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Argument Estimates of Certain Analytic Functions Associated with a Family of Multiplier Transformations

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Abstract. The purpose of the present paper is to derive some inclusion properties and argument estimates of certain normalized analytic functions in the open unit disk, which are defined by means of a class of multiplier transformations. Furthermore, the integral preserving properties in a sector are investigated for these multiplier transformations.

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1. Introduction

Let A denote the class of the functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$ written symbolically as follows:

$$f \prec g \ (z \in U) \text{ or } f(z) \prec g(z) \ (z \in U),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalent (cf., e.g., [2]; see also [10], [11, p. 4])

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0) \ \text{ and } \ f(U) \subset g(U).$$

Many essentially equivalent definitions of multiplier transformation have been given in literature (see [4], [5], and [20]). In [3] Catas defined the operator $I^m(\lambda, \ell)$ as follows:

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Definition 1. *[3] Let the function $f(z) \in A$. for $m \in N_0 = N \cup \{0\}$, where $N = \{1, 2, \dots\}$, $\lambda \geq 0$, $\ell \geq 0$. The extended multiplier transformation $I^m(\lambda, \ell)$ on A is defined by the following infinite series:*

$$I^m(\lambda,\ell)f(z) \quad = \quad z + \sum_{k=2}^{\infty} \left[\frac{\ell + 1 + \lambda(k-1)}{\ell + 1} \right]^m a_k z^k \tag{2}$$
$$(f \quad \in \quad A; \lambda \geq 0; \ell \geq 0; m \in N_0; z \in U).$$

We can write (2) as follows:

$$I^m(\lambda,\ell)f(z) = (\Phi_{\lambda,\ell}^{,m} * f)(z), \tag{3}$$

where

$$\Phi_{\lambda,\ell}^{m}(z) = z + \sum_{k=2}^{\infty} \left[\frac{\ell + 1 + \lambda(k-1)}{\ell + 1} \right]^m z^k.$$

It is easily verified from (2), that

$$\lambda z (I^m(\lambda,\ell)f(z))' = (1+\ell)I^{m+1}(\lambda,\ell)f(z) - [1-\lambda+\ell]I^m(\lambda,\ell)f(z) \; (\lambda > 0). \tag{4}$$

We note that:

$$I^0(\lambda,\ell)f(z) = f(z) \text{ and } I^1(1,0)f(z) = zf'(z).$$

Also by specializing the parameters λ, ℓ and m we obtain the following operators studied by various authors:

(i) $I^m(1,\ell) = I^m(\ell)f(z)$ (see Cho and Srivastava [4] and Cho and Kim [5]);

(ii) $I^m(\lambda,0)f(z) = D_\lambda^m f(z)$ (see AL-Oboudi [1]);

(iii) $I^m(1,0) = D^m f(z)$ (see Salagean [18]);

(iv) $I^m(1,1) = I^m f(z)$ (see Uralegaddi and Somanatha [20]).

Let the functions $g_1, ..., g_q$ be in the class A. Then we say that the functions $g_1, ..., g_q$ are in the class $\Omega_{m,\lambda,\ell}(q; A, B)$ if they satisfy the subordination condition:

$$\frac{z(I^m(\lambda,\ell)g_i(z))'}{\left(\frac{1}{q}\right) \sum\limits_{j=1}^{q} I^m(\lambda,\ell)g_j(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in U; i = 1, ..., q; -1 \leq B < A \leq 1), \tag{5}$$

where

$$\sum_{j=1}^{n} \frac{1}{z} I^m(\lambda,\ell)g_j(z) \neq 0 \quad (z \in U).$$

For $\lambda = 1$, $m = \ell = 0$ and

$$g_j(z) = w^{-j} f(w^j z) \quad (f \in A; j = 1, ..., q; w = e^{2\pi i/n}),$$

$\Omega_{m,\lambda,\ell}(q;A,B)$ reduces to the class of starlike functions in U with respect to q symmetric points [12] (see also [17]). If we take $\lambda = 1,\ \ell = 0,\ m = 0,\ q = 2,\ A = 1$ and $B = -1$ in (5), then we obtain the class of mutually adjoint close-to-convex functions in U considered by Lewandowski and Stankiewicz [9].

Let $C_{m,\lambda,\ell}(q;A,B)$ be the class of functions of functions $f \in A$ satisfying the argument inequality

$$\left|\arg\left(\frac{z(I^m(\lambda,\ell)f(z))'}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q} I^m(\lambda,\ell)g_j(z)}\right)\right| < \frac{\pi}{2}\alpha \tag{6}$$

$$(z \in U, m \in N_0, 0 < \alpha \le 1; g_j \in \Omega_{m,\lambda,\ell}(q;A,B); j = 1,...,q)\ .$$

If we take $m = \ell = 0,\ \lambda = 1,\ q = 1,\ \alpha = 1,\ A = 1$ and $B = -1$ in (6), $C_{m,\lambda,\ell}(q;A,B)$ becomes the familiar class of close-to-convex functions in U introduce by Kaplan [8]. Further, for $m = \ell = 0,\ \lambda = 1,\ q = 2,\ \alpha = 1,\ A = 1$ and $B = -1$, $C_{m,\lambda,\ell}(q;A,B)$ covers the class of close-to-convex functions in U with respect to symmetric points studied by Das and Singh [6].

In this present paper, we give some argument properties and estimates of analytic functions belonging to A, which contain the basic inclusion relationships among the classes $\Omega_{m,\lambda,\ell}(q;A,B)$ and $C_{m,\lambda,\ell}(q;A,B)$. The integral preserving properties in connection with the operator $I^m(\lambda,\ell)$ defined by (2) are also considered.

2. The Main Results And Their Consequences

Unless otherwise mentioned,we shall assume in the reminder of this paper that $\lambda > 0, \ell \ge 0$ and $m \in N_0$.

In proving our main results, we need the following lemmas.

Lemma 1. *[7] Let h be convex univalent in U with $h(0) = 1$ and*

$$R(\beta h(z) + \gamma) > 0 \quad (z \in U; \beta, \gamma \in \mathbb{C})\ .$$

If p is analytic in U with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (z \in U)\ ,$$

implies that

$$p(z) \prec h(z) \quad (z \in U)\ .$$

Lemma 2. *[10] Let h be convex univalent in U and ϕ be analytic in U with*

$$R(\phi(z)) \ge 0 \quad (z \in U)\ .$$

implies that

$$p(z) \prec h(z) \quad (z \in U)\ .$$

Lemma 3. *[14] Let p be analytic in U with $p(0)=1$ and $p(z)\neq 0$ in U. If there exists two points $z_1, z_2 \in U$ such that*

$$\frac{-\pi}{2}\alpha_1 = \arg(p(z_1)) < \arg(p(z)) < \arg(p(z_2)) = \frac{\pi}{2}\alpha_2 , \tag{7}$$

for some α_1 and α_2 $(\alpha_1, \alpha_2 > 0)$ and for all z $\left(|z| < |z_1| = |z_2|\right)$, then

$$\frac{z_1 p^{'}(z_1)}{p(z_1)} = -i\left(\frac{\alpha_1+\alpha_2}{2}\right) m \ \ and \ \ \frac{z_2 p^{'}(z_2)}{p(z_2)} = i\left(\frac{\alpha_1+\alpha_2}{2}\right) m , \tag{8}$$

where

$$m \geq \frac{1-|a|}{1+|a|} \quad and \quad a = i\tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\right) \tag{9}$$

First of all, with the help of Lemma 1 and 2, we obtain the following.

Proposition 1. *If $g_1, ..., g_q \in \Omega_{m+1,\lambda,\ell}(q;A,B)$, then $g_1, ..., g_q \in \Omega_{m,\lambda,\ell}(q;A,B)$.*

Proof. Let

$$p_i(z) = \frac{z(I^m(\lambda,\ell)g_i(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q} I^m(\lambda,\ell)g_j(z)} \quad (i=1,...,q) . \tag{10}$$

By using the identity (4), we get

$$\frac{1}{q}\sum_{j=1}^{q}(I^m(\lambda,\ell)g_j(z))p_i(z) + \frac{1-\lambda+\ell}{\lambda}(I^m(\lambda,\ell)g_i(z)) = \frac{1+\ell}{\lambda}(I^{m+1}(\lambda,\ell)g_i(z)) . \tag{11}$$

Differentiating both sides of (11) with respect to z, and simplifying, we obtain

$$p_i(z) + \frac{zp_i^{'}(z)}{\left(\frac{1}{q}\right)\sum\limits_{i=1}^{q} p_i(z) + \frac{1-\lambda+\ell}{\lambda}} = \frac{z(I^{m+1}(\lambda,\ell)g_i(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q} I^{m+1}(\lambda,\ell)g_j(z)} \prec \frac{1+Az}{1+Bz} \equiv h(z),$$

$$(z\in U; i=1,...,q) , \tag{12}$$

$g_1, ..., g_q \in \Omega_{m+1,\lambda,\ell}(q;A,B)$. Since h is convex, for any $z_0 \in U$, there exists a point $\zeta_0 \in U$ such that

$$X(z_0) + \frac{z_0 X^{'}(z_0)}{X(z_0) + \frac{1-\lambda+\ell}{\lambda}} = h(\zeta_0) ,$$

where

$$X(z) = \frac{1}{q}\sum_{i=1}^{q} p_i(z) .$$

Then we find from Lemma 1 that $X \prec h$. Applying Lemma 2 with

$$\phi(z) = \frac{1}{X(z) + \frac{1-\lambda+\ell}{\lambda}}$$

to (12) again, we find that $p_i \prec h$ for all $i (i = 1, ..., q)$. Next, we prove that

$$\sum_{j=1}^{q} \frac{1}{z} I^m(\lambda,\ell) g_j(z) \neq 0 \quad (z \in U) .$$

Since $g_1, ..., g_q \in \Omega_{m+1,\lambda,\ell}(q; A, B)$ and h is convex, we find that there exists a point $\zeta_0 \in U$ such that, for any $z_0 \in U$,

$$r(z_0) = \frac{z_0 \left(\sum\limits_{j=1}^{q} I^{m+1}(\lambda,\ell) g_j(z_0) \right)'}{\sum\limits_{j=1}^{q} I^{m+1}(\lambda,\ell) g_j(z_0)} = h(\zeta_0) ,$$

and hence, $r \prec h$. We note also that

$$\sum_{j=1}^{q} I^m(\lambda,\ell) g_j(z) = \frac{\frac{1-\lambda+\ell}{\lambda} + 1}{z^{(1-\lambda+\ell)/\lambda}} \int_0^z t^{\frac{1-\lambda+\ell}{\lambda} - 1} \sum_{j=1}^{q} I^{m+1}(\lambda,\ell) g_j(t) dt .$$

Thus, by applying Lemma A of [12], we conclude that

$$\sum_{j=1}^{q} \frac{1}{z} I^m(\lambda,\ell) g_j(z) \neq 0 \quad (z \in U) .$$

This evidently completes the proof of Proposition 1.

Proposition 2. *If $g_1, ..., g_q \in \Omega_{m,\lambda,\ell}(q; A, B)$, then $F_c(g_1), ..., F_c(g_q) \in \Omega_{m,\lambda,\ell}(q; A, B)$, where F_c is the integral operator defined by*

$$F_c(g_i) = F_c(g_i)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} g_i(t) dt \quad (i = 1, ..., q, c \geq 0) . \tag{13}$$

Proof. From (13), we have

$$z(I^m(\lambda,\ell) F_c(g_i)(z))' = (c+1) I^m(\lambda,\ell) g_i(z) - c I^m(\lambda,\ell) F_c(g_i)(z). \tag{14}$$

Let

$$p_i(z) = \frac{z(I^m(\lambda,\ell) F_c(g_i)(z))'}{\left(\frac{1}{q}\right) \sum\limits_{j=1}^{q} I^m(\lambda,\ell) F_c(g_j)(z)} \quad (i = 1, ..., q) .$$

Then by using (14), we obtain

$$\left(\frac{1}{q}\right)\sum_{j=1}^{q}(I^m(\lambda,\ell)F_c(g_j)(z))p_i(z)+cI^m(\lambda,\ell)F_c(g_i)(z)=(c+1)I^{m+1}(\lambda,\ell)g_i(z)\,. \qquad (15)$$

Differentiating both sides of (15) with respect to z, and simplifying, we obtain

$$p_i(z)+\frac{zp_i^{'}(z)}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}p_i(z)+c}=\frac{z(I^m(\lambda,\ell)g_i(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}I^m(\lambda,\ell)g_j(z)}.$$

Then, by the same arguments as in the proof of Proposition 1, it follows that Proposition 2 holds true as stated.

Remark 1.

(i) Putting $m=\ell=0,\lambda=1$ and $g_i(z)=w^{-1}f(w^iz)(f\in A;i=1,...,q;w=e^{\frac{2\pi i}{q}})$ in Proposition 2, we obtain the result obtained by Mocanu [12];

(ii) Putting $m=\ell=0,\lambda=1,q=2,g_1(z)=f(z)$, and $g_2(z)=-f(-z)$ in Proposition 2, we obtain the result obtained by Padmanabhan and Thangamani [16], which (in turn) includes the result given by Das and Singh [6] as a special case.

Next, we prove the following theorem.

Theorem 1. *Let $f\in A$ and $0<\delta_1,\ \delta_2\le 1$. If*

$$-\frac{\pi}{2}\delta_1<\arg\left(\frac{z(I^{m+1}(\lambda,\ell)f(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}I^{m+1}(\lambda,\ell)g_j(z)}\right)<\frac{\pi}{2}\delta_2\,,$$

where $g_1,...,g_q\in\Omega_{m+1,\lambda,\ell}(q;A,B)$, then

$$-\frac{\pi}{2}\alpha_1<\arg\left(\frac{z(I^{m}(\lambda,\ell)f(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}I^{m}(\lambda,\ell)g_j(z)}\right)<\frac{\pi}{2}\alpha_2\,,$$

where α_1 and α_2 $(0<\alpha_1,\ \alpha_2\le 1)$ are the solutions of the following equations:

$$\delta_1=\begin{cases}\alpha_1+\left(\frac{2}{\pi}\right)\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\left(\frac{\pi}{2}\right)t_1}{2\left(\frac{1+A}{1+B}+\frac{1-\lambda+\ell}{\lambda}\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\left(\frac{\pi}{2}\right)t_1}\right) & (B\neq-1)\,,\\ \alpha_1 & (B=-1)\,,\end{cases} \qquad (16)$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \left(\frac{2}{\pi}\right)\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\left(\frac{\pi}{2}\right)t_1}{2\left(\frac{1+A}{1+B}+\frac{1-\lambda+\ell}{\lambda}\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\left(\frac{\pi}{2}\right)t_1}\right) & (B \neq -1)\,, \\ \alpha_2 & (B = -1)\,, \end{cases} \tag{17}$$

a being given by (9), and

$$t_1 = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB+\frac{1-\lambda+\ell}{\lambda}(1-B^2)}\right)\,. \tag{18}$$

Proof. Let

$$p(z) = \frac{z(I^m(\lambda,\ell)f(z))'}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q} I^m(\lambda,\ell)g_j(z)} \quad \text{and} \quad Q(z) = \frac{1}{q}\sum_{i=1}^{q} Q_i(z)\,,$$

where

$$Q_i(z) = \frac{z(I^m(\lambda,\ell)g_i(z))'}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q} I^m(\lambda,\ell)g_j(z)} \qquad (i = 1,...,q)\,.$$

using (10) with g_i replaced by f, we have

$$\left(\frac{1}{q}\right)\sum_{j=1}^{q}(I^m(\lambda,\ell)g_j(z))p(z) + \frac{1-\lambda+\ell}{\lambda}I^m(\lambda,\ell)f(z) = \frac{1+\ell}{\lambda}I^{m+1}(\lambda,\ell)f(z)\,. \tag{19}$$

Differentiating (19) with respect to z, and simplifying, we obtain

$$\frac{z(I^{m+1}(\lambda,\ell)f(z))'}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q} I^{m+1}(\lambda,\ell)g_j(z)} = p(z) + \frac{zp'(z)}{Q(z)+\frac{1-\lambda+\ell}{\lambda}}\,.$$

Since $g_1,...,g_n \in \Omega_{m+1,\lambda,\ell}(q;A,B)$, by Proposition 1, we know that $g_1,...,g_q \in \Omega_{m,\lambda,\ell}(q;A,B)$, and so

$$Q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in U; -1 \le B < A \le 1)\,.$$

Hence, we observe from [19] that

$$\left|Q(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \qquad (z \in U; B \neq -1)\,, \tag{20}$$

and

$$\mathrm{Re}(Q(z)) > \frac{1-A}{2} \qquad (z \in U; B = -1)\,. \tag{21}$$

Then, by using (20) and (21), we have

$$Q(z)+\frac{1-\lambda+\ell}{\lambda}=\rho e^{\phi\pi i/2},$$

where

$$\frac{1-A}{1-B}+\frac{1-\lambda+\ell}{\lambda}<\rho<\frac{1+A}{1+B}+\frac{1-\lambda+\ell}{\lambda}\ ,$$
$$-t_1<\phi<t_1 \quad (B\neq -1)\,,$$

t_1 being given by (18), and

$$\frac{1-A}{2}+\frac{1-\lambda+\ell}{\lambda}<\rho<\infty$$
$$-1<\phi<1 \quad (B=-1)\,.$$

We note that p is analytic in U with $p(0)=1$. Let h be the function which maps U onto the angular domain

$$\left\{\phi : -\frac{\pi}{2}\delta_1<\arg(\phi)<\frac{\pi}{2}\delta_2\right\}, \text{ with } h(0)=1.$$

Applying Lemma 1 for this h with

$$\phi(z)=\frac{1}{Q(z)+\frac{1-\lambda+\ell}{\lambda}}\ ,$$

we see that

$$R(p(z))>0 \quad (z\in U)\,,$$

and hence, $p(z)\neq 0$ in U.

If there exist two points z_1 and z_2 in U such that condition (7) is satisfied, then (By Lemma 3) we obtain (8) under restriction (9). For the case $B\neq -1$, we first obtain

$$\begin{aligned}
&\arg\left(p(z_1)+\frac{z_1p^{'}(z_1)}{Q(z_1)+\frac{1-\lambda+\ell}{\lambda}}\right)\\
=\ & -\frac{\pi}{2}\alpha_1+\arg\left(1-i\frac{\alpha_1+\alpha_2}{2}m\left(\rho e^{\phi\pi i/2}\right)^{-1}\right)\\
\leq\ & -\frac{\pi}{2}\alpha_1-\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)m\sin\left(\frac{\pi}{2}\right)(1-\phi)}{2\rho+(\alpha_1+\alpha_2)m\cos\left(\frac{\pi}{2}\right)(1-\phi)}\right)\\
\leq\ & -\frac{\pi}{2}\alpha_1-\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\left(\frac{\pi}{2}\right)t_1}{2\left(\frac{1+A}{1+B}+\frac{1-\lambda+\ell}{\lambda}\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\left(\frac{\pi}{2}\right)t_1}\right)\\
=\ & -\frac{\pi}{2}\delta_1
\end{aligned}$$

and

$$\begin{aligned}
&\arg\left(p(z_2)+\frac{z_2p^{'}(z_2)}{Q(z_2)+\frac{1-\lambda+\ell}{\lambda}}\right)\\
\geq\ & \frac{\pi}{2}\alpha_2+\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\left(\frac{\pi}{2}\right)t_1}{2\left(\frac{1+A}{1+B}+\frac{1-\lambda+\ell}{\lambda}\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\left(\frac{\pi}{2}\right)t_1}\right)\\
=\ & \frac{\pi}{2}\delta_2\ ,
\end{aligned}$$

where we have used inequality (9), δ_1, δ_2 and t_1 being given by (16), (17), and (18), respectively. Similarly, for the case $B=-1$, we have

$$\arg\left(p(z_1)+\frac{z_1p^{'}(z_1)}{Q(z_1)+\frac{1-\lambda+\ell}{\lambda}}\right)\leq-\frac{\pi}{2}\alpha_1$$

and

$$\arg\left(p(z_2)+\frac{z_2p^{'}(z_2)}{Q(z_2)+\frac{1-\lambda+\ell}{\lambda}}\right)\geq\frac{\pi}{2}\alpha_2\ .$$

These obviously contradict the assumption of Theorem 1. The proof of Theorem 1 is thus completed.

Putting $\delta_1=\delta_2=\delta$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let $f\in A$ and $0<\delta\leq 1$. If*

$$\left|\arg\left(\frac{z(I^{m+1}(\lambda,\ell)f(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}I^{m+1}(\lambda,\ell)g_j(z)}\right)\right|<\frac{\pi}{2}\delta\ ,$$

where $g_1,...,g_q\in\Omega_{m,\lambda,\ell}(q;A,B)$, then

$$\left|\arg\left(\frac{z(I^{m}(\lambda,\ell)f(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}I^{m}(\lambda,\ell)g_j(z)}\right)\right|<\frac{\pi}{2}\alpha\ ,$$

where α $(0<\alpha\leq 1)$ is the solution of the equation

$$\delta=\begin{cases}\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\left(\frac{\pi}{2}\right)t_1}{\left(\frac{1+A}{1+B}+\frac{1-\lambda+\ell}{\lambda}\right)+\alpha\sin\left(\frac{\pi}{2}\right)t_1}\right) & (B\neq-1)\\ \alpha & (B=-1)\ ,\end{cases}$$

t_1 being given by (18).

From Corollary 1, we immediately obtain the following corollary.

Corollary 2. *The inclusion relation*

$$C_{m+1,\lambda,\ell}(q;A,B) \subset C_{m,\lambda,\ell}(q;A,B)$$

holds true for any integer m.

Remark 2. *For $m = \ell = 0, \lambda = 1, q = 1, \delta = 1, A = 1$ and $B = -1$, the class $C_{m,\lambda,\ell}(q;A,B)$ reduces to the class of quasiconvex functions in U introduced by Sakaguchi [17] (see also [13]). Hence, we see from Corollary 2 that every quasiconvex function in U is close-to-convex in U.*

Next, we prove the following theorem.

Theorem 2. *Let $f \in A$, $0 < \delta_1,\ \delta_2 \leq 1$ and $c \geq 0$. If*

$$-\frac{\pi}{2}\delta_1 < \arg\left(\frac{z(I^m(\lambda,\ell)f(z))'}{\left(\frac{1}{q}\right)\sum_{j=1}^{q} I^m(\lambda,\ell)g_j(z)}\right) < \frac{\pi}{2}\delta_2\,,$$

where $g_1, ..., g_q \in \Omega_{m,\lambda,\ell}(q;A,B)$, then

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z(I^m(\lambda,\ell)F_c(f)(z))'}{\left(\frac{1}{q}\right)\sum_{j=1}^{q} I^m(\lambda,\ell)F_c(g_j)(z)}\right) < \frac{\pi}{2}\alpha_2\,,$$

where F_c is the integral operator defined by (13), and α_1 and α_2 $(0 < \alpha_1,\ \alpha_2 \leq 1)$ are the solutions of the following equations:

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\left(\frac{\pi}{2}\right)t_2}{2\left(\frac{1+A}{1+B}+c\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\left(\frac{\pi}{2}\right)t_2}\right) & (B \neq -1) \\ \alpha_1 & (B = -1)\,, \end{cases}$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\left(\frac{\pi}{2}\right)t_2}{2\left(\frac{1+A}{1+B}+c\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\left(\frac{\pi}{2}\right)t_2}\right) & (B \neq -1)\,, \\ \alpha_1 & (B = -1)\,, \end{cases}$$

a being given by (9) and t_2 being the same as t_1 given by (18) with $c = \dfrac{1-\lambda+\ell}{\lambda}$.

Proof. Let

$$p(z)=\frac{z(I^m(\lambda,\ell)F_c(f)(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}I^m(\lambda,\ell)F_c(g_j)(z)} \quad \text{and} \quad Q(z)=\frac{1}{q}\sum_{k=1}^{q}Q_k(z)\ ,$$

$$Q_k(z)=\frac{z(I^m(\lambda,\ell)F_c(g_k)(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}I^m(\lambda,\ell)F_c(g_j)(z)}.$$

Using the relationship (14), we obtain

$$\left(\frac{1}{q}\right)\sum_{j=1}^{q}(I^m(\lambda,\ell)F_c(g_j)(z))p(z)+cI^m(\lambda,\ell)F_c(f)(z)=(c+1)I^m(\lambda,\ell)f(z)\ . \tag{22}$$

Differentiating (22) with respect to z, and simplifying, we get

$$\frac{z(I^m(\lambda,\ell)f(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}I^m(\lambda,\ell)g_j(z)}=p(z)+\frac{zp^{'}(z)}{Q(z)+c}\ .$$

Since $g_1,...,g_q\in\Omega_{m,\lambda,\ell}(q;A,B)$, by Proposition 2, we have $F_c(g_1),...,F_c(g_q)\in\Omega_{m,\lambda,\ell}(q;A,B)$. Hence, we find that

$$Q(z)\prec\frac{1+Az}{1+Bz}\qquad(z\in U;-1\leq B<A\leq 1)\ .$$

The remaining part of the proof is similar to that in the proof of Theorem 1, and so we omit the details involved.

Putting $\delta_1=\delta_2$ in Theorem 2 we obtain the following corollary.

Corollary 3. *Let* $f\in A,\ 0<\delta\leq 1$ *and* $c\geq 0$. *If*

$$\left|\arg\left(\frac{z(I^m(\lambda,\ell)f(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}I^m(\lambda,\ell)g_j(z)}\right)\right|<\frac{\pi}{2}\delta\ ,$$

where $g_1,...,g_q\in\Omega_{m,\lambda,\ell}(q;A,B)$*, then*

$$\left|\arg\left(\frac{z(I^m(\lambda,\ell)F_c(f)(z))^{'}}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q}I^m(\lambda,\ell)F_c(g_j)(z)}\right)\right|<\frac{\pi}{2}\alpha\ ,$$

where α $(0 < \alpha \leq 1)$ is the solution of the following equation

$$\delta = \begin{cases} \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\left(\frac{\pi}{2}\right)t_2}{\left(\frac{1+A}{1+B}+c\right)+\alpha\sin\left(\frac{\pi}{2}\right)t_2}\right) & (B \neq -1)\,, \\ \alpha & (B = -1)\,, \end{cases}$$

t_2 being the same as t_1 given by (18) with $c = \dfrac{1-\lambda+\ell}{\lambda}$.

From Corollary 3, we readily obtain the following corollary.

Corollary 4. *Let $f \in C_{m,\lambda,\ell}(q;A,B)$. Then $F_c(f) \in C_{m,\lambda,\ell}(q;A,B)$, where F_c is the integral operator defined by (13).*

Remark 3. *From Theorem 2 or Corollary 4, we see that every function in $C_{m,\lambda,\ell}(q;A,B)$ preserves the angles under the integral operator defined by (13). If we put $m = \ell = 0,\ \lambda = 1,\ q = 2,\ A = 1$ and $B = -1$ in Corollary 4, we are easily led to the result given earlier by Das and Singh [6].*

Finally, we state Theorem 3 below. The proof is much akin to that of Theorem 1, and so that details may be omitted.

Theorem 3. *Let $f \in A,\ 0 < \delta_1,\ \delta_2 \leq 1$ and $\gamma \geq 0$. If*

$$-\frac{\pi}{2}\delta_1 < \arg\left(\gamma\frac{z(I^{m+1}(\lambda,\ell)f(z))'}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q} I^{m+1}(\lambda,\ell)g_j(z)} + (1-\gamma)\frac{z(I^{m}(\lambda,\ell)f(z))'}{\left(\frac{1}{q}\right)\sum\limits_{j=1}^{q} I^{m}(\lambda,\ell)g_j(z)}\right) < \frac{\pi}{2}\delta_2\,,$$

where $g_1, ..., g_q \in \Omega_{m+1,\lambda,\ell}(q;A,B)$, then

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z(I^{m}(\lambda,\ell)f(z))'}{\frac{1}{q}\sum\limits_{j=1}^{q} I^{m}(\lambda,\ell)g_j(z)}\right) < \frac{\pi}{2}\alpha_2\,,$$

where α_1 and α_2 are the solutions of the following equation:

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\gamma\cos\left(\frac{\pi}{2}\right)t_1}{2\left(\frac{1+A}{1+B}+\frac{1-\lambda+\ell}{\lambda}\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\left(\frac{\pi}{2}\right)t_1}\right) & (B \neq -1)\,, \\ \alpha_1 & (B = -1)\,, \end{cases}$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\gamma\cos\left(\frac{\pi}{2}\right)t_1}{2\left(\frac{1+A}{1+B}+\frac{1-\lambda+\ell}{\lambda}\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\gamma\sin\left(\frac{\pi}{2}\right)t_1}\right) & (B \neq -1)\,, \\ \alpha_2 & (B = -1)\,, \end{cases}$$

a and t_1 being given by (9) and (18), respectively.

Remark 4. *For $m = \ell = 0$, $\lambda = 1$, $q = 2$, $A = 1$, $B = -1$ and $\delta = 1$, Theorem 3 reduces at once to the result given earlier by Padmanabhan and Thangamani [15].*

Remark 5. *Putting $\lambda = 1$ in the above results, we obtain the results obtained by Cho and Srivastava [5].*

Remark 6. *Putting $\ell = 0$ in the above results, we obtain the corresponding results for the operator D_λ^m.*

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