Characterizations of J-Total Dominating Sets of Some Graphs

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Abstract. Let G be a graph with no isolated vertex. A subset $M \subseteq V(G)$ is called a J-open set if $N_G(a) \setminus N_G(b) \neq \emptyset$ and $N_G(b) \setminus N_G(a) \neq \emptyset \ \forall \ a, b \in M$, where $a \neq b$. If in addition, $M$ is a total dominating in $G$, then we call $M$ a J-total dominating set in $G$. The maximum cardinality among all J-total dominating set in $G$, denoted by $\gamma_{Jt}(G)$, is called the J-total domination number of $G$. In this paper, we characterize J-total dominating sets in some special graphs and join of two graphs, and we use these results to obtain formulas for the parameters of these graphs. Moreover, we determine its relationships with other known parameters in graph theory. Finally, we derive the lower bound of the parameter for the corona of two graphs.

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1. Introduction

The study of domination in graphs came about partially as a result of the study of games and recreational mathematics. In particular, mathematicians studied how chess pieces of a particular type could be placed on a chessboard in such a way that they would attack, or dominate, every square on the board.

Domination in a graph was introduced by Oystein Ore in 1962 in his book on graph theory [10]. A subset $D$ of vertices of a graph $G$ is called a dominating of $G$ if for every $x \in V(G) \setminus D$, there exists $y \in D$ such that $xy \in E(G)$, that is, $N_G[D] = V(G)$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set.
in \( G \). A decade later, Cockayne and Hedetniemi [1] published a survey paper, in which the notation \( \gamma(G) \) was first used for the domination number of a graph \( G \). Since then, several mathematicians had studied and introduced new domination parameters in graphs. Some variants of domination were defined and further studied by researchers in [2–9, 11, 12].

In this paper, new variant of domination called \( J \)-total domination in a graph will be introduced and investigated. We will characterize \( J \)-total dominating sets in some classes of graphs and the join of two graphs, and we will use these results to determine the exact value of each of these graphs. Moreover, we will determine the bound of the parameter for the corona of two graphs. We believe that this new parameter would give additional insights to researchers in the field and would help them for more research directions in the future.

2. Terminology and Notation

Let \( G = (V(G), E(G)) \) be a graph. Two vertices \( a, b \) of \( G \) are said to be adjacent, or neighbors, if \( ab \) is an edge of \( G \). The open neighborhood of \( x \) in \( G \) is the set defined by \( N_G(x) = \{ y \in V(G) : xy \in E(G) \} \). The closed neighborhood of \( x \) in \( G \) is the set \( N_G[x] = N_G(x) \cup \{ x \} \). If \( X \subseteq V(G) \), then open neighborhood of \( X \) in \( G \) is the set \( N_G(X) = \bigcup_{x \in X} N_G(x) \). The closed neighborhood of \( X \) in \( G \) is the set \( N_G[X] = N_G(X) \cup X \).

A path graph is a non-empty graph with vertex-set \( \{x_1, x_2, \ldots, x_n\} \) and edge-set \( \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\} \), where the \( x_i \)'s are all distinct. The path of order \( n \) is denoted by \( P_n \). If \( G \) is a graph and \( u \) and \( v \) are vertices of \( G \), then a path from vertex \( u \) to vertex \( v \) is sometimes called a \( u-v \) path. The cycle graph is the graph of order \( n \geq 3 \) with vertex-set \( \{x_1, x_2, \ldots, x_n\} \) and edge-set \( \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\} \). The cycle graph of order \( n \) is denoted by \( P_n \).

A graph \( G \) is connected if every pair of its vertices can be joined by a path. Otherwise, \( G \) is disconnected. A maximal connected subgraph (not a subgraph of any connected subgraph) of \( G \) is called a component of \( G \).

The distance \( d_G(u, v) \) in \( G \) of two vertices \( u, v \) is the length of a shortest \( u-v \) path in \( G \). The greatest distance between any two vertices in \( G \), denoted by \( \text{diam}(G) \), is called the diameter of \( G \).

A subset \( S \) of \( V(G) \) is called a dominating of \( G \) if for every \( x \in V(G) \setminus S \), there exists \( y \in S \) such that \( xy \in E(G) \), that is, \( N_G[S] = V(G) \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set in \( G \). Any dominating set \( S \) with cardinality equal to \( \gamma(G) \) is called a \( \gamma \)-set of \( G \).

A subset \( T \) of \( V(G) \) is called a total dominating of \( G \) if for every \( x \in V(G) \), there exists \( y \in T \) such that \( xy \in E(G) \), that is, \( N_G(T) = V(G) \). The total domination number of \( G \), denoted by \( \gamma_t(G) \), is the minimum cardinality of a total dominating set in \( G \). Any total dominating set \( T \) with cardinality equal to \( \gamma_t(G) \) is called a \( \gamma_t \)-set of \( G \).

A graph \( G \) is complete if every pair of distinct vertices of \( G \) are adjacent. A complete graph of order \( n \) is denoted by \( K_n \).

A graph \( G \) is called a bipartite graph if its vertex-set \( V(G) \) can be partitioned into two
nonempty subsets $V_1$ and $V_2$ such that every edge of $G$ has one end in $V_1$ and one end in $V_2$. The sets $V_1$ and $V_2$ are called the partite sets of $G$. If each vertex in $V_1$ is adjacent to every vertex in $V_2$, then $G$ is called a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. A star graph of order $n + 1$ is the complete bipartite graph $K_{1,n}$.

Let $G$ and $H$ be any two graphs. The join of $G$ and $H$, denoted by $G + H$, is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

The fan $F_n$ of order $n + 1$, where $n \geq 1$, is given by $F_n = K_1 + P_n$.

The corona $G$ and $H$, denoted by $G \circ H$, the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$, and then Joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. We denote by $H^v$ the copy of $H$ in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $(\{v\} + H^v)$.

3. Results

We begin this section by introducing the concept of J-total domination in a graph.

**Definition 1.** Let $G$ be a graph with no isolated vertex. A subset $M \subseteq V(G)$ is called a J-open set in $G$ if $N_G(a) \setminus N_G(b) \neq \emptyset$ and $N_G(b) \setminus N_G(a) \neq \emptyset \ \forall \ a, b \in M, a \neq b$. If in addition, $M$ is a total dominating in $G$, then we call $M$ a J-total dominating set in $G$. The maximum cardinality among all J-total dominating sets in $G$, denoted by $\gamma_Jt(G)$, is called the J-total domination number of $G$. Any J-total dominating set $M$ with $|M| = \gamma_Jt(G)$ (resp. $|M| = \gamma_t(G)$), is called a $\gamma_Jt$-set or the maximum (resp. minimum) J-total dominating set in $G$.

**Remark 1.** Let $G$ be a graph with no isolated vertex. Then each of the following is true.

(i) A total dominating set $T$ of $G$ may not be a J-open set in $G$ (hence not a J-total dominating set).

(ii) A J-open set $Q$ in $G$ may not be a total dominating set in $G$ (hence not a J-total dominating set).

(iii) A vertex set $V(G)$ of $G$ may not be a J-total dominating set in $G$.

**Proposition 1.** Let $G$ be a graph with no isolated vertex. Then

(i) $\gamma_t(G) \leq \gamma_Jt(G)$.

(ii) $2 \leq \gamma_Jt(G) \leq |V(G)|$.

**Proof.** (i) Let $G$ be a graph with no isolated vertex and let $M$ be a maximum J-total dominating set of $G$. Then $M$ is a total dominating set of $G$ (by definition). Since
\(\gamma_t(G)\) is the minimum cardinality among all total dominating sets in \(G\), it follows that \(\gamma_Jt(G) = |M| \geq \gamma_t(G)\).

(ii) Since \(\gamma_t(G) \geq 2\) for any graph \(G\) with no isolated vertex, and so \(\gamma_Jt(G) \geq 2\) by (i). Since any \(J\)-total dominating set \(M\) is always a subset of vertices \(V(G)\) of \(G\), it follows that \(\gamma_Jt(G) \leq |V(G)|\). Consequently, \(2 \leq \gamma_Jt(G) \leq |V(G)|\).

**Remark 2.** The bound given in Proposition 1 is tight. Moreover, strict inequality is attainable.

For tightness, consider the graph \(G\) given in Figure 1 below.

![Figure 1: A graph \(G\) with \(\gamma_t(G) = 5 = \gamma_Jt(G)\)](image)

Let \(S = \{a, b, c, d, e\}\). Clearly, \(S\) is the minimum total dominating set of \(G\). Thus, \(\gamma_t(G) = 5\). Observe that \(x' \in N_G(x) \setminus N_G(y)\) and \(y' \in N_G(y) \setminus N_G(x)\) for every \(x, y \in S\), where \(x \neq y\). It follows that \(N_G(x) \setminus N_G(y) \neq \emptyset\) and \(N_G(y) \setminus N_G(x) \neq \emptyset\) for every \(x, y \in S, x \neq y\). Hence, \(S\) is a \(J\)-open set in \(G\), showing that \(S\) is a \(J\)-total dominating set of \(G\). Notice that \(N_G(d'), N_G(c') \subseteq N_G(b), N_G(b'), N_G(d') \subseteq N_G(c)\) and \(N_G(e') \subseteq N_G(d)\) and \(a, b, c, d, e\) must be in any total dominating set of \(G\). Consequently, \(S\) is the maximum \(J\)-total dominating set of \(G\), and so \(\gamma_Jt(G) = 5\).

For strict inequality, consider the graph \(G'\) given in Figure 2 below.

![Figure 2: A graph \(G'\) with \(\gamma_t(G') = 3 < 7 = \gamma_Jt(G')\)](image)

Let \(T_1 = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}\) and \(T_2 = \{u_3, u_4, u_5\}\). Then \(T_2\) is a minimum total dominating set of \(G'\). Hence, \(\gamma_t(G') = 3\). Since \(T_2 \subseteq T_1\), it follows that \(T_1\) is also a total
Let \( \gamma \) of \( G \) dominating set in \( K \) be a complete graph of order \( n \geq 2 \). Then \( M \subseteq V(K_n) \) is a \( J \)-total dominating set in \( K_n \) if and only if \( |M| \geq 2 \).

Proof. Let \( M \subseteq V(K_n) \) be a \( J \)-total dominating set in \( K_n \). Then \( M \) is a total dominating set in \( K_n \). Since \( \gamma_t(K_n) = 2 \) for all \( n \geq 2 \), it follows that \( |M| \geq \gamma_t(K_n) = 2 \).

Conversely, suppose that \( M \subseteq V(K_n) \) with \( |M| \geq 2 \). Let \( V(K_n) = \{v_1, v_2, \ldots, v_n\} \) and \( M = \{v_1, v_2, \ldots, v_s\} \subseteq V(K_n) \) where \( s \in \{2, 3, \ldots, n\} \). Observe that \( v_i \in N_{K_n}(v_j) \setminus N_{K_n}(v_i) \) for all \( i \neq j, j \in \{1, 2, \ldots, s\} \), showing that \( M \) is a \( J \)-open set in \( K_n \). Since any set \( \{v_i, v_j\}, i \neq j \), is a total dominating in \( K_n \), it follows that \( M \) is \( J \)-total dominating set in \( K_n \) \( \forall n \geq 2 \). \( \square \)

Corollary 1. Let \( n \geq 2 \) be any positive integer. Then

\[ \gamma_{jH}(K_n) = n. \]

Proof. Let \( M = V(K_n) = \{a_1, a_2, \ldots, a_n\} \). Then by Theorem 1, \( M \) is \( J \)-total dominating set in \( K_n \). Thus, \( \gamma_{jH}(K_n) \geq n \). By Proposition 1, \( \gamma_{jH}(K_n) = n \). \( \square \)

Theorem 2. Let \( m \) and \( n \) be positive integers with \( 2 \leq m \leq n \). Then there exists a connected graph \( H \) such that \( \gamma_t(H) = m \) and \( \gamma_{jH}(H) = n \). That is, \( \gamma_{jH}(H) - \gamma_t(H) \) can be made arbitrarily large.

Proof. For \( m = n \), consider the graph \( H \) in Figure 3 below.
Let \( M = \{a_1, a_2, \ldots, a_{m-1}, a_m\} \). Then \( M \) is a minimum total dominating set of \( H \), and so \( \gamma_t(H) = m \). Since \( b_i, c_i \in N_H(a_i) \setminus N_H(a_j) \) for every \( i \neq j, i, j \in \{1, 2, \ldots, m\} \), it follows that \( M \) is a J-open set in \( H \). Thus, \( M \) is a J-total dominating set of \( H \). Now, observe that \( N_H(b_i), N_H(c_i) \subseteq N_H(a_{i+1}), \forall i \in \{1, 2, \ldots, m-1\}, N_H(b_m), N_H(c_m) \subseteq N_H(a_{m-1}) \) and \( a_i \) must be in any total dominating set of \( H \) for each \( i \in \{1, 2, \ldots, m\} \). Therefore, \( M \) is the maximum J-total dominating set of \( H \), and so \( \gamma_{Jt}(H) = m \).

Suppose that \( m < n \). Let \( s = n - m \) and consider the graph \( H' \) in Figure 4 below, where \( \langle\{a_m, b_1, b_2, \ldots, b_s\}\rangle \) induced a complete graph for all positive integer \( s \geq 1 \).
dominating set of \( H' \). Consequently, \( \gamma_{Jt}(H') = s + m = n \).

**Theorem 3.** Let \( G \) be a graph with no isolated vertex. Then

(i) \( M \) is a \( \gamma_t \)-set in \( G \) if and only if \( M \) is a minimum \( J \)-total dominating set in \( G \).

(ii) If every component of \( G \) is non-trivial complete graph, then \( \gamma_{Jt}(G) = |V(G)| \). However, the converse is not true.

**Proof.** (i) Let \( M \) be a \( \gamma_t \)-set in \( G \). Then \( M \) is the minimum total dominating set in \( G \). Suppose that \( M \) is not a \( J \)-open set in \( G \). Then there exist \( a, b \in M \) such that \( N_G(a) \setminus N_G(b) = \emptyset \) or \( N_G(b) \setminus N_G(a) = \emptyset \). It follows that \( N_G(a) \subseteq N_G(b) \) or \( N_G(b) \subseteq N_G(a) \). Assume that \( N_G(a) \subseteq N_G(b) \), then \( M \setminus \{a\} \) is a total dominating set in \( G \). However, this is a contradiction to our assumption that \( M \) is the minimum total dominating set in \( G \). Hence, \( M \) is a \( J \)-open set in \( G \), and so \( M \) is a minimum \( J \)-total dominating set in \( G \).

The converse is clear.

(ii) Suppose that every component \( H \) of \( G \) is a non-trivial complete graph. Let \( H_1, \ldots, H_k, k \geq 2 \) be components of \( G \). Then by Corollary 1, \( \gamma_{Jt}(H_i) = |V(H_i)| \). It follows that
\[
\gamma_{Jt}(G) = \gamma_{Jt}(H_1) + \cdots + \gamma_{Jt}(H_k) = |V(H_1)| + \cdots + |V(H_k)| = |V(G)|.
\]

To see that the converse is not true, consider \( G = C_5 \) below.

![C5 graph](image)

**Figure 5:** A graph \( C_5 \) with \( \gamma_{Jt}(C_5) = 5 \)

Let \( M = \{a_1, a_2, \ldots, a_5\} = V(G) \). Observe that
\[
\begin{align*}
a_2 & \in N_G(a_1) \setminus N_G(a_i) \forall i \neq 3, \\
a_5 & \in N_G(a_1) \setminus N_G(a_j) \forall j \neq 4, \\
a_1 & \in N_G(a_2) \setminus N_G(a_k) \forall k \neq 5, \\
a_3 & \in N_G(a_2) \setminus N_G(a_e) \forall e \neq 4, \\
a_2 & \in N_G(a_3) \setminus N_G(a_s) \forall s \neq 1,
\end{align*}
\]
Theorem 4. Let $K_{m,n}$ be a complete bipartite graph where $m, n \geq 1$. Then $N \subseteq V(K_{m,n})$ is a $J$-total dominating in $K_{m,n}$ if and only if $N = \{a, b\}$ for some $a \in V(K_m)$ and $b \in V(K_n)$.

Proof. Let $N \subseteq V(K_{m,n})$ be a $J$-total dominating in $K_{m,n}, m, n \geq 1$. Then $N$ is a total dominating set in $K_{m,n}$. Thus, $|N| \geq 2$. Let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Observe that $N_{K_{m,n}}(u_i) = N_{K_{m,n}}(u_j) \forall i \neq j$, $i, j \in \{1, 2, \ldots, m\}$ and $N_{K_{m,n}}(v_r) = N_{K_{m,n}}(v_q) \forall r \neq q$, $r, q \in \{1, 2, \ldots, n\}$. This means that there are $m - 1$ and $n - 1$ vertices of $K_m$ and $K_n$, respectively, cannot be in any $J$-total dominating set of $K_{m,n}$. Hence, $|N| \leq 2$, and so $|N| = 2$. Thus, $N = \{a, b\}$ for some $a \in V(K_m)$ and $b \in V(K_n)$.

Conversely, let $N = \{a, b\}$ for some $a \in V(K_m)$ and $b \in V(K_n)$. Then $N_{K_{m,n}}(a) = V(K_n)$ and $N_{K_{m,n}}(b) = V(K_m)$. Hence, $N_{K_{m,n}}(N) = V(K_{m,n})$, and $N_{K_{m,n}}(a) \setminus N_{K_{m,n}}(b) = V(K_n) \neq \emptyset$ and $N_{K_{m,n}}(b) \setminus N_{K_{m,n}}(a) = V(K_m) \neq \emptyset$. Consequently, $N = \{a, b\}$ is a $J$-total dominating set in $K_{m,n}$.

The following result follows immediately for Theorem 4.

Corollary 2. Let $m, n \geq 1$ be positive integers. Then $\gamma_{Jt}(K_{m,n}) = 2$.

Theorem 5. Let $m \geq 2$ be positive integer. Then

$$\gamma_{Jt}(P_m) = \begin{cases} 2 & \text{if } m = 2, 3, 4 \\ 4 & \text{if } m = 5 \\ m - 2 & \text{if } m \geq 6. \end{cases}$$

Proof. Clearly, $\gamma_{Jt}(P_2) = 2$. For $m = 3$, let $V(P_3) = \{v_1, v_2, v_3\}$ and let $S = \{v_1, v_2\}$. Then $v_2 \in N_{P_3}(v_1) \setminus N_{P_3}(v_2)$ and $v_1 \in N_{P_3}(v_2) \setminus N_{P_3}(v_1)$. Thus, $S$ is a $J$-open set in $P_3$. Since $N_{P_3}(S) = V(P_3)$, it follows that $S$ is a $J$-total dominating set of $P_3$. Notice that $N_{P_3}(v_1) = N_{P_3}(v_3)$. Hence, $v_1$ and $v_3$ cannot be both in any $J$-open set of $P_3$. Therefore, $S = \{v_1, v_2\}$ is a maximum $J$-total dominating set in $P_3$, showing that $\gamma_{Jt}(P_3) = 2$.

For $m = 4$, let $V(P_4) = \{a_1, a_2, a_3, a_4\}$ and $S' = \{a_2, a_3\}$. Then

$$a_1, a_3 \in N_{P_4}(a_2) \setminus N_{P_4}(a_3)$$

and

$$a_2, a_4 \in N_{P_4}(a_3) \setminus N_{P_4}(a_2).$$
Thus, $S'$ is a $J$-open set in $P_3$. Observe that $N_{P_3}(S') = V(P_3)$. Therefore, $S'$ is a $J$-total dominating set in $P_3$. Notice that $N_{P_3}(a_1) \subseteq N_{P_3}(a_3)$ and $N_{P_3}(a_4) \subseteq N_{P_3}(a_2)$. This means that $a_1$ and $a_3$ (resp. $a_2$ and $a_4$) cannot be both in any $J$-open set of $P_3$. Consequently, $S' = \{v_2, v_3\}$ is a maximum $J$-total dominating set of $P_3$, and so $\gamma_{Jt}(P_3) = 2$.

For $m = 5$, let $V(P_5) = \{u_1, u_2, u_3, u_4, u_5\}$ and consider $C = \{u_1, u_2, u_4, u_5\}$. Then $u_1 \in N_{P_5}(u_2) \setminus N_{P_5}(u_1) \quad \forall \ i \neq 2, u_2 \in N_{P_5}(u_1) \setminus N_{P_5}(u_j) \quad \forall j \neq 1, u_4 \in N_{P_5}(u_5) \setminus N_{P_5}(u_r) \quad \forall r \neq 5$ and $u_5 \in N_{P_5}(u_1) \setminus N_{P_5}(u_q) \quad \forall q \neq 4$. Hence, $C$ is a $J$-open set in $P_5$. Since $N_{P_5}(C) = V(5)$, it follows that $C$ is a $J$-total dominating set of $P_5$. Notice that $N_{P_5}(u_1) \subseteq N_{P_5}(u_3)$. Thus, $u_1$ and $u_3$ cannot be both in any $J$-open set of $P_5$. Consequently, $C$ is a maximum $J$-total dominating set of $P_5$, and so $\gamma_{Jt}(P_5) = 4$.

Next, suppose that $m \geq 6$. Let $V(P_m) = \{w_1, w_2, \ldots, w_m\}$ and consider $C' = \{w_2, w_3, \ldots, w_{m-2}, w_{m-1}\}$. Notice that $w_{i-1} \in N_{P_m}(w_i) \setminus N_{P_m}(w_j)$ and $w_{j+1} \in N_{P_m}(w_j) \setminus N_{P_m}(w_i) \quad \forall i < j, i, j \in \{2, 3, \ldots, m-1\}$. It follows that $N_{P_m}(w_i) \setminus N_{P_m}(w_j) \neq \emptyset \quad \forall i \neq j, i, j \in \{2, 3, \ldots, m-1\}$.

Thus, $C'$ is a $J$-open set in $P_m$ for all $m \geq 6$. Since $N_{P_m}(C') = V(P_m)$, $C'$ is a $J$-total dominating set of $P_m$. Now, observe that $N_{P_m}(w_1) \subseteq N_{P_m}(w_3)$ and $N_{P_m}(w_m) \subseteq N_{P_m}(w_{m-2})$. Hence, $w_1$ and $w_3$ (resp. $w_{m-2}$ and $w_m$) cannot be both in any $J$-open set of $P_m$. Therefore, $C'$ is a maximum $J$-total dominating set of $P_m$, and so $\gamma_{Jt}(P_m) = m - 2$ for all $m \geq 6$.

\begin{align*}
\gamma_{Jt}(F_n) = \begin{cases} 
2 & \text{if } n = 1 \\
3 & \text{if } n = 2, 3, 4 \\
5 & \text{if } n = 5 \\
n - 1 & \text{if } n \geq 6
\end{cases}
\end{align*}

\textbf{Proof.} Since $F_1$ and $F_2$ are complete graphs, $\gamma_{Jt}(F_1) = 2$ and $\gamma_{Jt}(F_2) = 3$ by Corollary 1. For $n = 3$, let $V(F_3) = \{v_0, v_1, v_2, v_3\}$, where $v_0$ is the dominating vertex of $F_3$. Consider $M = \{v_0, v_1, v_2\}$. Then $v_1 \in N_{F_3}(v_0) \setminus N_{F_3}(v_i)$ and $v_2 \in N_{F_3}(v_i) \setminus N_{F_3}(v_0) \quad \forall i \neq 0$, and $v_3 \in N_{F_3}(v_1) \setminus N_{F_3}(v_2)$ and $v_2 \in N_{F_3}(v_2) \setminus N_{F_3}(v_1)$. Thus, $M$ is a $J$-open set in $F_3$. Since $N_{F_3}(M) = V(F_3)$, it follows that $M$ is a $J$-total dominating set of $F_3$. Since $N_{F_3}(v_1) = N_{F_3}(v_3)$, $v_1$ and $v_3$ cannot be both in any $J$-open set of $F_3$. Therefore, $M$ is a maximum $J$-total dominating set of $F_3$, and so $\gamma_{Jt}(F_3) = 3$. Similarly, if $n = 4$, then $\gamma_{Jt}(F_4) = 3$.

For $n = 5$, let $V(F_5) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where $v_0$ is the dominating vertex of $F_5$. Let $M' = \{v_0, v_1, v_2, v_4, v_5\}$. Then $v_j \in N_{F_5}(v_0) \setminus N_{F_5}(v_j)$ and $v_0 \in N_{F_5}(v_j) \setminus N_{F_5}(v_0) \quad \forall j \neq 0$. Since $\{v_1, v_2, v_4, v_5\}$ is a $J$-open set in $F_5$ by Theorem 5, it follows that $M'$ is a $J$-open set in $F_5$. Notice that $N_{F_5}(M') = V(F_5)$. Thus, $M'$ is a $J$-total dominating set of $F_5$. Since $N_{F_5}(v_1) \subseteq N_{F_5}(v_3)$, it follows that $v_1$ and $v_3$ cannot be both in any $J$-open set of $F_5$. Therefore, $M'$ is a maximum $J$-total dominating set of $F_5$, and so $\gamma_{Jt}(F_5) = 5$. 

\begin{align*}
\gamma_{Jt}(F_n) = \begin{cases} 
2 & \text{if } n = 1 \\
3 & \text{if } n = 2, 3, 4 \\
5 & \text{if } n = 5 \\
n - 1 & \text{if } n \geq 6
\end{cases}
\end{align*}
Next, suppose that \( n \geq 6 \). Let \( V(F_n) = \{u_0, u_1, \ldots, u_n\} \), where \( u_0 \) is the dominating vertex of \( F_n \). Let \( C = \{u_0, u_2, u_3, \ldots, u_{n-1}\} \). Observe that \( u_r \in N_{F_n}(u_0) \setminus N_{F_n}(u_r) \) and \( u_0 \in N_{F_n}(u_r) \setminus N_{F_n}(u_o) \) \( \forall r \neq 0 \). Since \( \{u_2, u_3, \ldots, u_{n-1}\} \) is a \( J \)-open set in \( P_n \) by Theorem 5, it follows that \( C \) is a \( J \)-open set in \( F_n \). Observe further that \( N_{F_n}(C) = V(F_n) \). Hence, \( C \) is a \( J \)-total dominating set of \( F_n \). Since \( N_{F_n}(u_1) \subseteq N_{F_n}(u_3) \) and \( N_{F_n}(u_n) \subseteq N_{F_n}(u_{n-2}) \), \( u_1 \) and \( u_3 \) (resp. \( u_{n-2} \) and \( u_n \)) cannot be both in any \( J \)-open set of \( F_n \). Therefore, \( C \) is a maximum \( J \)-total dominating set of \( F_n \), showing that \( \gamma_J(F_n) = n - 1 \) for all \( n \geq 6 \). \( \square \)

**Theorem 7.** Let \( G \) and \( H \) be two graphs with no isolated vertices. A subset \( M \) of vertices of \( G + H \) is a \( J \)-total dominating set of \( G + H \) if and only if one of the following conditions holds:

(i) \( M \) is a \( J \)-total dominating set of \( G \).

(ii) \( M \) is a \( J \)-total dominating set of \( H \).

(iii) \( M = M_G \cup M_H \), where \( M_G \) and \( M_H \) are \( J \)-open sets in \( G \) and \( H \), respectively.

**Proof.** Let \( M \) be a \( J \)-total dominating set of \( G + H \). If \( M_H = \emptyset \), then \( M = M_G \) is a \( J \)-total dominating set in \( G \). Thus, (i) holds. If \( M_G = \emptyset \), then \( M = M_H \) is a \( J \)-total dominating set in \( H \), and hence (ii) holds. Next, assume that \( M_G \) and \( M_H \) are both non-empty. Suppose on the contrary that \( M_G \) is not a \( J \)-open set in \( G \). Then there exist \( a, b \in M_G \subseteq M \) such that either \( N_G(a) \setminus N_G(b) = \emptyset \) or \( N_G(b) \setminus N_G(a) = \emptyset \). Thus, \( N_{G+H}(a) \setminus N_{G+H}(b) = \emptyset \) or \( N_{G+H}(b) \setminus N_{G+H}(a) = \emptyset \), a contradiction to the fact that \( M \) is a \( J \)-open set in \( G + H \). Therefore, \( D_G \) is a \( J \)-open set in \( G \). Similarly, \( M_H \) is a \( J \)-open set in \( H \). Consequently, (iii) holds.

Conversely, if (i) or (ii) holds, then the assertion follows. Next, suppose that (iii) holds. Since \( M_G \) and \( M_H \) are both non-empty, it follows that \( M \) is a total dominating set in \( G + H \). Let \( a, b \in M \). Suppose that \( a, b \in M_G \subseteq M \). Since \( M_G \) is a \( J \)-open set in \( G \), we have \( N_G(a) \setminus N_G(b) = \emptyset \) and \( N_G(b) \setminus N_G(a) = \emptyset \). It follows that \( N_{G+H}(a) \setminus N_{G+H}(b) = \emptyset \) and \( N_{G+H}(b) \setminus N_{G+H}(a) = \emptyset \). Hence \( M \) is a \( J \)-open set in \( G + H \). Similarly, if \( a, b \in M_H \subseteq M \), then \( M \) is a \( J \)-open set in \( G + H \). Now, assume that \( a \in M_G \) and \( b \in M_H \). If \( a \) is a dominating vertex of \( G \), then we are done. Similarly, if \( b \) a dominating vertex of \( H \). Suppose that \( a \) and \( b \) are not dominating vertices of \( G \) and \( H \), respectively. Let \( x \in V(G) \) and \( y \in V(H) \), where \( x \notin N_G(a) \) and \( y \notin N_H(b) \). Then \( y \in N_{G+H}(a) \setminus N_{G+H}(b) \) and \( x \in N_{G+H}(b) \setminus N_{G+H}(a) \). Thus, \( M \) is a \( J \)-open set in \( G + H \). Consequently, \( D \) is a \( J \)-total dominating set in \( G + H \). \( \square \)

The following result follows immediately from Theorem 7.

**Corollary 3.** Let \( G \) and \( H \) be two graphs with no isolated vertices. Then

\[
\gamma_J(G + H) = \gamma_J(G) + \gamma_J(H).
\]
**Theorem 8.** Let $G$ be a connected non-trivial graph and $H$ be a graph with no isolated vertex. If $M = V(G) \cup \left( \bigcup_{v \in V(G)} M_v \right)$, where $M_v$ is a $J$-total dominating set in $H^v$ for each $v \in V(G)$, then $M$ is a $J$-total dominating set in $G \circ H$. Moreover,

$$\gamma_{Jt}(G \circ H) \geq |V(G)| + \gamma_{Jt}(H) \cdot |V(G)|.$$ 

**Proof.** Let $M = V(G) \cup \left( \bigcup_{v \in V(G)} M_v \right)$, where $M_v$ is a $J$-total dominating set in $H^v$ for each $v \in V(G)$. Since $G$ is connected, it follows that $V(G)$ is a total dominating set in $G$. Moreover, since $M_v$ is a total dominating set in $H^v$ for each $v \in V(G)$, $M$ is a total dominating set in $G \circ H$. Now, let $a, b \in M$. If $a, b \in M_u$ for some $u \in V(G)$, then $N_H(a) \setminus N_H(b) \neq \emptyset$ and $N_H(b) \setminus N_H(a) \neq \emptyset$. It follows that $N_{G \circ H}(a) \setminus N_{G \circ H}(b) \neq \emptyset$ and $N_{G \circ H}(b) \setminus N_{G \circ H}(a) \neq \emptyset$. Thus, $M$ is a $J$-open set in $G \circ H$. Similarly, if $a, b \in V(G)$, then $M$ is a $J$-open set in $G \circ H$. Assume that $a \in M_u$ and $b \in M_t$ for some $s, t \in V(G), s \neq t$. Then $s \in N_{G \circ H}(a) \setminus N_{G \circ H}(b)$ and $t \in N_{G \circ H}(b) \setminus N_{G \circ H}(a)$, hence we are done. Suppose that $a \in M_w$ for some $w \in V(G)$ and $b \in V(G)$. If $w = b$, then $b \in N_{G \circ H}(a) \setminus N_{G \circ H}(b)$ and $a \in N_{G \circ H}(b) \setminus N_{G \circ H}(a)$, and we are done. Suppose $w \neq b$. Since $H$ is graph with no isolated vertex, there exists $q \in H^w$ such that $q \in N_{G \circ H}(a) \setminus N_{G \circ H}(b)$. Clearly, $N_{G \circ H}(b) \setminus N_{G \circ H}(a) = M_b \neq \emptyset$. Therefore, $M$ is a $J$-open set in $G \circ H$, showing that $M$ is a $J$-total dominating set in $G \circ H$. Consequently,

$$\gamma_{Jt}(G \circ H) \geq |V(G)| + \gamma_{Jt}(H) \cdot |V(G)|.$$ 

\[ \square \]

4. Conclusion

The concept of $J$-total domination has been introduced and investigated in this study. Characterizations of $J$-total dominating sets in some graphs and join of two graphs are formulated and were used to solve exact values of the parameters of these graphs. Some bounds and relationships of this newly defined parameter have been established. Other graphs that were not considered in this study could be an interesting topic to consider by researchers for further investigation of the concept. They may also consider the bounds of the parameter with respect to other well known parameters in graph theory.

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