_1 = (v_1, v_5, \ldots, v_{n-5}, v_{n-1}, v_3, v_7, \ldots, v_{n-7}, v_2, v_6, \ldots, v_{n-4}, v_n, v_4, v_8, \ldots, v_{n-6}).$$

Then $S_1$ is a Grundy total hop dominating sequence of $C_n$. Hence, $\gamma_{gr}^th(C_n) \geq n - 2$. On the other hand, let $S = (w_1, w_2, \ldots, w_k)$ be a Grundy total hop dominating sequence of $C_n$. Then applying the same arguments with the first part, one can show that one of the
Let \( v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1}, v_3, v_7, \ldots, v_{n-7}, v_{n-3} \) and \( v_2, v_6, v_{10}, \ldots, v_{n-4}, v_n, v_4, v_8, \ldots, v_{n-6}, v_{n-2} \) is not in \( \hat{S}_1 \), respectively. Hence, \( \gamma_{gr}^h(C_n) = k \leq n - 2 \). Consequently, \( \gamma_{gr}^h(C_n) = n - 2 \).

Let \( n \geq 7 \) and odd. Clearly, \( \gamma_{gr}^h(C_7) = 6 \). Suppose \( n \geq 9 \) and odd. For \( n \in \{9, 13, 17, \ldots\} \), let

\[
S_2 = (v_1, v_5, \ldots, v_n, v_4, \ldots, v_{n-2}, v_2, \ldots, v_{n-7}).
\]

Then \( S_2 \) is a Grundy total hop dominating sequence of \( C_n \). Hence, \( \gamma_{gr}^h(C_n) \geq n - 1 \). Next, for \( n \in \{11, 15, 19, \ldots\} \), let

\[
S_3 = (v_1, v_5, \ldots, v_{n-2}, v_2, \ldots, v_{n-1}, v_3, \ldots, v_n, v_4, \ldots, v_{n-7}).
\]

Then \( S_3 \) is a Grundy total dominating sequence of \( C_n \). Hence, \( \gamma_{gr}^h(C_n) \geq n - 1 \). On the other hand, since \( |N(C_n(v))| = 2 \) for every \( v \in V(C_n) \), it follows that \( \gamma_{gr}^h(C_n) \leq n - 1 \) by Corollary 2. Therefore, \( \gamma_{gr}^h(C_n) = n - 1 \).

**Theorem 5.** Let \( G \) be a graph of order \( n \) with \( \gamma(C) \neq 1 \) for each component \( C \) of \( G \). Then \( \gamma_{gr}^h(G) \leq 2\gamma_{gr}^h(G) \).

**Proof.** Let \( S = (v_1, v_2, \ldots, v_k) \) be a Grundy total hop dominating sequence of \( G \), where \( k = \gamma_{gr}^h(G) \). We will prove that at most \( k/2 \) vertices can be removed from \( S \) in such a way the resulting sequence \( S' \) forms a legal closed hop neighborhood sequence of \( G \). Notice that a vertex \( v_i \in \hat{S} \) prevents \( S \) from being a legal closed hop neighborhood sequence only if \( N_G(v_i) \setminus \bigcup_{j=1}^{k-1} N_G(v_j) = \emptyset \) for each \( i \in \{1, \ldots, k\} \). Since \( S \) is a Grundy total dominating sequence, \( v_i \) hop footprinted only vertices from \( S \) that precedes \( v_i \).

That is, \( h_{\hat{S}}^{-1}(v_i) \subseteq \{v_1, \ldots, v_{i-1}\} \), where \( h_{\hat{S}} : V(G) \rightarrow \hat{S} \) is a hop footprinter function, mapping each vertex to its hop footprinter. Set \( T = \{v_i \in \hat{S} : h_{\hat{S}}^{-1}(v_i) \subseteq \{v_1, \ldots, v_{i-1}\}\} \). Since \( v_1 \notin T \), \( T \neq \hat{S} \). Suppose that \( h_{\hat{S}}^{-1}(v_j) \cap T \neq \emptyset \) for some \( v_j \in T \). Let \( v_{\ell} \in h_{\hat{S}}^{-1}(v_j) \cap T \). Since \( v_j \in T \), the vertex \( v_i \) that is hop footprinted by \( v_j \) satisfies \( i < j \). Since \( v_j \in T \), \( v_i \) hop footprints some vertex \( v_{t} \), where \( t < i \). This means that \( h_{\hat{S}}(v_{\ell}) = v_i \), where \( 1 \leq \ell < i - 1 \).

It follows that \( v_j \notin \bigcup_{k=1}^{i-1} h_{\hat{S}}^{-1}(v_k) \), that is, \( h_{\hat{S}}^{-1}(v_j) \neq v_j \), contrary to the assumption that \( v_j \in h_{\hat{S}}^{-1}(v_j) \). Therefore, \( h_{\hat{S}}^{-1}(v_j) \cap T = \emptyset \) for every vertex \( v_j \in T \). Now, suppose that \( v_i, v_j \in T \), where \( i < j \). By definition,

\[
h_{\hat{S}}^{-1}(v_i) \subseteq \{v_1, \ldots, v_{i-1}\} \text{ and } h_{\hat{S}}^{-1}(v_j) \subseteq \{v_1, \ldots, v_{j-1}\}.
\]

Since every vertex is hop footprinted by a unique vertex in \( \hat{S} \), it follows that \( h_{\hat{S}}^{-1}(v_i) \cap h_{\hat{S}}^{-1}(v_j) = \emptyset \). Since \( h_{\hat{S}}^{-1}(v_j) \cap T = \emptyset \) for every vertex \( v_j \in T \), \( \{h_{\hat{S}}^{-1}(v_i) : v_i \in T\} \) forms a partition of a subset of \( \hat{S} \setminus T \). Note that for each \( v_i \in T \), \( |h_{\hat{S}}^{-1}(v_i)| \geq 1 \), and so \( |T| \leq \bigcup_{v_i \in T} h_{\hat{S}}^{-1}(v_i) \leq |\hat{S}| - |T| \) implying that \( 2|T| \leq |\hat{S}| \). Hence, \( |T| \leq \frac{k}{2} \). Let \( S' \) be a sequence obtained from \( S \) by deleting vertices from \( T \). Then \( S' \) is a legal closed hop neighborhood sequence of \( G \). Thus, \( \gamma_{gr}^h(G) \geq |S'| = k - |T| \geq k - \frac{k}{2} = \frac{k}{2} = \frac{1}{2}\gamma_{gr}^h(G) \).

Consequently, \( \gamma_{gr}^h(G) \leq 2\gamma_{gr}^h(G) \).
Remark 1. The bound given in Theorem 5 is tight.

To see this, consider $C_4$ in Fig. 1. Let $S = (u_1, u_2, u_3, u_4)$. Then $S$ is a $\gamma_{gr}^{th}$-sequence of $C_4$. Thus, $\gamma_{gr}^{th}(C_4) = 4$. Next, let $S^* = (u_1, u_2)$. Then $S^*$ is a $\gamma_{gr}^{th}$-sequence of $C_4$. Hence, $\gamma_{gr}^{th}(C_4) = 2$. Consequently, $\gamma_{gr}^{th}(C_4) = 4 = 2\gamma_{gr}^{th}(C_4)$.

Remark 2. Let $G$ be a graph of order $n$ with $\gamma(C) \neq 1$ for each component $C$ of $G$. Then $\gamma_{gr}^{th}(G) \geq \gamma_{gr}^{th}(G)$ does not hold in general.

To see this, consider $K_5 \circ K_2$ in Fig. 2. Let $S = (v_1, v_2, \ldots, v_{10})$. Then $S$ is a $\gamma_{gr}^{th}$-sequence of $K_5 \circ K_2$, that is, $\gamma_{gr}^{th}(K_5 \circ K_2) = 10$. Next, let $S' = (v_1, v_{12}, v_3, v_{13})$. Then $S'$ is a $\gamma_{gr}^{th}$-sequence of $K_5 \circ K_2$. Thus, $\gamma_{gr}^{th}(K_5 \circ K_2) = 4$.

Lemma 1. Let $G$ be a non-trivial connected graph and let $G_1$ and $G_2$ be two copies of $G$ in the graph $S(G)$. If $v \in V(G_1)$ and $v' \in V(G_2)$ is the corresponding vertex of $v$, then
Remark 3. The bound given in Theorem 6 is tight. Moreover, strict inequality can also be attained.

For equality, consider $C_4$. Then $\gamma^{th}(C_4) = 4$. Now, consider the shadow graph of $C_4$ given in Fig. 3. Let $S = \langle a, a', b, b' \rangle$. Observe that $S$ is a Grundy total hop dominating sequence of $\hat{S}(C_4)$. Moreover, it can be verified that $\gamma^{th}(S(C_4)) = 4$. Consequently, $\gamma^{th}(S(C_4)) = 4 = \gamma^{th}(S(C_4))$. 

Theorem 6. Let $G$ be a graph of order $n$ with $\gamma(C) \neq 1$ for each component $C$ of $G$. If $S$ is a Grundy total hop dominating sequence of $G_1$ or $G_2$, then $S$ is a Grundy total hop dominating sequence of $S(G)$. Moreover, $\gamma^{th}(G) \leq \gamma^{th}(S(G))$.

Proof. Let $G_1$ and $G_2$ be two copies of $G$. Let $S = \langle v_1, v_2, \ldots, v_k \rangle$ be a Grundy total hop dominating sequence in $G_1$ and let $v' \in V(G_2)$. Then

$$N^2_{G_1}(v_i) \setminus \bigcup_{j=1}^{i-1} N^2_{G_1}(v_j) \neq \emptyset \text{ for each } i \in \{2, 3, \ldots, k\}.$$ 

Thus, by Lemma 1

$$N^2_{\hat{S}(G)}(v_i) \setminus \bigcup_{j=1}^{i-1} N^2_{\hat{S}(G)}(v_j)$$

$$= \left[ N^2_{G_1}(v_i) \cup N^2_{G_2}[v'_i] \right] \setminus \bigcup_{j=1}^{i-1} \left[ N^2_{G_1}(v_j) \cup N^2_{G_2}[v'_j] \right]$$

$$= \left[ (N^2_{G_1}(v_i) \cup N^2_{G_2}[v'_i]) \setminus \bigcup_{j=1}^{i-1} N^2_{G_1}(v_j) \right] \cup \left[ (N^2_{G_1}(v_i) \cup N^2_{G_2}[v'_i]) \setminus \bigcup_{j=1}^{i-1} N^2_{G_2}[v'_j] \right]$$

$$\neq \emptyset \text{ for each } i \in \{2, 3, \ldots, k\}.$$ 

Since $\hat{S}$ is a total hop dominating set of $G_1$, there exists $w \in \hat{S} \cap N^2_{G_1}(v)$. By Lemma 1, $w \in \hat{S} \cap N^2_{\hat{S}(G)}(v')$, i.e., $w \in \hat{S}$ and $d_{S(G)}(v', w) = 2$. Consequently, $\hat{S}$ is a Grundy total hop dominating sequence of $S(G)$. 


For strict inequality, consider $P_5$. Then $\gamma_{gr}^{th}(P_5) = 4$. Now, consider the shadow graph of $P_5$ given in Fig. 4. Let $S = (a, a', e, e', d, d')$. Observe that $S$ is a Grundy total hop dominating sequence of $S(P_5)$. Moreover, it can be verified that $\gamma_{gr}^{th}(S(P_5)) = 6$. Hence, $\gamma_{gr}^{th}(S(P_5)) = 4 < 6 = \gamma_{gr}^{th}(S(P_5))$.

**Lemma 2.** Let $G$ be a graph of order $n$ with no isolated vertices. If $|N_G(v)| \geq l$ for every $v \in V(G)$, then $\gamma_{gr}^t(G) \leq n - (l - 1)$.

**Proof.** Suppose $\gamma_{gr}^t(G) = k$, say $S = (w_1, w_2, \ldots, w_k)$ is a Grundy total dominating sequence of $G$. Assume $w_1 = v_i$ for some $i \in \{1, \ldots, n\}$. Then $|N_G(w_1)| = |N_G(v_i)| \geq l$ for some $i \in \{1, \ldots, n\}$. It follows that there are at most $n - l$ remaining vertices of $G$ that could be footprinted by the next terms of $S$. Therefore,

$$\gamma_{gr}^t(G) = k \leq n - l + |\{v_i\}| = n - l + 1 = n - (l - 1).$$

**Proposition 4.** Let $n \geq 4$ be any positive integer. Then

$$\gamma_{gr}^t(P_n) = 4 = \gamma_{gr}^t(C_n).$$
**Theorem 7.** Let \( G \) and \( H \) be any two graphs such that \( \gamma(G) \neq 1 \) and \( \gamma(H) \neq 1 \). A sequence \( S \) of distinct vertices of \( G + H \) is a legal open hop neighborhood sequence if and only if one of the following holds:

(i) \( S \) is a co-legal open neighborhood sequence in \( G \) (legal open neighborhood sequence in \( G \)).

(ii) \( S \) is a co-legal open neighborhood sequence in \( H \) (legal open neighborhood sequence in \( H \)).

(iii) \( S \) is loh-identical to \( S' = S_G \oplus S_H \), where \( S_G \) and \( S_H \) are co-legal open neighborhood sequences in \( G \) and \( H \), respectively.
Proof. Suppose that \( S = (u_1, \ldots, u_t) \) is a legal open hop neighborhood sequence in \( G + H \) and let \( \hat{S} \) be the corresponding set of \( S \). Suppose further that \( \hat{S} \subseteq V(G) \). Then by the legality condition in \( S \), we have

\[
N^2_{G+H}(u_i) \setminus \bigcup_{j=1}^{t-1} N^2_{G+H}(u_j) \neq \emptyset \quad \text{for each } i \in \{2, 3, \ldots, t\}.
\]

Since \( N^2_{G+H}(u_i) = V(G) \setminus N_G[u_i] \) for each \( i \in [t] \), it follows that

\[
[V(G) \setminus N_G[u_i]] \setminus \bigcup_{j=1}^{t-1}[V(G) \setminus N_G[u_j]] \neq \emptyset \quad \text{for each } i \in \{2, 3, \ldots, t\}.
\]

Hence, \( S \) is a co-legal open neighborhood sequence in \( G \), and so (i) holds. Similarly, (ii) holds if \( \hat{S} \subseteq V(H) \).

Next, suppose that \( \hat{S} \cap V(G) \neq \emptyset \) and \( \hat{S} \cap V(H) \neq \emptyset \). Since \( N^2_{G+H}(u_j) \subseteq V(G) \) for all \( u_j \in \hat{S} \cap V(G) \) and \( N^2_{G+H}(u_s) \subseteq V(H) \) for all \( u_s \in \hat{S} \cap V(H) \), \( S \) is loh-identical to \( S' = S_G \oplus S_H \), where \( S_G = \hat{S} \cap V(G) = \{u_1, u_2, \ldots, u_m\} \), \( S_H = \hat{S} \cap V(H) = \{w_1, w_2, \ldots, w_t\} \), \( |S| = m + t \), and the orders of appearances of the terms of both sequences in \( S \) are retained.

Since \( S \) is a legal open hop neighborhood sequence, it follows that

\[
[V(G) \setminus N_G[u_i]] \setminus \bigcup_{j=1}^{t-1}[V(G) \setminus N_G[u_j]] = N^2_{G+H}(u_i) \setminus \bigcup_{j=1}^{t-1} N^2_{G+H}(u_j) \neq \emptyset
\]

for each \( i \in \{2, 3, \ldots, m\} \). Thus, \( S_G \) is a co-legal open neighborhood sequence in \( G \). Similarly, \( S_H \) is a co-legal open neighborhood sequence in \( H \), and so (iii) holds.

The converse is clear.

The next result follows from Lemma 3 and Theorem 7.

**Corollary 3.** Let \( G \) and \( H \) be any two graphs such that \( \gamma(G) \neq 1 \) and \( \gamma(H) \neq 1 \). A sequence \( S \) of distinct vertices of \( G + H \) is a Grundy total hop dominating sequence in \( G + H \) if and only if \( S \) is loh-identical to \( S' = S_G \oplus S_H \), where \( S_G \) and \( S_H \) are co-Grundy total dominating sequences in \( G \) and \( H \), respectively (Grundy total dominating sequences in \( G \) and \( H \), respectively). Moreover,

\[
\gamma_{gr}^{th}(G + H) = \gamma_{cogr}^{t}(G) + \gamma_{cogr}^{t}(H) = \gamma_{gr}^{t}(G) + \gamma_{gr}^{t}(H).
\]

In particular, we have

(i) \( \gamma_{gr}^{th}(K_{m,n}) = \gamma_{gr}^{th}(\overline{K}_m + \overline{K}_n) = \gamma_{gr}^{t}(K_m) + \gamma_{gr}^{t}(K_n) = 4 \) for any \( m, n \geq 2 \),

(ii) \( \gamma_{gr}^{th}(\overline{K}_m + P_m) = \gamma_{gr}^{t}(K_m) + \gamma_{gr}^{t}(P_m) = 6 \) for any \( n \geq 2 \) and \( m \geq 4 \),

(iii) \( \gamma_{gr}^{th}(\overline{K}_n + C_m) = \gamma_{gr}^{t}(K_n) + \gamma_{gr}^{t}(C_m) = 6 \) for any \( n \geq 2 \) and \( m \geq 4 \),

(iv) \( \gamma_{gr}^{th}(P_n + P_m) = \gamma_{gr}^{t}(\overline{P}_n) + \gamma_{gr}^{t}(\overline{P}_m) = 8 \) for any \( n, m \geq 4 \),

(v) \( \gamma_{gr}^{th}(P_n + C_m) = \gamma_{gr}^{t}(\overline{P}_n) + \gamma_{gr}^{t}(\overline{C}_m) = 8 \) for any \( n, m \geq 4 \),

(vi) \( \gamma_{gr}^{th}(C_n + C_m) = \gamma_{gr}^{t}(\overline{C}_n) + \gamma_{gr}^{t}(\overline{C}_m) = 8 \) for any \( n, m \geq 4 \).
Theorem 8. Let $G$ be a non-trivial connected graph on $n$ vertices and let $H$ be any graph such that $\gamma(H) \neq 1$. Then $\gamma^{th}_{gr}(G \circ H) \geq n \cdot \gamma_{cogr}(H) = n \cdot \gamma^{t}_{gr}(\overline{H})$.

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let $S_{u_i} = (w_{u_i}^1, w_{u_i}^2, \ldots, w_{u_i}^k)$ be a co-Grundy total dominating sequence in $H^{u_i}$ for each $i \in [n]$, where $k = \gamma_{cogr}(H)$. Let $\hat{S} = S_{u_1} \oplus S_{u_2} \oplus \cdots \oplus S_{u_n}$. Let $v \in V(G \circ H) \setminus \hat{S}$ and let $u_t \in V(G)$ such that $v \in V(u_t + H^{u_t})$ for some $t \in [n]$. Suppose first that $v = u_t$. Let $u_s \in N_G(u_t)$ and pick any $w_{u_s}^d \in \hat{S}_{u_s}$ for some $s \in [n]$. Then $w_{u_s}^d \in \hat{S} \cap N^2_{GoH}(u_t)$. Suppose $v \neq u_t$. Then $v \in V(H^{u_t}) \setminus \hat{S}_{u_t}$. Since $\hat{S}_{u_t}$ is a co-Grundy total dominating sequence in $H^{u_t}$, it follows that there exists $w_{u_t}^d \in \hat{S}_{u_t} \subseteq \hat{S}$ such that $d_{H^{u_t}}(v, w_{u_t}^d) \neq 1$. It follows that $d_{GoH}(v, w_{u_t}^d) = 2$. Therefore, $\hat{S}$ is a total hop dominating set in $G \circ H$. Now, we relabel the terms in $\hat{S}$, say $\hat{S} = (v_1, v_2, \cdots, v_k, \cdots, v_{nk})$. Let $i \in [nk] \setminus \{1\}$ and let $v_i = w_{u_t}^d$ for some $r \in [n]$ and $t \in [k]$. Then

$$N^2_{GoH}(v_i) \setminus \bigcup_{j=1}^{i-1} N^2_{GoH}(v_j) = N^2_{GoH}(w_{u_t}^d) \setminus \bigcup_{s=1}^{j} N^2_{GoH}(w_{u_s}^d) \cup \{N^2_{GoH}(w_{u_t}^d) : p \in [k] \text{ and } 1 \leq q \leq r-1\}.$$  

If $t = 1$, then $N^2_{GoH}(w_{u_t}^d) \setminus \bigcup_{s=1}^{j} N^2_{GoH}(w_{u_s}^d) = N^2_{GoH}(w_{u_t}^d)$. Clearly, $w_{u_t}^d \in N^2_{GoH} w_{u_t}^d \setminus \bigcup \{N^2_{GoH}(w_{u_s}^d) : p \in [k] \text{ and } 1 \leq q \leq r-1\}$.

Suppose $t \neq 1$. Since $S_{u_t}$ is a co-legal open neighborhood sequence in $H^{u_t}$,

$$N^2_{GoH}(w_{u_t}^d) \setminus \bigcup_{s=1}^{j} N^2_{GoH}(w_{u_s}^d) = [V(H^{u_t}) \setminus N_{H^{u_t}}(w_{u_t}^d)] \setminus \bigcup_{s=1}^{j} [V(H^{u_t}) \setminus N_{H^{u_t}}(w_{u_s}^d)] \neq \emptyset.$$  

Observe that

$$N^2_{GoH}(w_{u_t}^d) \setminus \bigcup_{s=1}^{j} N^2_{GoH}(w_{u_s}^d) \cap \bigcup \{N^2_{GoH}(w_{u_i}^d) : p \in [k] \text{ and } 1 \leq q \leq r-1\} = \emptyset.$$  

Hence, $N^2_{GoH}(v_i) \setminus \bigcup_{j=1}^{i-1} N^2_{GoH}(v_j) \neq \emptyset$ for all $i \in [nk] \setminus \{1\}$ and so $\hat{S}$ is a Grundy total hop dominating sequence in $G \circ H$. Consequently,

$$\gamma^{th}_{gr}(G \circ H) \geq |\hat{S}| = \sum_{i=1}^{n} |\hat{S}_{v_i}| = n \cdot \gamma_{cogr}(H) = n \cdot \gamma^{t}_{gr}(\overline{H}).$$  

Remark 4. The bound given in Theorem 8 is tight.

To see this, consider the graph $K_5 \circ P_4$ in Fig. 5. Let $S = (a_1, a_2, \cdots, a_{20})$. Then $S$ is a Grundy total hop dominating sequence of $K_5 \circ P_4$. Moreover, it can be verified that $\gamma^{th}_{gr}(K_5 \circ P_4) = 20$. Since $\gamma^{t}_{gr}(\overline{P_4}) = 4$, the assertion follows.
Figure 5: A graph $K_5 \circ P_4$ with $\gamma_{\text{hop}}(K_5 \circ P_4) = |K_5|\gamma_{\text{gr}}(P_4)$.

Acknowledgements

The authors would like to thank the referees for the comments and suggestions they made in the initial manuscript. Also, special thanks must go to the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP)-Philippines, MSU-Iligan Institute of Technology (Philippines), and MSU-Tawi-Tawi College of Technology and Oceanography (Philippines) for funding this research.

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