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Certified Perfect Domination in Graphs

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Abstract. Let $G = (V, E)$ be a simple connected graph. A set $S \subseteq V(G)$ is called a certified perfect dominating set of G if every vertex $v \in V(G) \setminus S$ is dominated by exactly one element $u \in S$, such that u has either zero or at least two neighbors in $V(G) \setminus S$. The minimum cardinality of a certified perfect dominating set of G is called the certified perfect domination number of G and denoted by $\gamma_{cerp}(G)$. A certified perfect dominating set S of G with $|S| = \gamma_{cerp}(G)$ is called a γ_{cerp}-set. In this paper, the author focuses on several key aspects: a characterization of the certified perfect dominating set, determining the exact values of the certified perfect domination number for specific graphs, and investigating the certified perfect domination number of graphs resulting from the join and corona of graphs. Furthermore, the relationship between the perfect dominating set, and the certified perfect dominating set of a graph G are established.

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Key Words and Phrases: Certified domination, perfect domination, certified perfect domination

1. Introduction

In 1970, dominating sets were investigated in the context of social networks, where vertices represented individuals and edges represented relationships between them. The concept of dominating set helps to identify key individuals who could exert influence or control over the entire network. However, the concept quickly found applications in various other fields, including computer science, operations research, and biology.

In 2018, Dettlaff et. al, introduced the concept of certified dominating set in a graphs. Therein, they presented the exact values of the certified domination number for some classes of graphs as well as provided some upper bounds on this parameter for arbitrary graphs. They then characterised a wide class of graphs with equal domination and certified domination numbers and characterise graphs with large values of certified domination numbers [4]. Moreover, several authors investigated further in this concept. They obtained the certified domination number of Cartesian product, Corona product of some standard graphs and special Subdivision graphs of certain families of graphs (see [9] [1], [8] and [10]).

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In 1990, Livingston M. et.al, investigated the concept of perfect dominating set of a graph. They studied the existence and construction of perfect dominating sets in families of a graphs arising from the interconnection networks of parallel computers. These include trees, dags, series-parallel graphs, meshes, tori, hypercubes, cube-connected cycles, cube-connected paths, and de Bruijn graphs [7]. In 2014, Kwon Y.S. et. al., got some results related to perfect domination sets of Cayley graphs. They showed that if a Cayley graph $C(\mathcal{A}, X)$ has a perfect dominating set S which is a normal subgroup of $\mathcal{A}$ and whose induced subgraph is F, then there exists an F-bundle projection $p : C(\mathcal{A}, X) \to K_m$ for some positive integer m [6]. Moreover, numerous classes of graphs have been investigated to explore variations and parameters of the perfect dominating set (see [3], [11] and [2]).

Now, let us consider a scenario, denoted as G, where we are given a set of officials, denoted as $S \subseteq V(G)$, and a set of civilians, denoted as $H = V(G) \setminus S$. For each civilian, represented by $x \in H$, it is necessary to have precisely one official, denoted as $u \in S$, who can serve that civilian. Furthermore, whenever such an official u serves a civilian x, there must exist another civilian, denoted as $y \in H$, who only observes the service provided by the official u to civilian x. In other words, y acts as a witness, ensuring that there is no abuse or misconduct from official u.

The question arises: What is the minimum number of officials required to guarantee such a service, considering a given social network? This problem leads us to introduce the concept of a certified perfect dominating set of a graph G.

2. Terminology and Notation

This section comprises essential definitions required for the study.

Let $G = (V, E)$, where V represents the *vertex set* of G and E represents the *edge set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of V is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of E is the *size* of G. The *degree* of a vertex v, denoted as $\deg(v)$, refers to the number of edges incident with v. The *maximum degree* among all vertices in G is denoted as $\Delta(G)$. The *open neighborhood* of a vertex u in G is the set of its neighboring vertices and is denoted as $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of u in G is the open neighborhood of u along with the vertex u itself, expressed as $N_G[u] = N_G(u) \cup \{u\}$. Similarly, the *closed neighborhood of a subset* S of $V(G)$, denoted as $N_G[S] = \cup_{v \in S} N_G[v]$, represents the set of vertices in G that are either in S or are adjacent to a vertex in S.

The *join* of two graphs G and H, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of graphs G and H, $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$

corresponding to the join $\langle\{v\}\rangle + H^v$, $v \in V(G)$ [5].

A set $S \subseteq V(G)$ is called *dominating set* if $N_G[S] = V(G)$. A dominating set S is a minimal dominating set if no proper subset $S' \subset S$ is a dominating set. A minimum cardinality of a dominating set of G is called *domination number* of G, and is denoted by $\gamma(G)$. A dominating set S with $|S| = \gamma(G)$ is called a γ-set.

A dominating set $S \subseteq V(G)$ is called *certified dominating set* of G if every vertex $v \in S$ has either zero or at least two neighbors in $V(G) \setminus S$. A minimum cardinality of a certified dominating set of G is called *certified domination number* of G and denoted by $\gamma_{cer}(G)$. A certified dominating set of S with $|S| = \gamma_{cer}(G)$ is called a γ_{cer}-set [4].

A set $S \subseteq V(G)$ is called *perfect dominating set* if every vertex $v \in V(G) \setminus S$ is dominated by exactly one element in S. The minimum cardinality of a perfect dominating set of G is called *perfect domination number*, and is denoted by $\gamma_p(G)$. A perfect dominating set S with $|S| = \gamma_p(G)$ is said to be a γ_p-set [7].

A perfect dominating set $S \subseteq V(G)$ is called *certified perfect dominating set* of G if every $u \in S$ has either zero or at least two neighbors in $V(G) \setminus S$. A minimum cardinality of a certified perfect dominating set of G is called *certified perfect domination number* of G and denoted by $\gamma_{cerp}(G)$. A certified perfect dominating set S of G with $|S| = \gamma_{cerp}(G)$ is called a γ_{cerp}-set.

Example 1. *Consider the three subsets S_1, S_2, and S_3 of the graph G shown in Figure 1. First, let $S_1 = \{a, e, g, j, n, r\} \subseteq V(G)$ be a first dominating set of G. Observe that all vertices $a, e, g, j, n, r \in S_1$ have at least two neighbors in $V(G) \setminus S_1$. Thus, S_1 is a certified dominating set of G, and $\gamma_{cer}(G) = |S_1| = 6$. However, S_1 is not a perfect dominating set since there exist vertices $b, c, d \in V(G) \setminus S_1$ dominated by two vertices $a, e \in S$, and vertex $f \in V(G) \setminus S_1$ is dominated by $e, f \in S_1$. Therefore, S_1 is not a certified perfect dominating set of G.*

Second, let $S_2 = \{a, b, g, j, n, r\} \subseteq V(G)$ be a second dominanting set of G. Observe that all vertices in $V(G) \setminus S_2$ are dominated by exactly one vertex in S_2. Thus, S_2 is a perfect dominating set of G, and $\gamma_p(G) = |S_2| = 6$. However, S_2 is not a certified dominating set of G since there exists vertex $b \in S_2$ that has only one neighbor in $V(G) \setminus S_2$. Therefore, S_2 is not a perfect certified dominating set of G.

Lastly, let $S_3 = \{a, b, c, d, e, f, g, j, n, r\} \subseteq V(G)$ be a third dominating set of G. Observe that all vertices in $V(G) \setminus S_3$ are dominated by exactly one vertex in S_3, and all vertices in S_3 have either zero or at least two neighbors in $V(G) \setminus S_3$. Therefore, S_3 is a certified perfect dominating set of G, and $\gamma_{cerp}(G) = |S_3| = 10$.

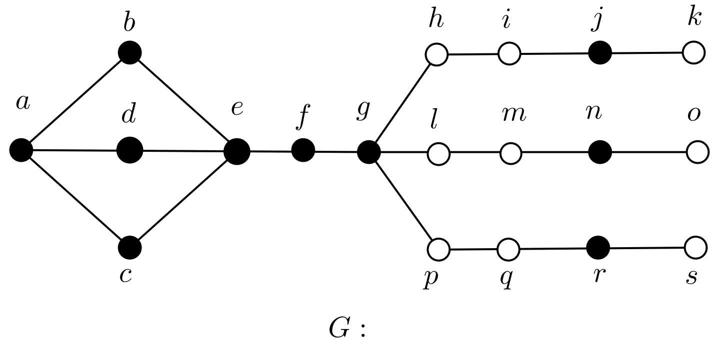

Figure 1: Graph G with $\gamma_{cerp}(G) = 10$

3. Main Results

Proposition 1. *Let G be a connected graph of order n. Then every support vertex of G belongs to every certified perfect dominating set of G.*

Proof. Assume that S is a certified perfect dominating set of G. Let u be a support vertex of G, and v be a leaf adjacent to u. If $u \notin S$, then $v \in S$. However, since v would have only one neighbor in $V(G) \setminus S$, S is not a certified dominating set of G. Therefore, we can conclude that S cannot be a certified perfect dominating set. This contradicts the initial assumption that S is a certified perfect dominating set. □

Theorem 1. *For any graph G of order n, $\gamma_p(G) \leq \gamma_{cp}(G) \leq n$.*

Proof. Let G be a graph of order n. Let us first show its upper bounds. Let S_1 be a perfect dominasting set, and S_2 be a certified perfect dominating set of G. Since every certified perfect dominating set $S_1 \subseteq S_2$, $\gamma_p(G) \leq \gamma_{cerp}(G)$. So, we are left to show its lower bounds. Since every certified dominating set $S_2 \subseteq V(G)$, $\gamma_{cerp}(G) \leq |V(G)| = n$. Therefore, the assertion holds. □

Theorem 2. *Let a and b positive integers with $1 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_p(G) = a$ and $\gamma_{cerp}(G) = b$.*

Proof. Consider the following cases:
Case 1: $a = b$
Let G_1 be the graph shown in Figure 2. Let $S = \{x_1, x_2, \ldots, x_{c-1}, x_c, \} \subseteq V(G_1)$. Then S is both γ_p-set and γ_{cerp}-set of G. Therefore, $a = \gamma_p(G_3) = \gamma_{cerp}(G_3) = b$.

Case 2: $a < b$.
Let G_2 be the graph shown in Figure 3 and Figure 4 for $\gamma_p(G_2)$ and $\gamma_{cerp}(G_2)$, respectively. Let $n = b-a$ and $a = c+n$ with $c \geq 2$ and $n \geq 2$. Let $S = \{x_1, x_2, \ldots, x_{c-1}, x_c, y_1, y_2, \ldots, y_n\}$ and $S^* = S \cup \{z_1, z_2, \ldots, z_n\}$. Hence, $\gamma_p(G_2) = |S| = c + n = a$ and $\gamma_{cerp}(G_2) = |S^*| = c + n + n = a + n = b$.

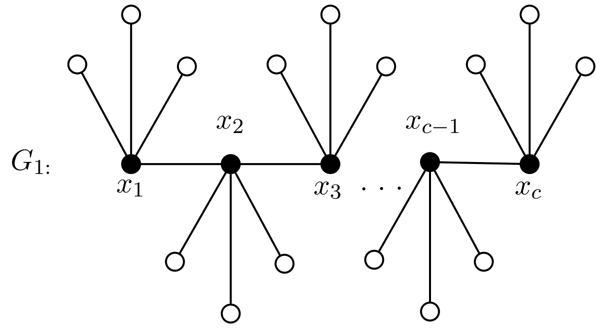

Figure 2: Graph G_1 with $\gamma_p(G_1) = \gamma_{cerp}(G_1)$

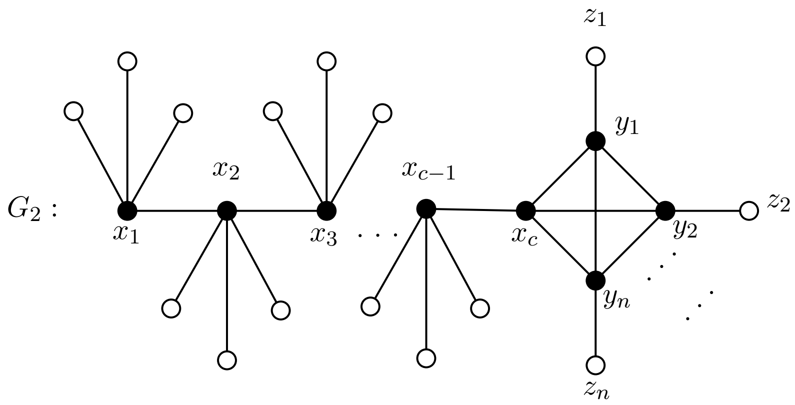

Figure 3: Graph G_2 with $\gamma_p(G_2) = a$

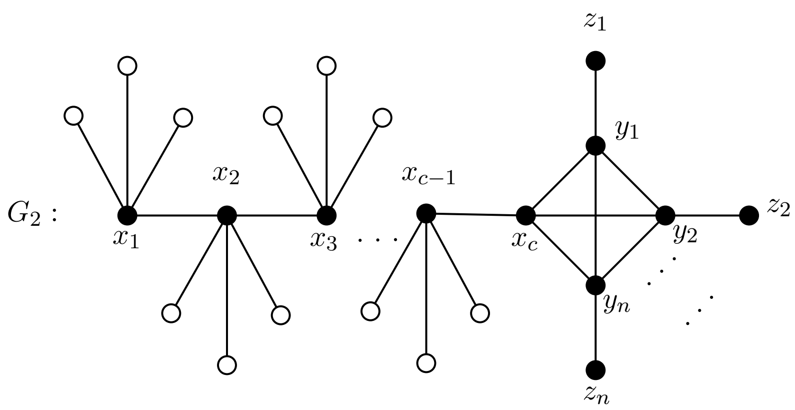

Figure 4: Graph G_2 with $\gamma_{cerp}(G_2) = b$

Consequently, The statement is substantiated by this evidence. Therefore, this completes the proof. $\square$

Theorem 3. *Let G be a graph with components $G_1, G_2, \ldots, G_k$, where $k \geq 2$. Then*

$$\gamma_{cp}(G) = \sum_{i=1}^{k} \gamma_{cp}(G_i).$$

Proof. Let S_i be a γ_{cerp}-set of G_i for each $i \in \{1, 2, \ldots, k\}$. Then $S = \bigcup_{i=1}^{k} S_i$ forms a γ_{cerp}-set of G. Therefore, we have

$$\gamma_{cerp}(G) \leq |S| = \sum_{i=1}^{k} |S_i| = \sum_{i=1}^{k} \gamma_{cerp}(G_i).$$

Conversely, suppose S^* is a γ_{cerp}-set of G. For each $i \in \{1, 2, \ldots, k\}$, let $S_i^* = S^* \cap V(G_i)$. Since S^* is a γ_{cerp}-set of G, S_i^* is a γ_{cerp}-set of G_i for each $i \in \{1, 2, \ldots, k\}$. This implies that

$$\gamma_{cerp}(G) = |S^*| = \sum_{i=1}^{k} |S_i^*| \geq \sum_{i=1}^{k} \gamma_{cerp}(G_i).$$

Therefore, $\gamma_{cerp}(G) = \sum_{i=1}^{k} \gamma_{cerp}(G_i)$. □

Theorem 4. *For a path P_n of order $n \geq 1$,*

$$\gamma_{cerp}(P_n) = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod 3; \\ n & \text{otherwise.} \end{cases}$$

Proof. Suppose that $V(P_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ such that $deg(v_1) = deg(v_n) = 1$ and $deg(v_i) = 2$ for each $i \in \{2, 3, \ldots, n-1\}$. Consider the following cases:

Case 1. Suppose that $n \equiv 0 \pmod 3$. Suppose that $n = 3$. Then $\gamma_{cerp}(P_3) = 1 = \frac{3}{3}$. Suppose that $n > 3$. Let $q = \frac{n}{3}$ and $r \in \{1, 2, \ldots, q-1, q\}$. Then let us denote a group of vertices of P_n into q disjoint subsets G_r, these are,

$$\begin{aligned} G_1 &= \{v_1, v_2, v_3\} \\ G_2 &= \{v_4, v_5, v_6\} \\ &\vdots \\ G_q &= \{v_{n-2}, v_{n-1}, v_n\} \end{aligned}$$

Clearly, the set $S = \{v_2, v_5, \ldots, v_{n-4}, v_{n-1}\} \subseteq V(P_n)$ is a γ_{cerp}-set of P_n since $N_G[S] = V(P_n)$ and every vertex $v_j \in V(P_n)$, $j \in \{2, 5, \ldots, n-4, n-1\}$ has two neighbors in $V(P_n) \setminus S$. It follows that all other vertices $v_i \in V(P_n) \setminus S$, $i \in \{1, 3, 4, \ldots, n-3, n-2, n\}$ are dominated by exactly one vertex in S. Therefore, $\gamma_{cerp}(P_n) = |S| = \frac{n}{3}$.

Case 2. Suppose that $n \equiv 1(\text{mod}3)$. Clearly if $n = 1$, $\gamma_{cerp}(P_1) = 1$. Suppose that $n = 4$. Let $S_1 = \{v_1, v_2, v_3, v_4\}$. Observe that every vertex in S_1 has zero neighbor in $V(P_4) \setminus S_1$, this means that $\gamma_{cerp}(S_1) = |S_1| = 4$. Suppose that $n \geq 4$. Let $q = \frac{n}{4}$ and $r \in \{1, 2, \ldots, q-1, q\}$. Then let us denote a group of vertices of P_n into q disjoint subsets as G_r, these are,

$$\begin{aligned} G_1 &= \{v_1, v_2, v_3, v_4\} \\ G_2 &= \{v_5, v_6, v_7, v_8\} \\ &\vdots \\ G_q &= \{v_{n-3}, v_{n-2}, v_{n-1}, v_n\}. \end{aligned}$$

Clearly, the set $S^{'} = \bigcup_{r=1}^{n} S_r$ is a γ_{cerp}-set of P_n since $N_G[S^{'}] = V(P_n)$ and every vertex $v_j \in V(P_n)$, $j \in \{1, 2, \ldots, n-1, n\}$ has zero neighbor in $V(P_n) \setminus S^{'}$. Consequently, $\gamma_{cerp}(P_n) = |S^{'}| = |V(P_n)| = n$.

Case 3. Suppose that $n \equiv 2 \pmod 3$. Clearly, if $n = 2$, then $\gamma_{cerp}(P_2) = 2$. Suppose that $n = 5$. Let $S_1 = \{v_1, v_2, v_3, v_4, v_5\}$. Observe that every vertex in S_1 has zero neighbor in $V(P_5) \setminus S_1$. Thus, $\gamma_{cerp}(S_1) = |S_1| = 5$. Let $q = \frac{n}{5}$ and $r \in \{1, 2, \ldots, q-1, q\}$. Then let us denote a group of vertices of P_n into q disjoint subsets as G_r, these are,

$$\begin{aligned} G_1 &= \{v_1, v_2, v_3, v_4, v_5\} \\ G_2 &= \{v_6, v_7, v_8, v_9, v_{10}\} \\ &\vdots \\ G_q &= \{v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}. \end{aligned}$$

Clearly, the set $S^{''} = \bigcup_{r=1}^{n} S_r$, $r \in \{1, 2, \ldots, q-1, q\}$ is a γ_{cerp}-set of P_n since $N_G[S^{''}] = V(P_n)$ and every vertex $v_j \in V(P_n)$, $j \in \{1, 2, \ldots, n-1, n\}$ has zero neighbor in $V(P_n) \setminus S^{''}$. Consequently, $\gamma_{cerp}(P_n) = |S^{''}| = |V(P_n)| = n$.

Theorem 5. *For a cycle C_n of order $n \geq 3$,*

$$\gamma_{cerp}(C_n) = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod 3; \\ n & \text{otherwise.} \end{cases}$$

Proof. Suppose that $V(C_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ such that $\deg(v_i) = 2$, $i \in \{1, 2, \ldots, n-1, n\}$. Consider the following cases:

Case 1: Suppose that $n \equiv 0 \pmod 3$. If $n = 3$. then $\gamma_{cerp}(C_3) = 1 = \frac{3}{3}$. Suppose that $n > 3$. Let $q = \frac{n}{3}$ and $r \in \{1, 2, \ldots, q-1, q\}$. Then let us denote a group of vertices of C_n into q disjoint subsets H_r, these are,

$$H_1 = \{v_1, v_2, v_3\}$$

$$\begin{aligned} H_2 &= \{v_4, v_5, v_6\} \\ &\vdots \\ H_q &= \{v_{n-2}, v_{n-1}, v_n\} \end{aligned}$$

Clearly, the set $S = \{v_2, v_5, \ldots, v_{n-4}, v_{n-1}\} \subseteq V(C_n)$ is a γ_{cerp}-set of C_n since $N_G[S] = V(C_n)$ and every vertex $v_j \in V(C_n)$, $j \in \{2, 5, \ldots, n-4, n-1\}$ has two neighbors in $V(C_n) \backslash S$. Furthermore, every vertex $v_i \in V(C_n) \backslash S$, $i \in \{1, 3, 4, 6, \ldots, n-5, n-3, n-2, n\}$ is dominated by exactly one vertex in S. Therefore, $\gamma_{cerp}(C_n) = |S| = \frac{n}{3}$.

Case 2. Suppose that $n \equiv 1(\text{mod}3)$. Suppose that $n = 4$. Let $S_1 = \{v_1, v_2, v_3, v_4\}$. Observe that every vertex in $\in S_1$ has zero neighbor in $V(C_4) \setminus S_1$. This means that $\gamma_{cerp}(S_1) = |S_1| = 4$. Suppose that $n > 4$. Let $q = \frac{n}{4}$ and $r \in \{1, 2, \ldots, q-1, q\}$. Then let us denote a group of vertices of C_n into q disjoint subsets as H_r, these are,

$$\begin{aligned} H_1 &= \{v_1, v_2, v_3, v_4\} \\ H_2 &= \{v_5, v_6, v_7, v_8\} \\ &\vdots \\ H_q &= \{v_{n-3}, v_{n-2}, v_{n-1}, v_n\}. \end{aligned}$$

Clearly, the set $S^* = \bigcup_{r=1}^{n} S_r$ is a γ_{cerp}-set of C_n since $N_G[S^*] = V(C_n)$ and every vertex $v_j \in V(C_n)$, $j \in \{1, 2, \ldots, n-1, n\}$ has zero neighbor in $V(C_n) \setminus S^*$. Consequently, $\gamma_{cerp}(C_n) = |S^*| = |V(C_n)| = n$.

Case 3. Suppose that $n \equiv 2(\text{mod}3)$. Suppose that $n = 5$. Let $S_1 = \{v_1, v_2, v_3, v_4, v_5\}$. Observe that every vertex in $\in S_1$ has zero neighbor in $V(C_5) \setminus S_1$. Thus, $\gamma_{cerp}(S_1) = |S_1| = 5$. Suppose that $n > 5$. Let $q = \frac{n}{5}$ and $r \in \{1, 2, \ldots, q-1, q\}$. Further, let us denote a group of vertices of C_n into q disjoint subsets as H_r, these are,

$$\begin{aligned} H_1 &= \{v_1, v_2, v_3, v_4, v_5\} \\ H_2 &= \{v_6, v_7, v_8, v_9, v_{10}\} \\ &\vdots \\ H_q &= \{v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}. \end{aligned}$$

Clearly, the set $S^{**} = \bigcup_{r=1}^{n} S_r$ is a γ_{cerp}-set of C_n since $N_G[S^{**}] = V(C_n)$ and every vertex $v_j \in V(C_n)$, $j \in \{1, 2, \ldots, n-1, n\}$ has zero neighbor in $V(C_n) \setminus S^{**}$. Consequently, $\gamma_{cerp}(C_n) = |S^{**}| = |V(C_n)| = n$. $\square$

Theorem 6. *For a complete K_n of order n,*

$$\gamma_{cerp}(K_n) = \begin{cases} 1 & \text{if } n = 1 \text{or } n \geq 3; \\ 2 & \text{if } n = 2. \end{cases}$$

Proof. Suppose that K_n be a complete graph of order $n \geq 1$ such that every pair of distinct vertices are adjacent. Consider the following cases:

Case 1. Suppose that $n = 1$ or $n \geq 3$. If $n = 1$, then $\gamma_{cerp}(K_1) = 1$. Suppose that $n \geq 3$. Let $S = \{v_1\} \subseteq V(K_n)$. Since $N_{K_n}[S] = V(K_n)$, all vertices in $V(K_n) \setminus S$ are dominated by exactly one vertex $v \in S$ and vertex $v \in S$ has at least two neighbors in $V(K_n) \setminus S$. Therefore, $\gamma_{cerp}(K_n) = |S| = 1$.

Case 2. Suppose that $n = 2$. Then $\gamma_{cerp}(K_2) = 2$. □

Theorem 7. *For a complete bipartite $K_{m,n}$ with m, n vertices,*

$$\gamma_{cerp}(K_{m,n}) = \begin{cases} 1, & \textit{if } m = 1 \textit{or } n = 1; \\ 4, & \textit{if } m = n = 2. \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. Let m and n be a positive integers. Suppose that $K_{m,n}$ be a complete bipartite graph whose vertices can be partitioned into two disjoint sets such that every vertex in one set U of order m is connected to every vertex in the other set V of order n. Consider the following cases:

Case 1. Suppose that $m = 1$ or $n = 1$. Clearly, If $m = 1$, then $\gamma_{cerp}(K_{1,m}) = 1$. Similarly, if $n = 1$, then $\gamma_{cerp}(K_{m,1}) = 1$.

Case 2. Suppose that $m = n = 2$. Then $K_{m,n} = K_{2,2}$. Then $K_{2,2} \cong C_4$. Therefore, by Theorem 5, $\gamma_{cerp}(K_{2,2}) = 4$.

Case 3. Suppose that $m, n \geq 3$. Write $K_{m,n} = \overline{K_m} + \overline{K_n} = U + V$. Let $S = \{u_1, v_1\} \subseteq V(K_{m,n})$, where $u_1 \in U$ and $v_1 \in V$. Observe that for every vertex $u_i \in U \setminus S$, $i \in \{1, 2, \ldots, m-1, m\}$ is dominated by exactly one vertex $v_1 \in S$ and v_1 has at least two neighbors in $V(K_{m,n}) \setminus S$ since $m \geq 3$. Similarly, for every vertex $v_j \in V \setminus S$, $j \in \{1, 2, \ldots, n-1, n\}$is dominated by exactly one $u_1 \in S$ and u_1 has at least two neighbors in $V(K_{m,n}) \setminus S$ since $n \geq 3$. Furthermore, $N_G[S] = V(K_{m,n})$. Therefore, $\gamma_{cerp}(K_{m,n}) = |S| = 2$. □

4. Certified Perfect Domination Number in the Join of two Graphs

This section presents the outcomes obtained when the graph $G+H$ possesses a γ_{cerp}-set along with its certified perfect domination number.

Theorem 8. *Let G be a graph of order $n \geq 3$. Then $\gamma_{cerp}(G) = 1$ if and only if $G = K_1 + H$ for some graph H of order $n \geq 2$.*

Proof. Suppose $\gamma_{cerp}(G) = 1$. Then there exists a dominating set $S \subseteq V(G)$ consisting of a single vertex, that is, $S = \{v\}$. Consequently, every vertex in $V(G) \setminus S$ is dominated by $v \in S$. Therefore, S is a perfect dominating set of G. This means that $N_G(v) = |V(H)|$ for some graph H of order $n \geq 2$. Thus, S is a certified dominating set of G. From these observations, we conclude that S is a certified perfect dominating set of $G = K_1 + H$.

Conversely, suppose that $G = K_1 + H$ for some H is a graph of order $n \geq 2$. Let $S = V(K_1)$. Then $S = \{v\}$. Since every vertex in $V(H)$ is dominated by exactly one vertex $v \in S$ and vertex $v \in S$ has at least two neighbors. Therefore, $\gamma_{cerp}(G) = |S| = 1$.

Proposition 2. *If S is a dominating set or a perfect dominating set of G with $|S| = 1$, then S is a certified perfect dominating set of G. In particular, $\gamma(G) = \gamma_p(G)$ if and only if $\gamma_{cerp}(G) = 1$.*

The next result follows from Theorem 8 and Proposition 2

Corollary 1. *The following are graphs having $\gamma_{cerp}(G) = 1$:*

i. *star graph $S_n = K_1 + \overline{K_n}$, $n \geq 2$.*

ii. *fan graph $F_n = K_1 + P_n$, $n \geq 2$.*

iii. *wheel $W_n = K_1 + C_n$, $n \geq 3$.*

iv. *friendship graph $Fr_n = K_1 + nP_1$, $n \geq 2$.*

v. *windmill graph $W_n^m = K_1 + mK_{n-1}$, $m \geq 2$ and $n \geq 3$.*

vi. *complete bipartite graph $K_{m,n} = \overline{K_m} + \overline{K_n}$, $m = 1$ or $n = 1$.*

Corollary 2. *Let G and H be any graph of order m and n, respectively with $\gamma(G) = 1$ or $\gamma(H) = 1$. Then $\gamma_{cerp}(G + H) = 1$.*

Proposition 3. *Let G and H be a trivial graphs. Then $\gamma_{cerp}(G + H) = 2$.*

Proof. Clearly, If G and H are graphs of order $m = 1$ and $n = 1$, respectively. Then $\gamma_{cerp}(G + H) = 2$.

Proposition 4. *Let G and H be any connected non-trivial graph of order m and n, respectively with $\gamma(G) \neq 1$ or $\gamma(H) \neq 1$. Then $\gamma_{cerp}(G + H) = |V(G + H)|$.*

5. The Certified Perfect Domination in the Corona of Graphs

In this section presents the γ_{cerp}-set of $G \circ H$ and its certified perfect domination number.

Theorem 9. *Let G be a connected graph of order m and H be any graph of order $n \geq 2$. Then a subset S of $V(G \circ H)$ is a certified perfect dominating set of $G \circ H$ if and only if $S \cap V(v + H^v)$ is a certified perfect dominating set of $v + H^v$ for every $v \in V(G)$.*

Proof. Let $S \subseteq G \circ H$ be a certified perfect dominating set of $G \circ H$ and let $v \in V(G)$. If $v \in S$, then v is a certified perfect dominating set of $v + H^v$ since H is any graph with vertices $n \geq 2$. Suppose that $v \notin S$. Let $a \in V(v + H^v) \setminus S$, where $a \neq v$. Since S is a certified perfect dominating set of $G \circ H$, there exist $b \in S$ such that $ab \in E(G \circ H)$. This means that $b \in V(v + H^v) \cap S$ and $ab \in E(v + H^v)$. This proves that $S \cap V(v + H^v)$ is a certified perfect dominating set of $v + H^v$.

Conversely, suppose that $S \cap V(v + H^v)$ is a certified perfect dominating set of $v + H^v$ for every $v \in V(G)$. Indeed, S is a certified perfect dominating set of $G \circ H$. □

Corollary 3. *If G is a connected graph of order m and H be any graph of order $n \geq 2$. Then $\gamma_{cerp}(G \circ H) = m$.*

Proof. Let $S \subseteq V(G \circ H)$. Suppose that $S = V(G)$. Then $S \cap V(v + H^v) = v$ is a certified perfect dominating set of $v + H^v$ for every $v \in V(G)$ since H is any graph with vertices $n \geq 2$. This implies that S is a certified perfect dominating set of $G \circ H$ by Theorem 9. Consequently, $\gamma_{cerp}(G \circ H) \leq |S| = m$. Moreover, if S^* is a minimum certified perfect dominating set of $G \circ H$, then $V(S^* \cap v + H^v)$ is a certified perfect dominating set of $v + H^v$ for every $v \in V(G)$ since H is any graph with vertices $n \geq 2$ by Theorem 9. This means that $\gamma_{cerp}(G \circ H) = |S^*| \geq m$. Consequently, $\gamma_{cerp}(G \circ H) = m$. □

Theorem 10. *Let G be a connected graph of order m and H be a trivial graph. Then $\gamma_{cerp}(G \circ H) = 2m$.*

Proof. Let S_1 be a vertex set of a connected graph G of order m and S_2 be a vertex set of a trivial graph H. Since $H \cong K_1$, $V(H)$ are pendant vertices of $G \circ H$. This means that $|S_2| = |S_1| = m$. Furthermore, every vertex in $v \in V(v + H^v)$ has only one neighbor in $V(v + H^v) \setminus v$ for every $v \in S_1$. Hence, $S_1 \cup S_2$ is a certified perfect dominating set of $G \circ H$. Therefore $\gamma_{cerp}(G \circ H) = |V(G \circ H)| = |S_1| + |S_2| = m + m = 2m$ □

6. Conclusion and Recommendation

The study has introduced and examined the concept of a certified perfect dominating set, denoted as S, within the context of graph G. The author's primary focus has been on several critical areas: characterizing the certified perfect dominating set, determining precise values for the certified perfect domination number in specific graphs, and exploring the certified perfect domination number in graphs resulting from join and corona operations. Furthermore, the study has established relationships between the perfect dominating set and the certified perfect dominating set of graph G.

Researchers interested in this concept can further investigate it in various graph products that were not addressed in this paper. Additionally, they may explore and analyze its bounds in relation to other well-established parameters in graph theory.

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